Nonparametric Analysis of Dynamical Systems

From Recurrent Sets to Generalized Lyapunov and Barrier Conditions

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ESE Fall Colloquium, UPenn

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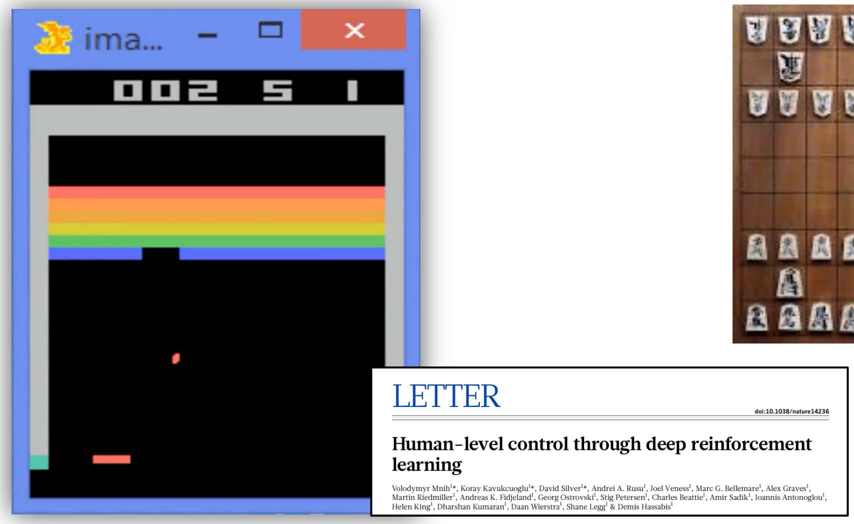
A World of Success Stories

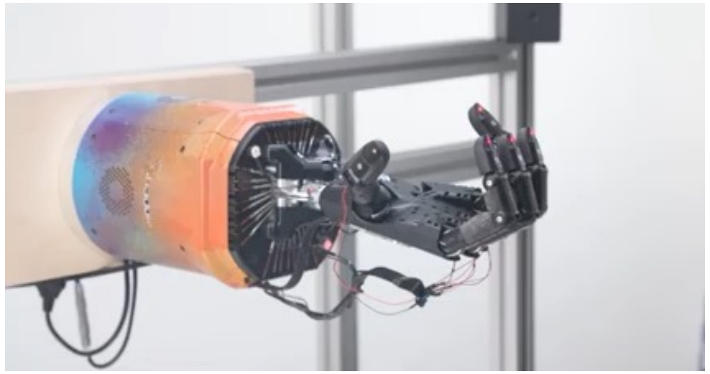
2017 Google DeepMind's DQN



2017 AlphaZero – Chess, Shogi, Go

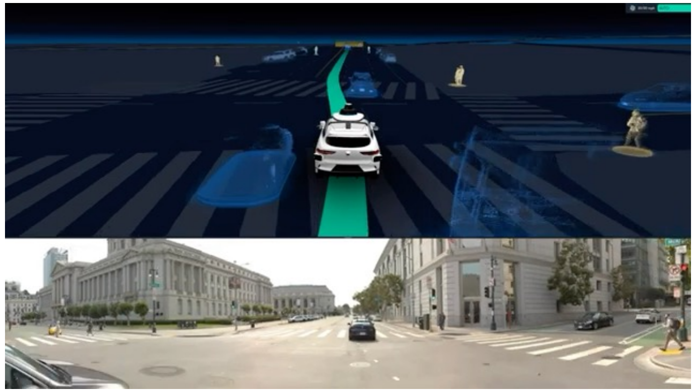
2019 AlphaStar – Starcraft II

OpenAI – Rubik's Cube



Boston Dynamics

Waymo



The Need for Safety Guarantees

Angry Residents, Abrupt Stops: Waymo Vehicles Are Still Causing Problems in Arizona

RAY STERN | MARCH 31, 2021 | 8:26AM

GARY MARCUS BUSINESS 08.14.2019 09:00 AM

DeepMind's Losses and the Future of Artificial Intelligence

Alphabet's DeepMind unit, conqueror of Go and other games, is losing lots of money. Continued deficits could imperil investments in AI.

AARIAN MARSHALL BUSINESS 12.07.2020 04:06 PM

Uber Gives Up on the Self-Driving Dream

The ride-hail giant invested more than $1 billion in autonomous vehicles. Now it's selling the unit to Aurora, which makes self-driving tech.

Tesla Recalls Nearly All Vehicles Due to Autopilot Failures

Tesla disagrees with feds' analysis of glitches

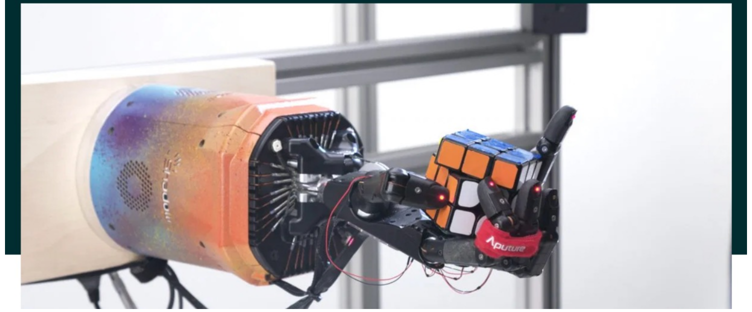
BY LINA FISHER, 2:54PM, WED. DEC. 13, 2023

CRUISE KNEW ITS SELF-DRIVING CARS HAD PROBLEMS RECOGNIZING CHILDREN — AND KEPT THEM ON THE STREETS

According to internal materials reviewed by The Intercept, Cruise cars were also in danger of driving into holes in the road.

OpenAI disbands its robotics research team

Kyle Wiggers @Kyle_L_Wiggers July 16, 2021 11:24 AM

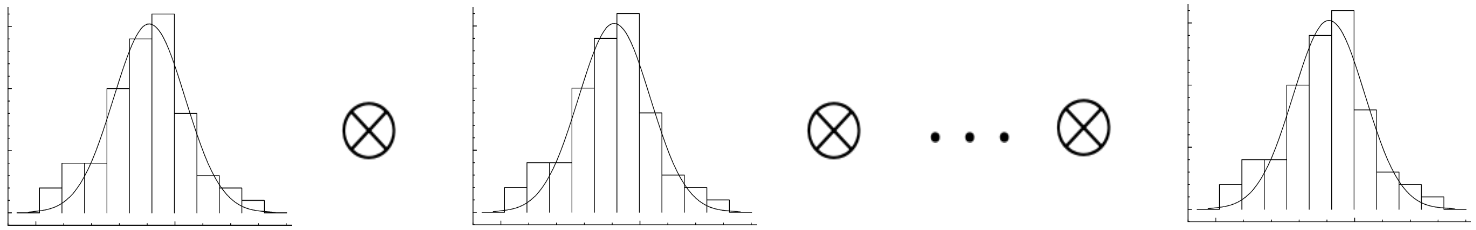


Self-driving Uber car that hit and killed woman did not recognize that pedestrians jaywalk

The automated car lacked "the capability to classify an object as a pedestrian unless that object was near a crosswalk," an NTSB report said.

Core challenge: The curse of dimensionality

- **Statistical: Sampling in d dimension with resolution ϵ**



Sample complexity:

$$O(\varepsilon^{-d})$$

For $\epsilon = 0.1$ and $d = 100$, we would need 10^{100} points.

Atoms in the universe: 10^{78}

- **Computational: Verifying non-negativity of polynomials**

Copositive matrices:

$$[x_1^2 \dots x_d^2] A [x_1^2 \dots x_d^2]^{\mathrm{T}} \geq 0$$

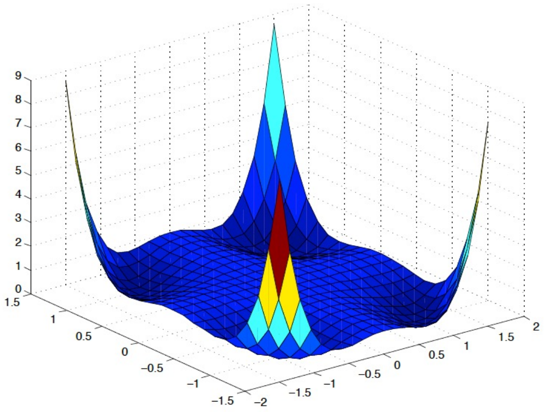
Murty&Kadabi [1987]: Testing co-positivity is NP-Hard

Sum of Squares (SoS):

$$z(x)^T Q z(x) \geq 0, \quad z_i(x) \in \mathbb{R}[x], \; x \in \mathbb{R}^d, Q \succcurlyeq 0$$

Artin [1927] (Hilbert's 17th problem):

Non-negative polynomials are sum of square of *rational* functions



Motzkin [1967]:

$p = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$

is nonnegative,

not a sum of squares,

but $(x^2 + y^2)^2 p$ is SoS

Question: Are we asking too much?

- Analysis tools build on a strict and exhaustive notion of ***invariance***

  Q: Can we substitute invariance with less restrictive notions?

  [arXiv '22] Shen, Bichuch, M - [CDC '23] Siegelmann, Shen, Paganini, M – [Allerton '24] Shen, Sibai, M

- Certificates impose conditions on the entire duration of the trajectory

  Q: Can we provide guarantees using time-localized trajectory information?

  [arXiv '22] Shen, Bichuch, M - [CDC '23] Siegelmann, Shen, Paganini, M – [Allerton '24] Shen, Sibai, M

- Analysis/synthesis usually aims for the ***best*** (optimal) certificate/controller

  Q: Is there any gain in focusing on weaker requirements from the get-go?

  [HSCC 24] Sibai, M - [CDC '23] Siegelmann, Shen, Paganini, M

[arXiv 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* **CDC 2022,** journal preprint arXiv:2204.10372.
[CDC 23] Siegelmann, Shen, Paganini, M, *A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions*, **CDC 2023**
[HSCC 24] Sibai, M, *Recurrence of nonlinear control systems: Entropy and bit rates,* **HSCC, 2024**
[Allerton 24] Shen, Sibai, M, *Generalized Barrier Functions: Integral Conditions & Recurrent Relaxations,* **Allerton 2024**

Outline

- Invariance: Merits and trade-offs
- Letting things go and come back: Recurrent sets
  - Approximating regions of attractions via recurrent sets
- Non-parametric analysis of dynamical systems
  - Stability analysis via non-monotonic Lyapunov conditions
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Problem setup

Continuous time dynamical system: $\dot{x}(t) = f(x(t))$

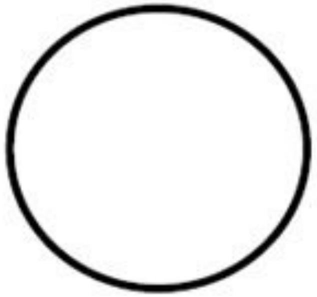
- Initial condition $x_0 = x(0)$, solution at time t: $\phi(t, x_0)$.

Ω-Limit Set $\Omega(f)$:

$$x \in \Omega(f) \iff \exists\, x_0, \{t_n\}_{n \geq 0}, \text{ s.t. } \lim_{n\to\infty} t_n = \infty \text{ and } \lim_{n\to\infty} \phi(t_n, x_0) = x$$

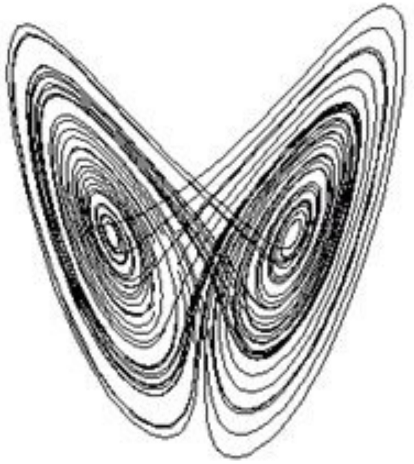
Types of Ω-limit set

equilibrium



limit cycle

limit torus



chaotic attractor

Problem setup

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- Initial condition $x_0 = x(0)$, solution at time t: $\phi(t, x_0)$.
- The ω-limit set of the system: $\Omega(f)$

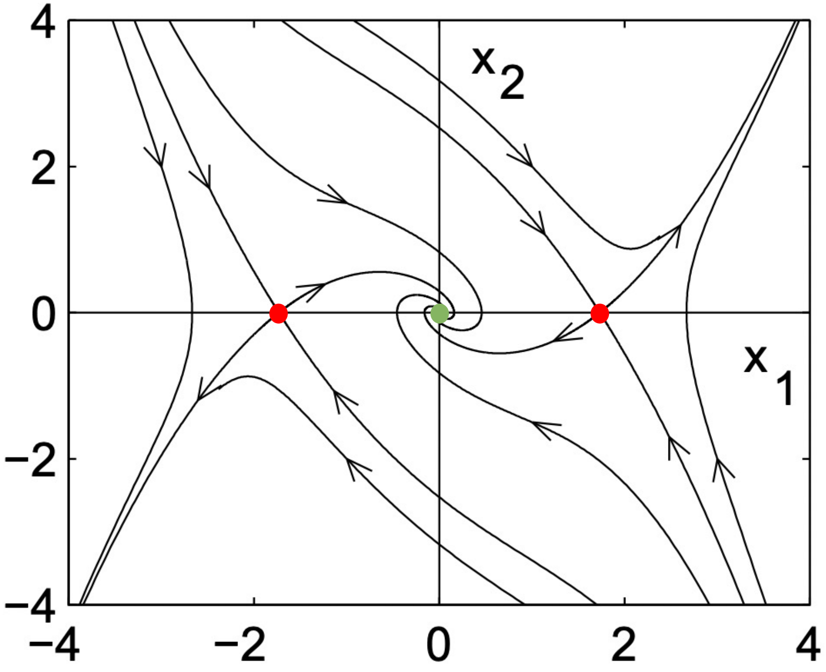
Region of attraction (ROA) of a set $S \subseteq \Omega(f)$:

$$\mathcal{A}(S) := \left\{ x \in \mathbb{R}^d \middle| \liminf_{t \to \infty} d(\phi(t, x), S) = 0 \right\}$$

Illustrative Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 + \frac{1}{3}x_1^3 - x_2 \end{bmatrix}$$

$$\Omega(f) = \{(0,0), (-\sqrt{3}, 0), (\sqrt{3}, 0)\}$$



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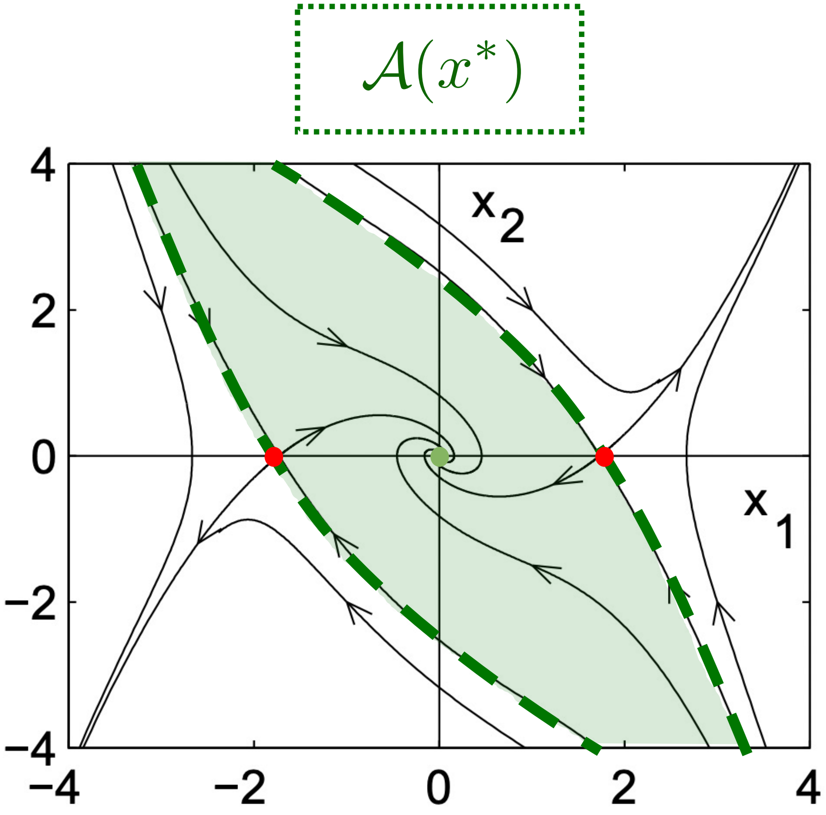
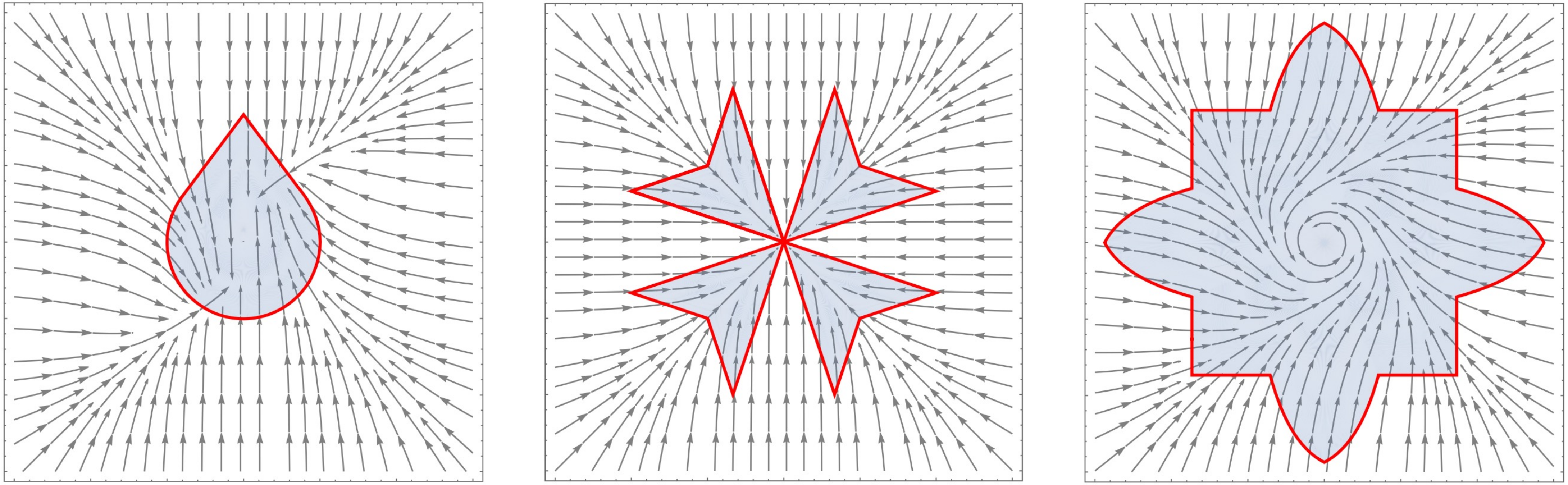
Asymptotically stable equilibrium at $x^* = (0,0)$



Analysis of Dynamical Systems via Invariant sets

A set $\mathcal{S} \subseteq \mathbb{R}^d$ is **positively invariant** if and only if: $x_0 \in \mathcal{S} \rightarrow \phi(t, x_0) \in \mathcal{S},\ \forall t \geq 0$

Any trajectory starting in the set remains in inside it for all times



Source: K. Ghorbal, K. and A. Sogokon, Characterizing positively invariant sets: Inductive and topological methods. Journal of Symbolic Computation, 2022

Invariant sets: Merits

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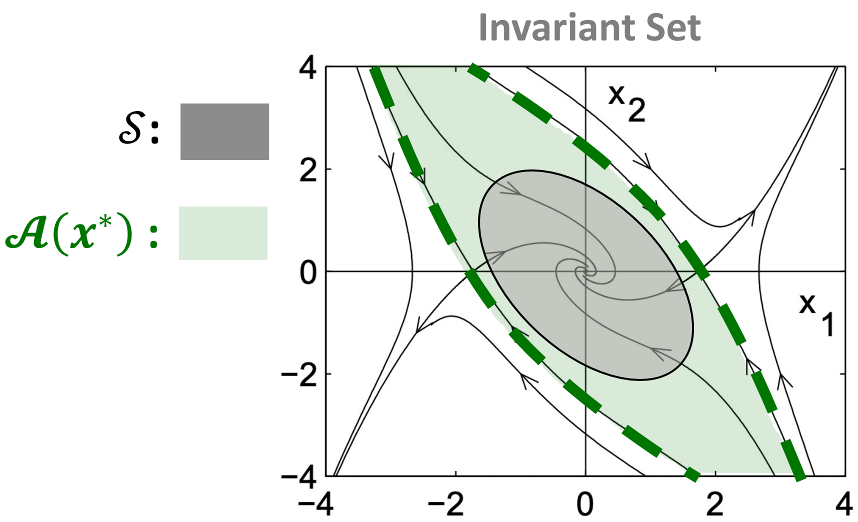
- **Invariant sets approximate regions of attraction**
  Compact invariant set $\mathcal{S}$, containing **only** $\{x^*\} = \Omega(f) \cap \mathcal{S}$ must be in the region of attraction $\mathcal{A}(x^*)$ ($\mathcal{S} \subset \mathcal{A}(x^*)$)



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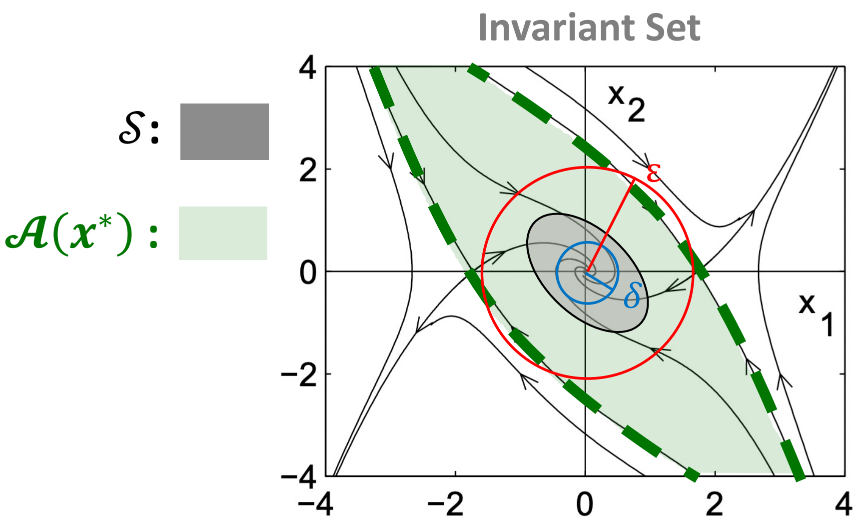

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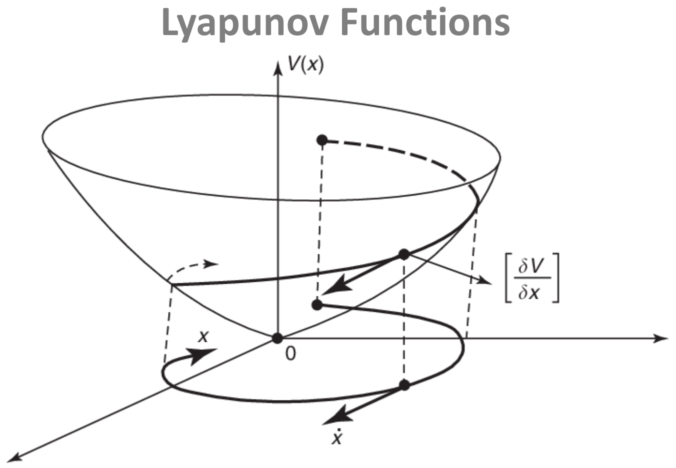
- **Invariant sets guarantee stability**
  Lyapunov stability: solutions starting "close enough" to the equilibrium (within a distance δ) remain "close enough" forever (within a distance ε)



- **Invariant sets further certify asymptotic stability via Lyapunov's direct method**
  Asymptotic stability: solutions that start close enough, remain close enough, and eventually converge to equilibrium.

Invariant sets: Challenges

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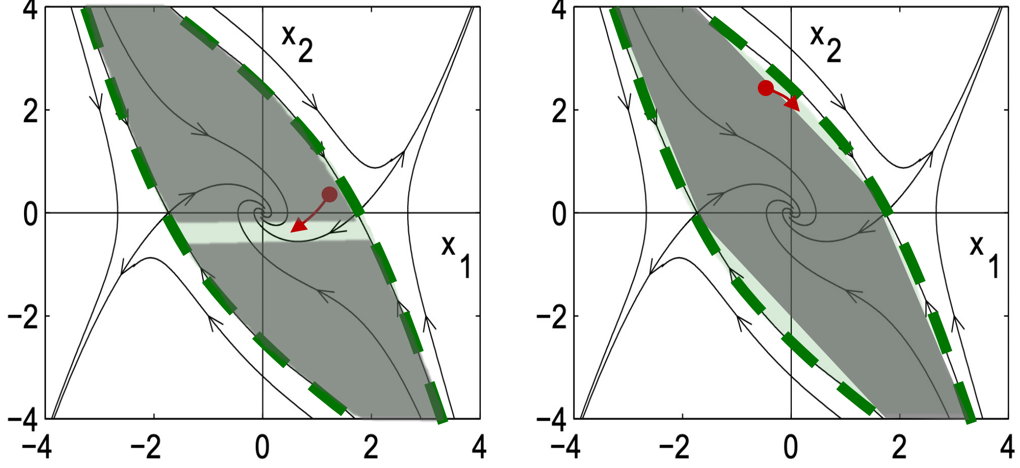
$\mathcal{S}$:

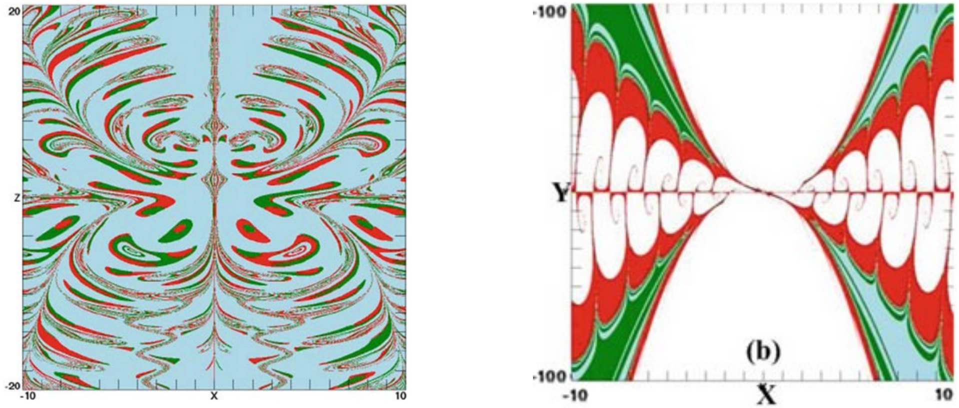
$\mathcal{A}(x^*)$:



A not invariant trajectory:

- **$\mathcal{S}$ is topologically constrained**
  - If $\mathcal{S} \cap \Omega(f) = \{x^*\}$, then $\mathcal{S}$ is connected
- **$\mathcal{S}$ is geometrically constrained**
  - f should not point outwards for $x \in \partial\mathcal{S}$
- **$\mathcal{S}$ geometry can be wild**
  - $\mathcal{A}(\Omega(f))$ is not necessarily analytic!

Basin of $\Omega(f)$



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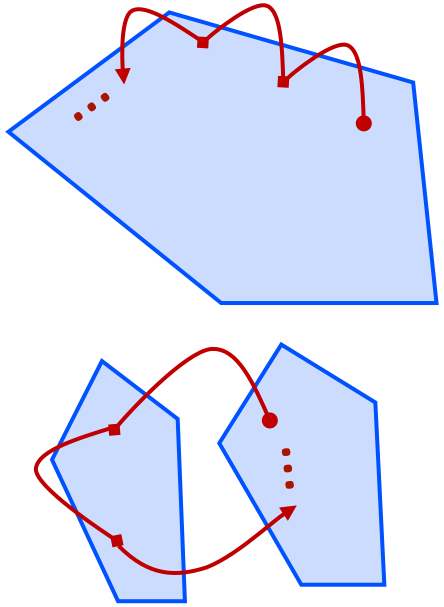
Recurrent sets: Letting things go, and come back

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \geq t$ s.t. $\phi(t', x_0) \in \mathcal{R}$.



Property of Recurrent Sets

- $\mathcal{R}$ need **not** be **connected**
- $\mathcal{R}$ does **not** require f to **point inwards** on all $\partial\mathcal{R}$

> Recurrent sets, while not invariant, guarantee that solutions that start in this set, will come back **infinitely often, forever!**

Recurrent set $\mathcal{R}$:

A recurrent trajectory:

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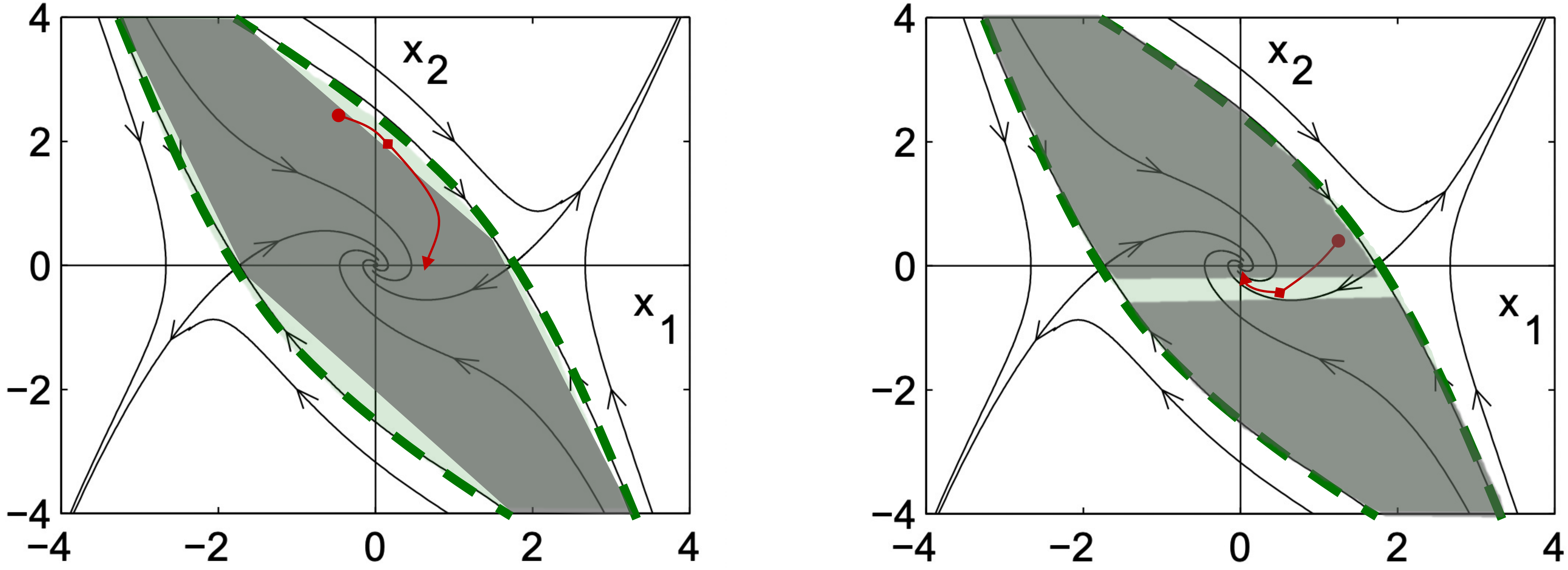
Previous two good inner approximations of $\mathcal{A}(x^*)$ are recurrent sets



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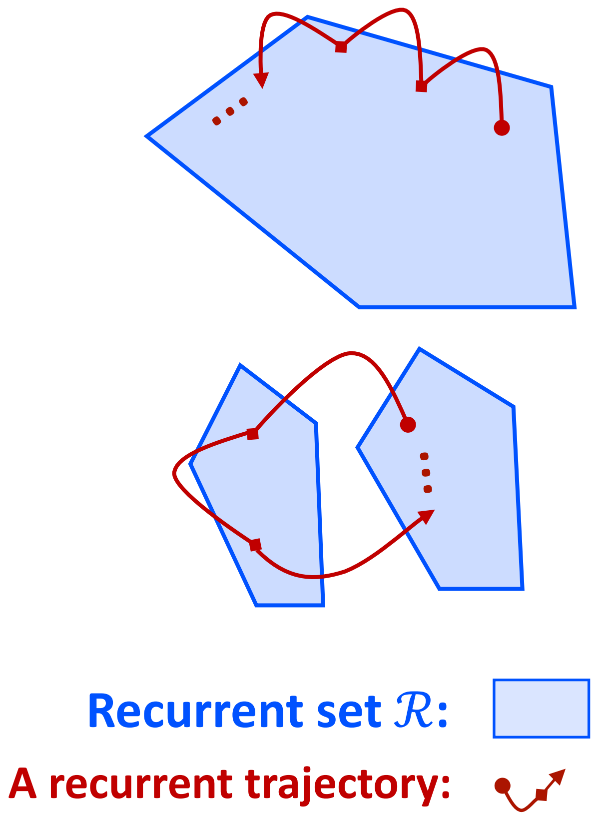

Question: Can we use recurrent sets as functional substitutes of invariant sets?

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University at Buffalo

Model-free Learning of Regions of Attractions via Recurrent Sets

Y Shen, M. Bichuch, and E Mallada, “Model-free Learning of regions of attraction via recurrent sets.” CDC 2022.

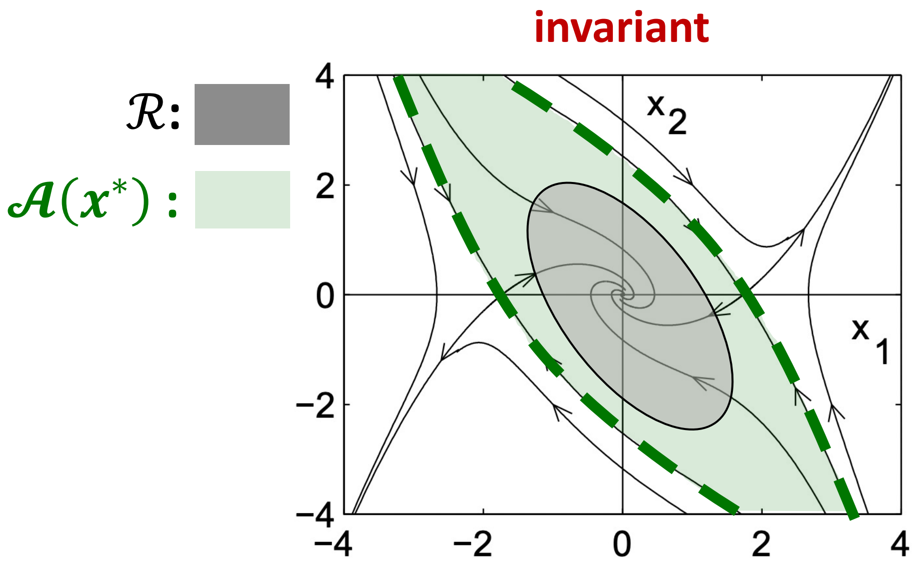
Recurrent sets are subsets of the region of attraction

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \geq t$ s.t. $\phi(t', x_0) \in \mathcal{R}$.

***Theorem**. Let $\mathcal{R} \subset \mathbb{R}^d$ be a compact set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$.*

Then:

$\mathcal{R}$ is *invariant* ➡ $\mathcal{R} \cap \Omega(f) \neq \emptyset$, $\mathcal{R} \subset \mathcal{A}(\mathcal{R} \cap \Omega(f))$



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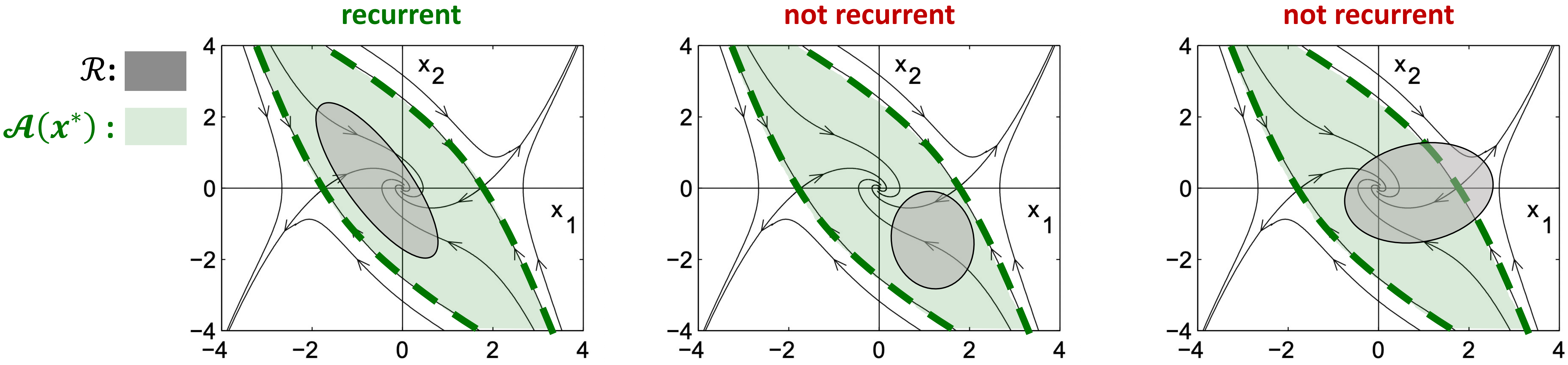

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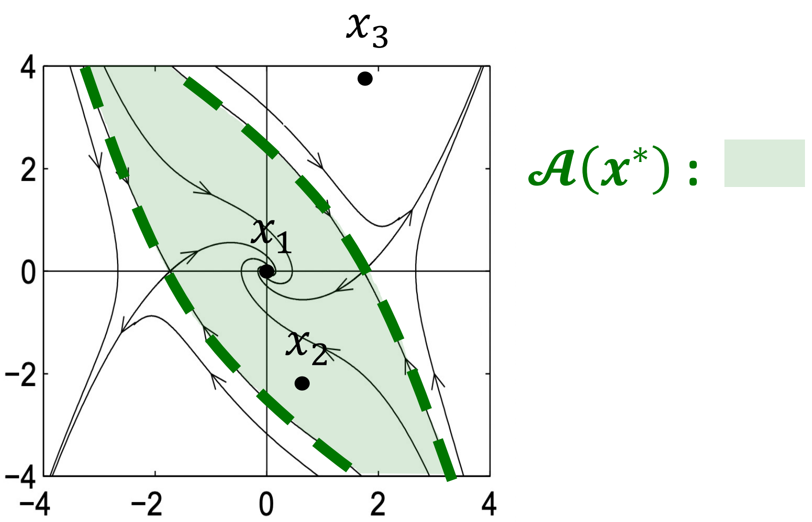
Learning Regions of Attractions via Recurrent Sets

Algorithm: Given h centers, param. τ, and $\varepsilon > 0$:

- *Build approximation using unions of balls centered at* $x_1, \dots, x_h$, *with* $x_1 = x^*$



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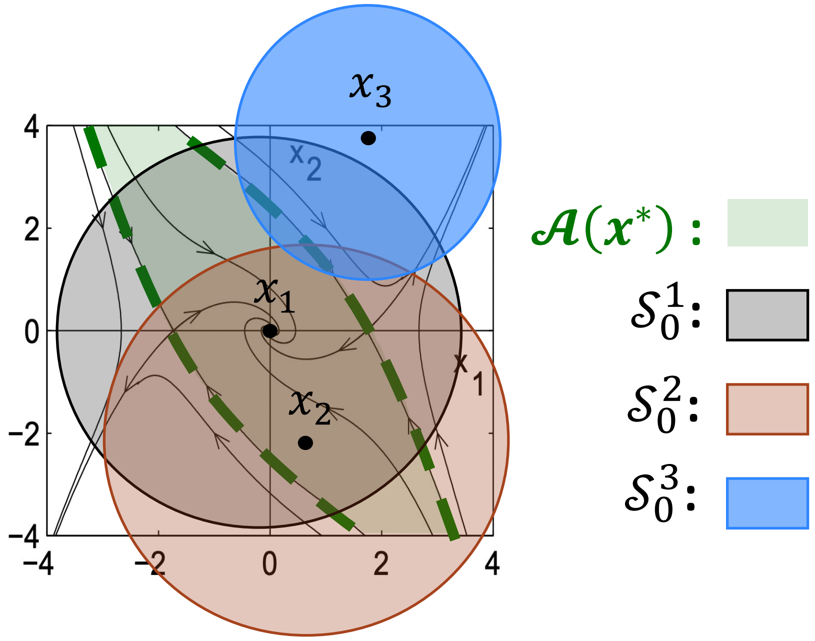

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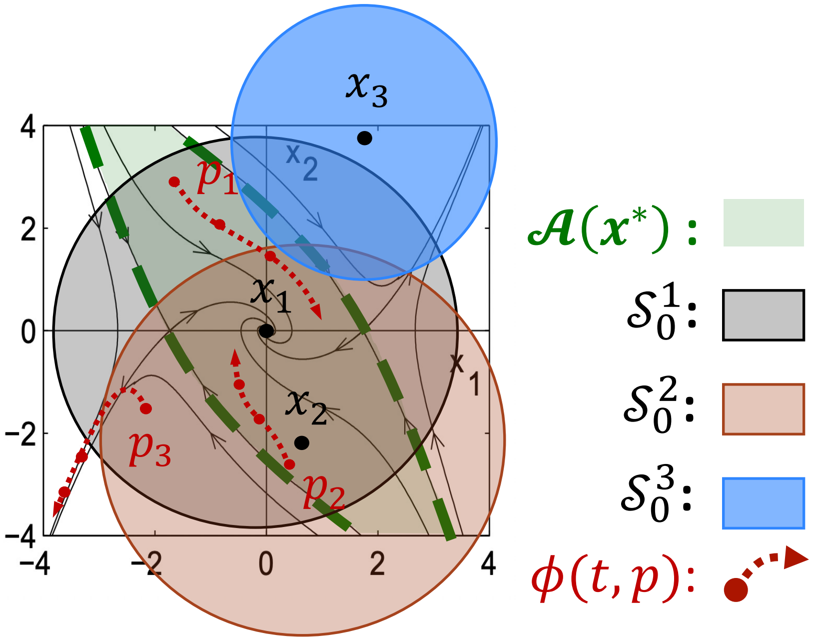
At each iteration l

- *Sample trajectories of duration* τ *from* $\mathcal{S}_l$ *until recurrence is violated (counter-example)*



p_3: counter example

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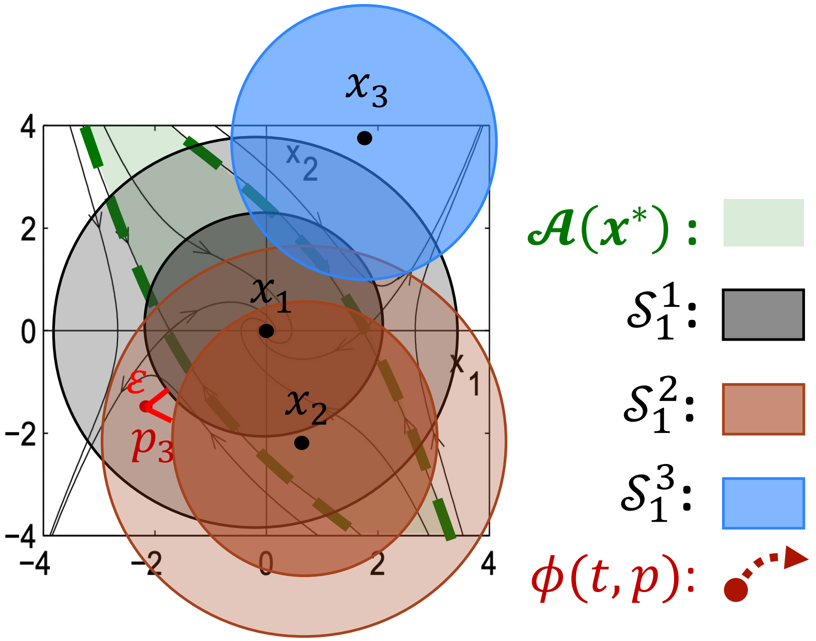
At each iteration l

- *Sample trajectories of duration* τ *from* $\mathcal{S}_l$ *until recurrence is violated (counter-example)*
- *Update approximation* $\mathcal{S}_{l+1}$ *to exclude counter-example neighborhood:* $p_j + B_\varepsilon$

Sample complexity: $m \geq \frac{\sum_l V(\mathcal{S}_l + B_\varepsilon)}{V(B_\varepsilon)} \log\left(\frac{1}{\delta}\right)$



p_3: counter example

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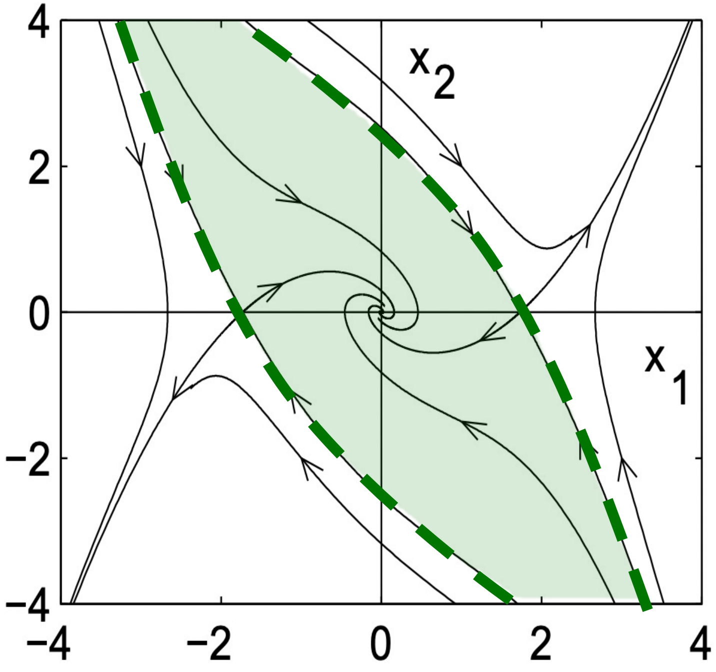
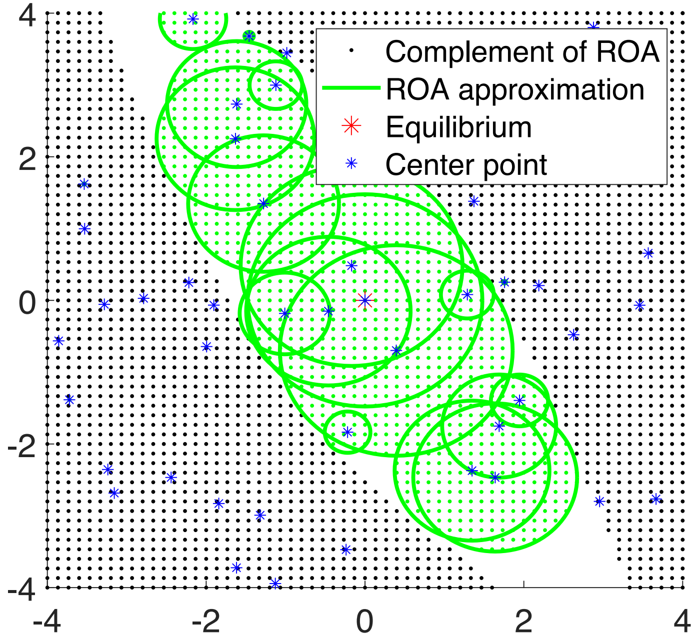
Example: Using 50 Center Points



$\mathcal{A}(x^*)$:

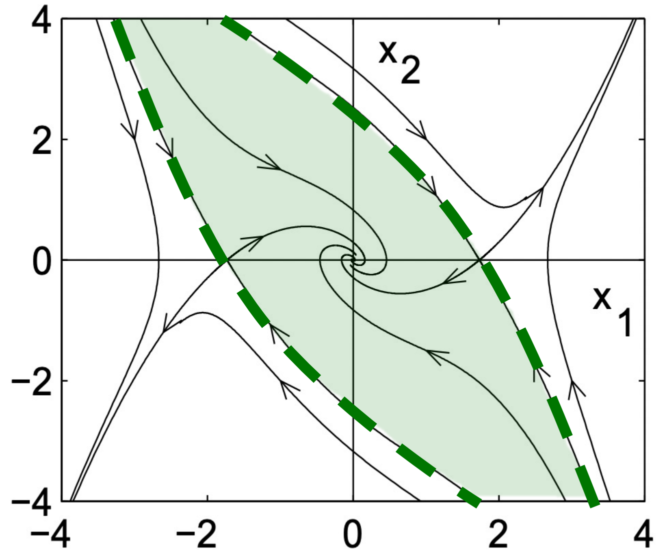


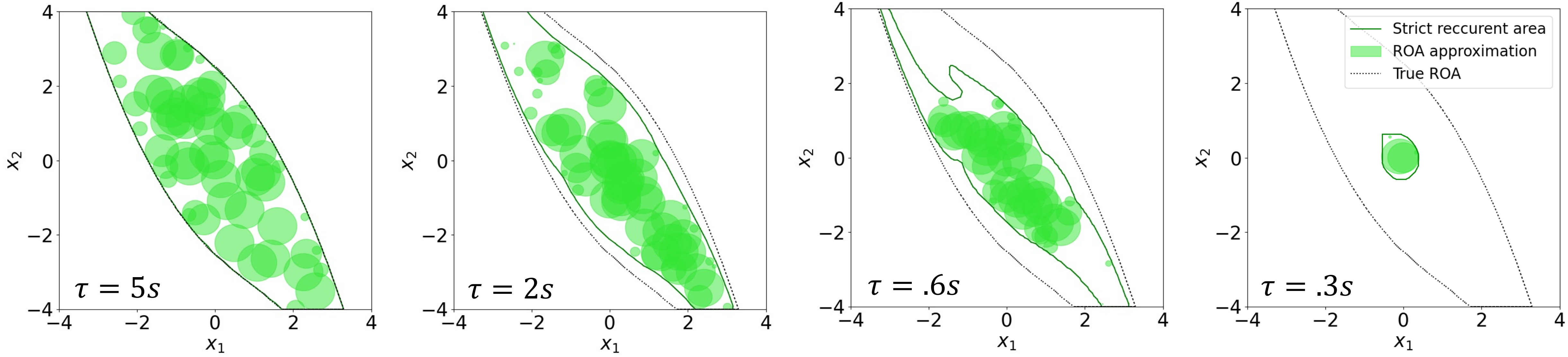
50 sphere approximation

Example: Changing trajectory duration τ

- **Run:** 200 center points sampled (uniformly)
- **Stopping criteria:** $\delta = 10^{-5}$

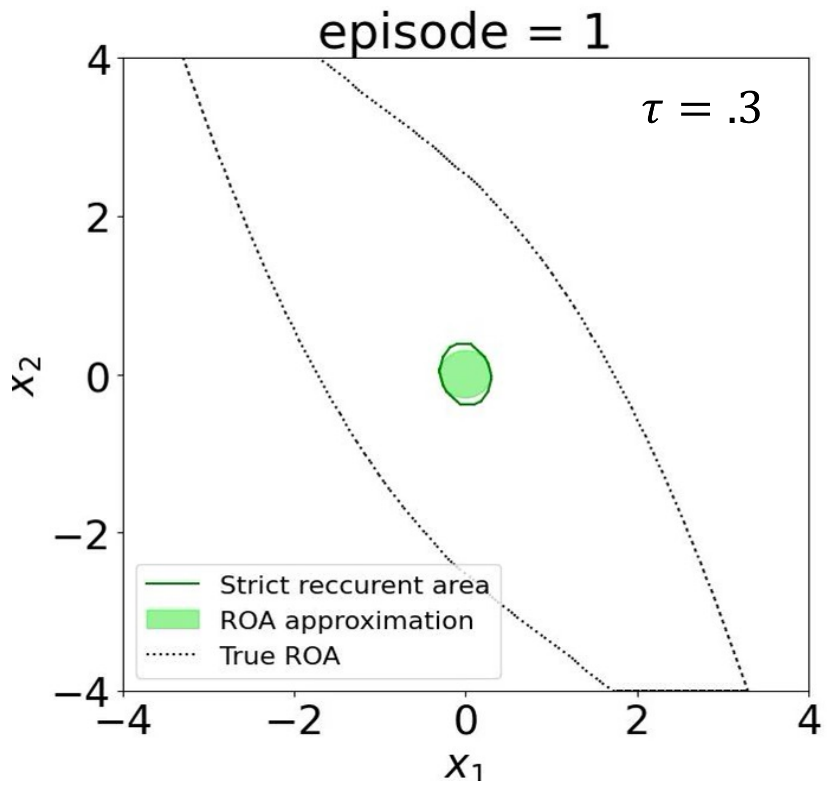
τ (s)	Running time	Volume %
5	57.7	72.0%
2	55.8	51.2%
.6	47.1	31.2%
.3	28.7	3.24%

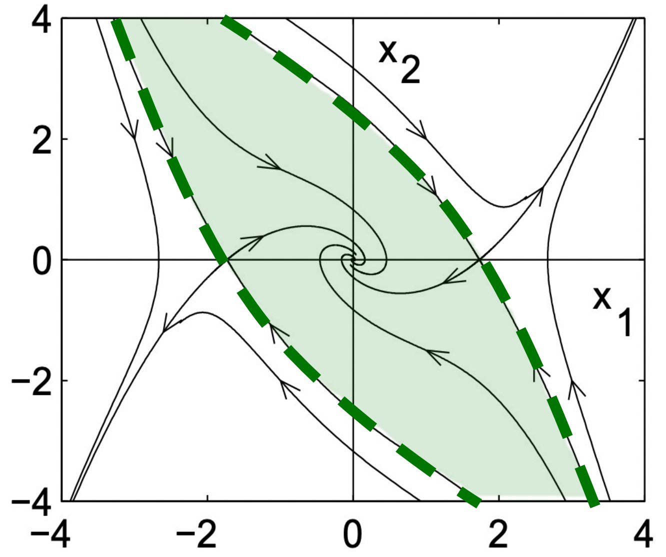


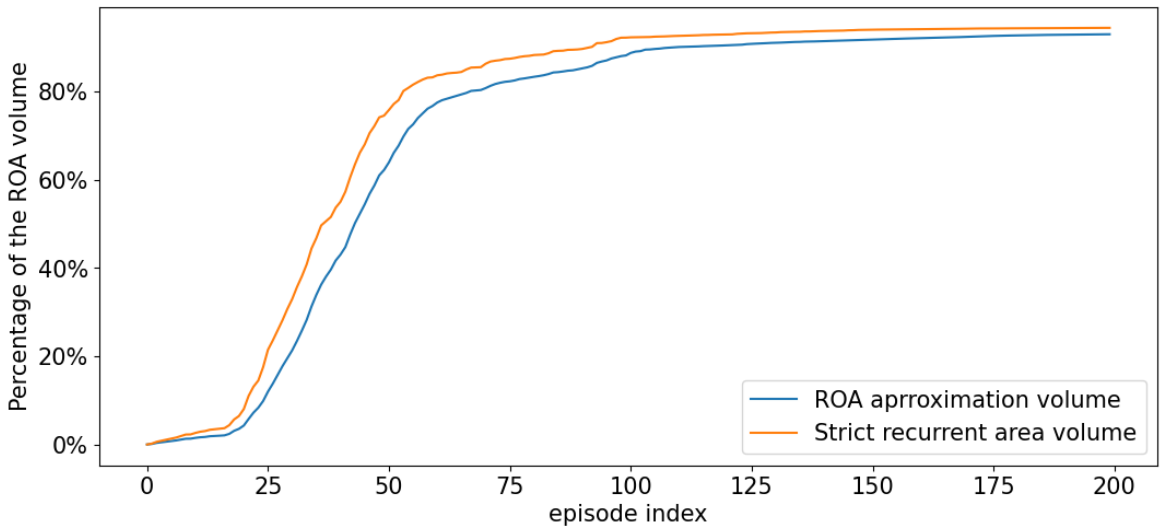


Example: Episodic Expansions of Approximation

- At Each Episode:
  - **Sample 50 new** center points (uniformly)
  - **Stopping criteria:** $\delta = 10^{-5}$



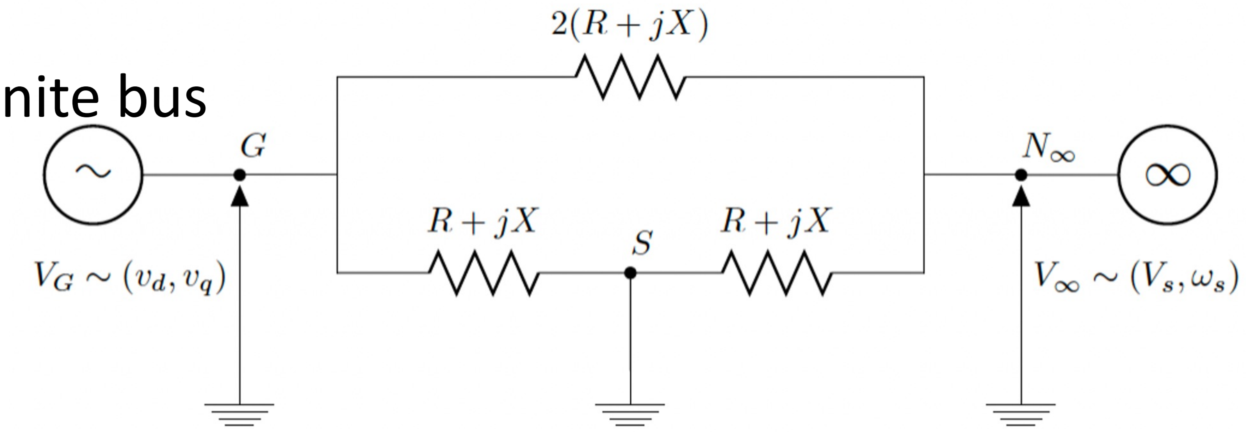




[CDC 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets*, **CDC 2022,** journal version submitted.

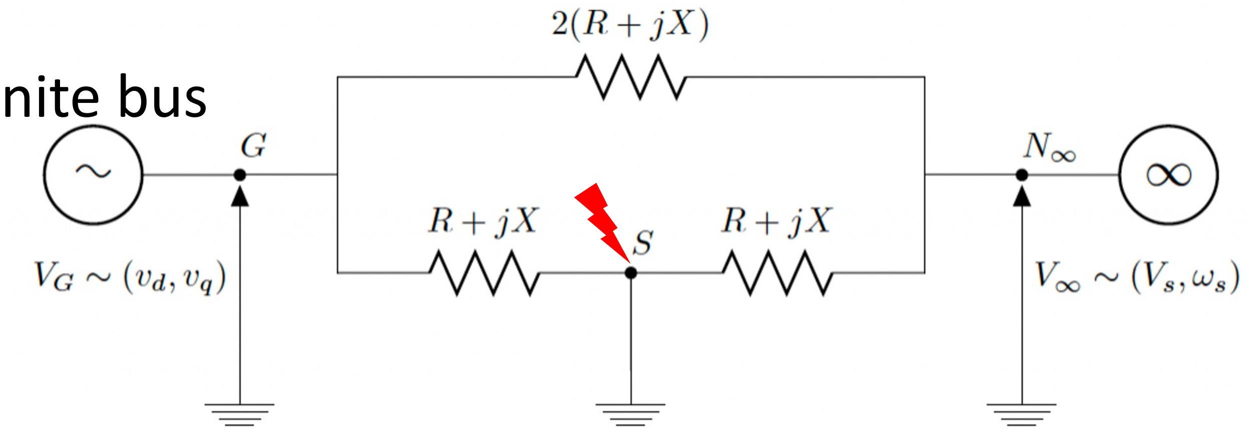
Transient Stability Analysis

- Synchronous machine connected to infinite bus



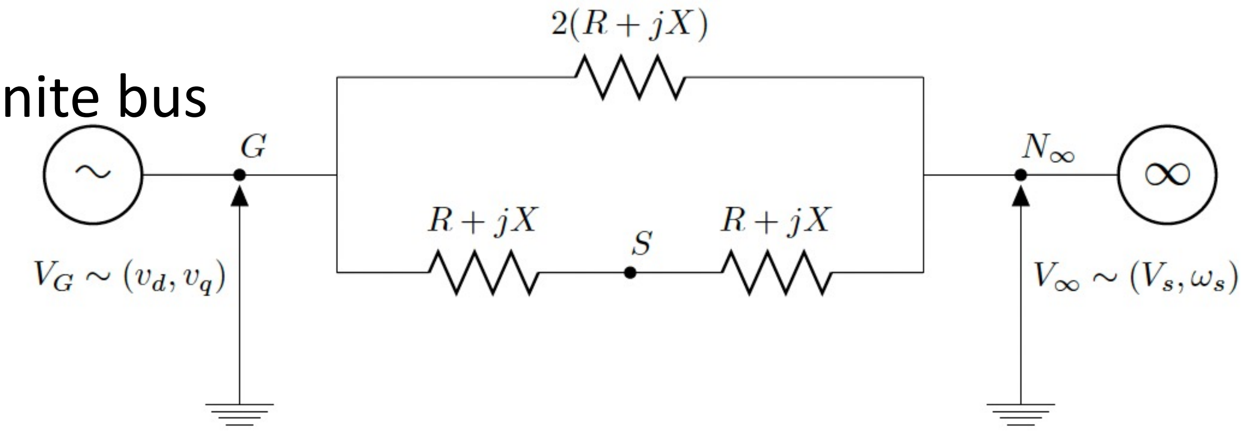
Transient Stability Analysis

- Synchronous machine connected to infinite bus
- t_1 lower line is short-circuited



Transient Stability Analysis

- Synchronous machine connected to infinite bus
- t_1 lower line is short-circuited
- t_2 fault is cleared



Transient Stability Analysis

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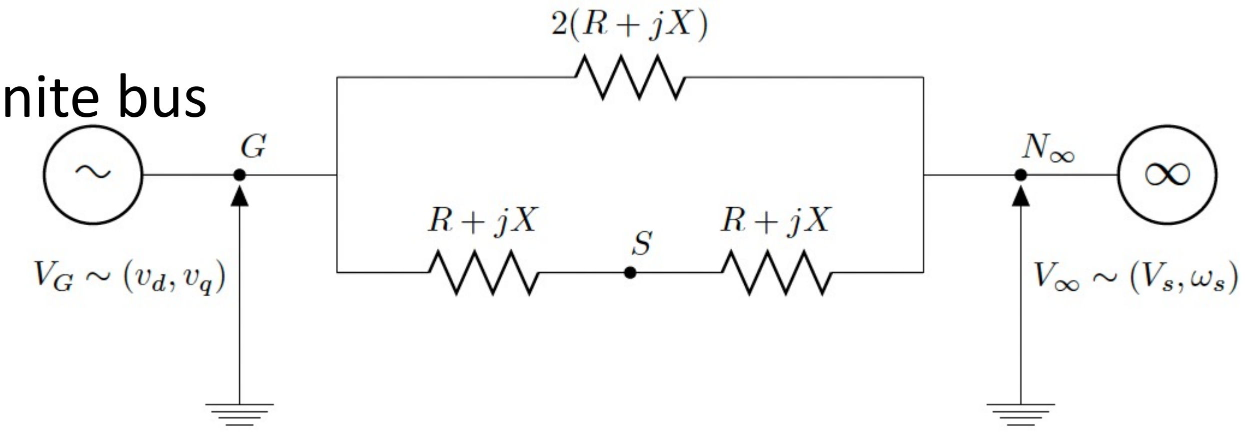
- t_1 lower line is short-circuited
- t_2 fault is cleared



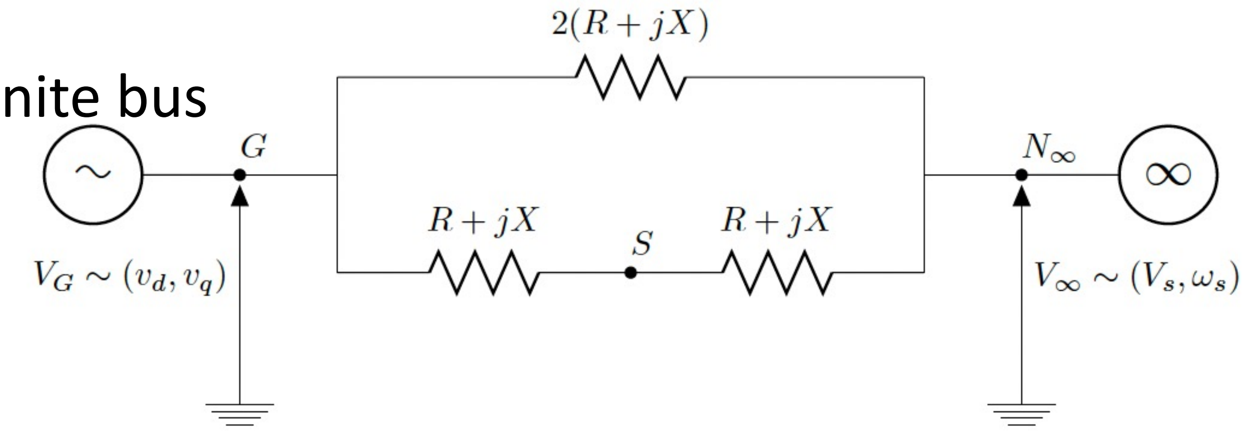
$$
\begin{aligned}
\frac{d\delta}{dt} &= \omega - \omega_s \\
2H\frac{d\omega}{dt} &= P_m - (v_d i_d + v_q i_q + e i_d^2 + r i_q^2) \\
T'_{d_0}\frac{de'_q}{dt} &= -e'_q - (x_d - x'_d)i_d + E_{fd} \\
T_a\frac{dE_{fd}}{dt} &= -E_{fd} + K_a(V_{ref} - V_t) \\
T_g\frac{dP_m}{dt} &= -P_m + P_{ref} + K_g(\omega_{ref} - \omega) \\
i_q &= \frac{(X - x'_d)V_s\sin(\delta) - (R + r)(V_s\cos(\delta) - e'_q)}{(R + r)^2 + (X + x'_d)(X + x_q)}
\end{aligned}
$$

$$
\begin{aligned}
i_d &= \frac{X - x_q}{R + r}i_q - \frac{1}{R + r}V_s\sin(\delta) \\
v_d &= x_q i_q - r - i_d \\
v_q &= R i_q + X i_d + V_s\cos(\delta) \\
V_t &= \sqrt{v_d^2 + v_q^2}
\end{aligned}
$$

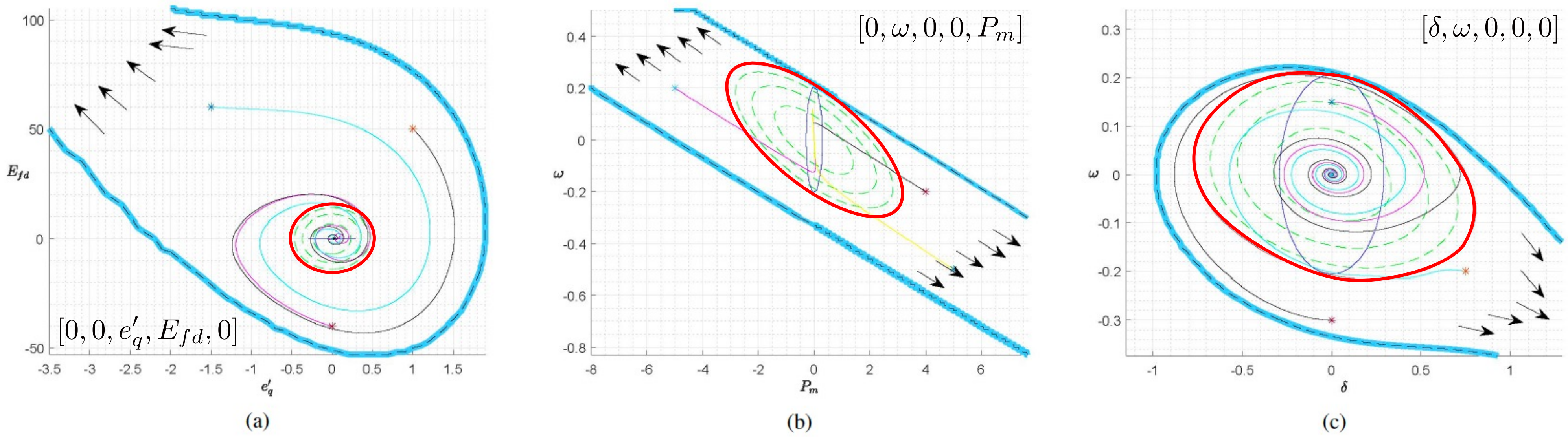
$T'_{d_0} = 9.67$	$x_d = 2.38$	$x'_d = 0.336$	$x_q = 1.21$
$H = 3$	$r = 0.002$	$\omega_s = \omega_{ref} = 1$	$R = 0.01$
$X = 1.185$	$V_s = 1$	$T_a = 1$	$K_a = 70$
$V_{ref} = 1$	$T_g = 0.4$	$K_g = 0.5$	$P_{ref} = 0.7$

Transient Stability Analysis

- Synchronous machine connected to infinite bus
- t_1 lower line is short-circuited
- t_2 fault is cleared



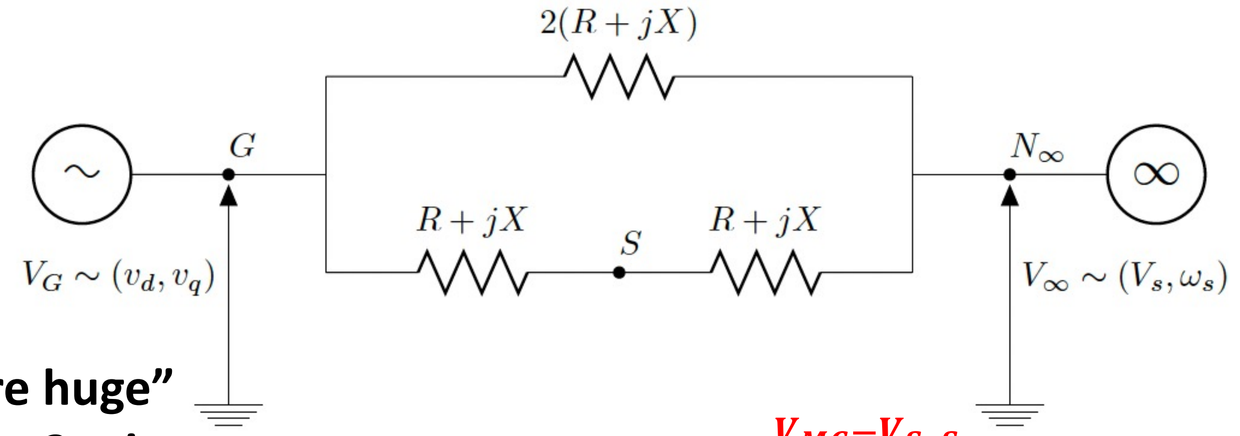
SoS approx. in red (2d-sections)



M. Tacchi et al "Power system transient stability analysis using SoS programming" Power System Computation Conference (PSCC) 2018

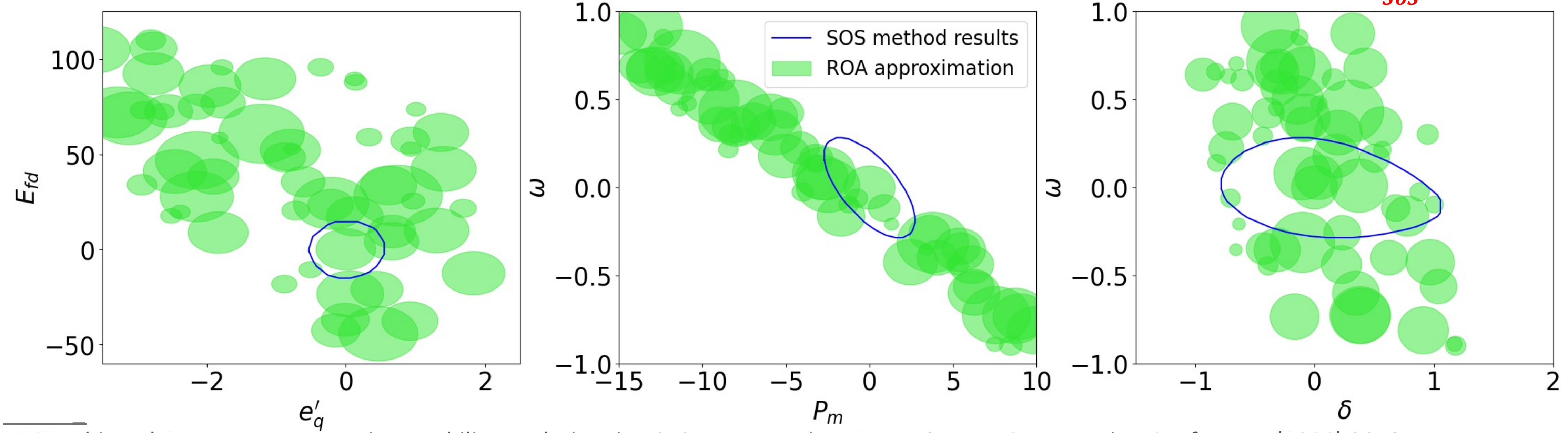
Transient Stability Analysis

- **Algorithm parameters:**
  - **Centers:** 1000 per episode
  - **Failure prob.:** $\delta = 10^{-5}$
  - **Time constant:** $\tau = 100$ s



SoS in **blue**: [Tacchi 18] **vol = 0.05%,** run time **“they are huge”**
Multi-center in **green**: **vol = 0.23%, 1** episode, run time **3 min**

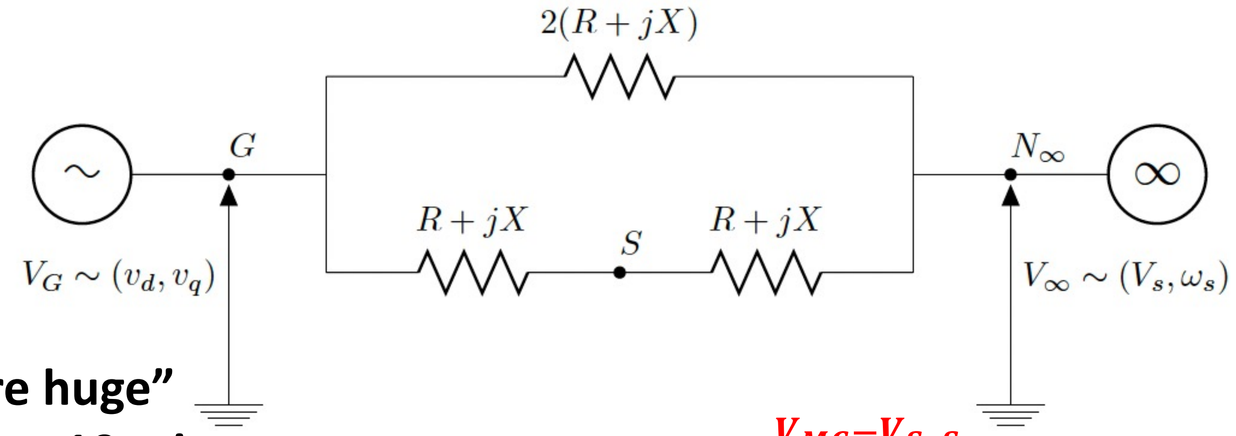
Percent vol. gain: $\frac{V_{MC}-V_{SoS}}{V_{SoS}} = 360\%$



M. Tacchi et al *Power system transient stability analysis using SoS programming,* Power System Computation Conference (PSCC) 2018
Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* Control and Decision Conference (**CDC) 2022**

Transient Stability Analysis

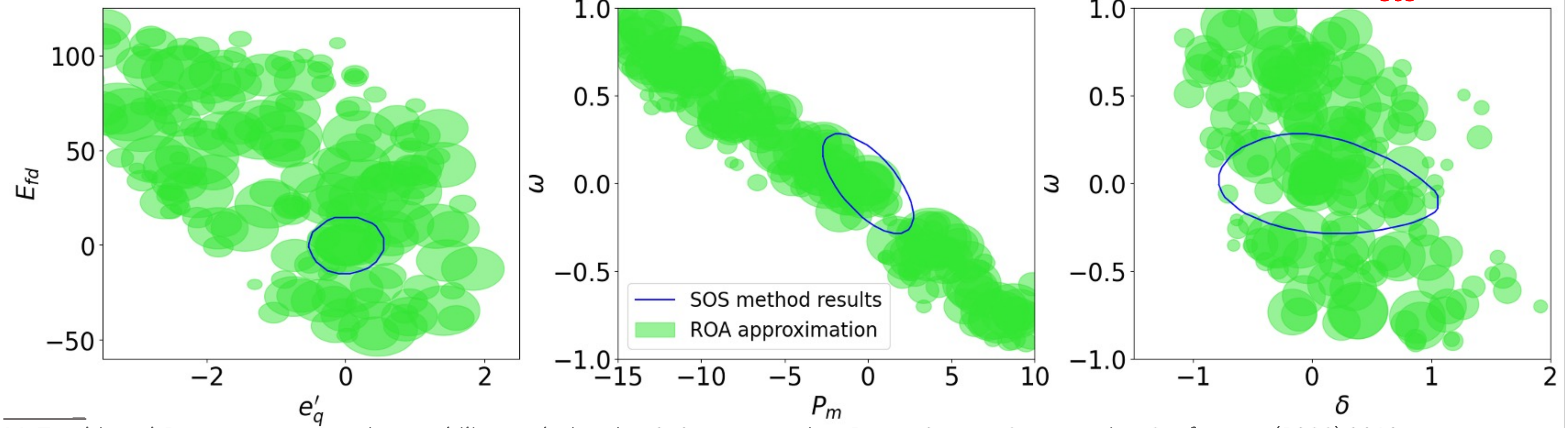
- **Algorithm parameters:**
  - **Centers:** 1000 per episode
  - **Failure prob.:** $\rho = 10^{-5}$
  - **Time constant:** $\tau = 100$ s



SoS in **blue**: [Tacchi 18] **vol = 0.05%,** run time **"they are huge"**

Multi-center in **green**: **vol = 0.45%,** **3** episodes, run time **10 min**

Percent vol. gain: $\frac{V_{MC} - V_{SoS}}{V_{SoS}} = 800\%$



M. Tacchi et al *Power system transient stability analysis using SoS programming,* Power System Computation Conference (PSCC) 2018

Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* Control and Decision Conference (**CDC) 2022**

Transient Stability Analysis

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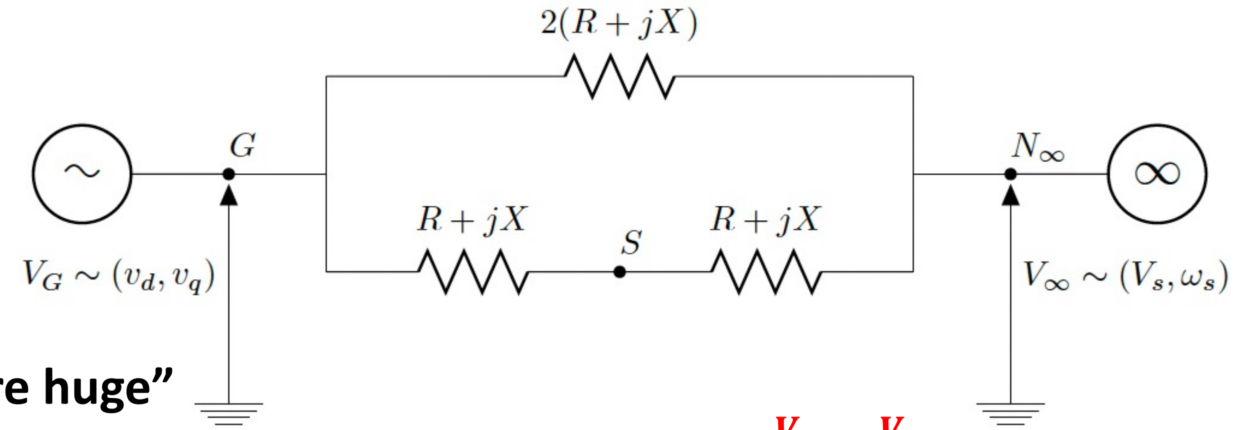
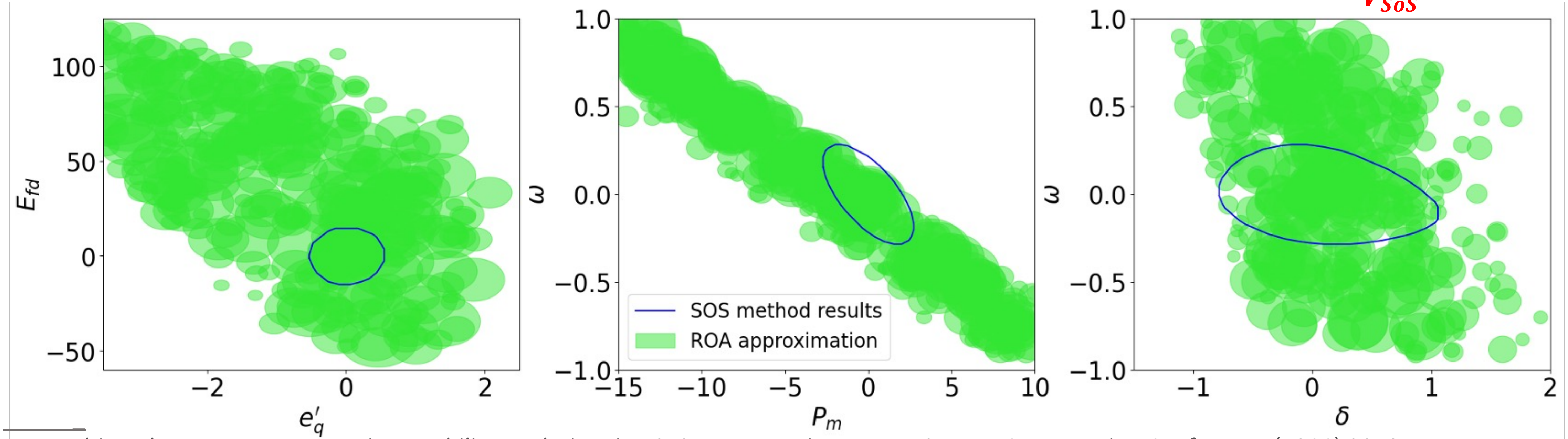
  - **Time constant:** $\tau = 100$ s



SoS in **blue**: [Tacchi 18] **vol = 0.05%,** run time **"they are huge"**
Multi-center in **green**: **vol = 0.74%,** **5** episode, run time **17.5 min**

Percent vol. gain: $\frac{V_{MC} - V_{SoS}}{V_{SoS}} = 1380\%$



M. Tacchi et al *Power system transient stability analysis using SoS programming,* Power System Computation Conference (PSCC) 2018
Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* Control and Decision Conference (**CDC) 2022**

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- **Algorithm parameters:**
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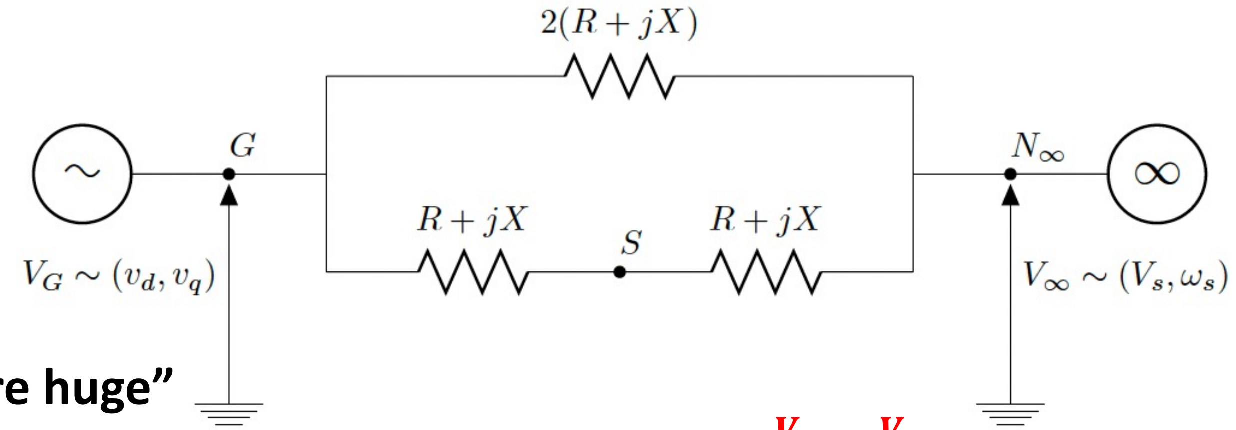
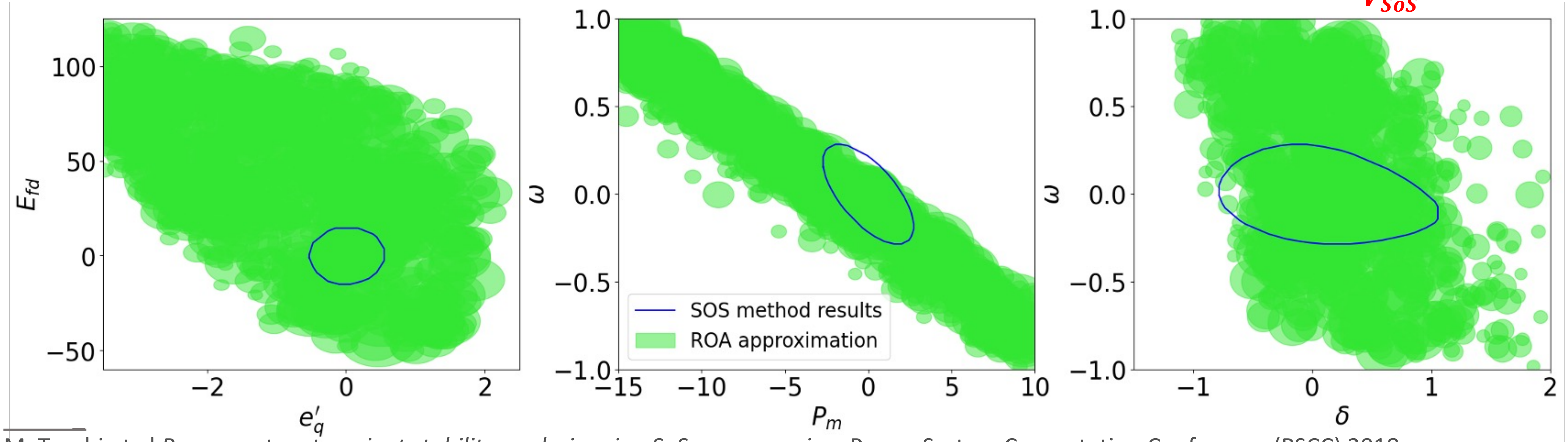
  - **Failure prob.:** $\rho = 10^{-5}$
  - **Time constant:** $\tau = 100$ s



SoS in **blue**: [Tacchi 18] **vol = 0.05%,** run time **"they are huge"**

Multi-center in **green**: **vol = 1.56%,** **10** episodes, run time **39.5 min**

Percent vol. gain: $\frac{V_{MC} - V_{SoS}}{V_{SoS}} = 3020\%$



M. Tacchi et al *Power system transient stability analysis using SoS programming,* Power System Computation Conference (PSCC) 2018

Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* Control and Decision Conference (**CDC) 2022**

Outline

- Invariance: Merits and trade-offs
- **Letting things go and come back: Recurrent sets**
  - **Approximating regions of attractions via recurrent sets**
- Non-parametric analysis of dynamical systems
  - Stability analysis via non-monotonic Lyapunov conditions
  - Safety verification via generalized Barrier functions

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- **Non-parametric analysis of dynamical systems**
  - **Stability analysis via non-monotonic Lyapunov conditions**
  - Safety verification via generalized Barrier functions

Roy Siegelmann — Johns Hopkins University

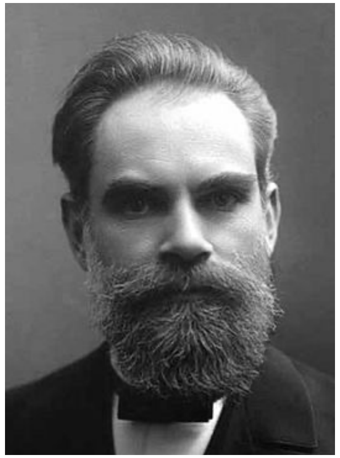
Yue Shen — Johns Hopkins University

Fernando Paganini — Universidad ORT Uruguay

Nonparametric Stability Analysis

R. Siegelmann, Y. Shen, F. Paganini, and E. Mallada, “A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions”, CDC 2023

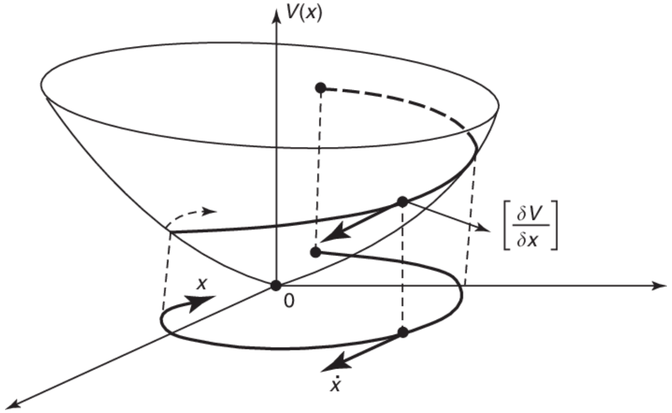
Lyapunov's Direct Method



Key idea: Make sub-level sets invariant to trap trajectories

***Theorem [Lyapunov '1892]**. Given $V: \mathbb{R}^d \to \mathbb{R}_{\geq 0}$, with $V(x) > 0, \forall x \in \mathbb{R}^d \backslash \{x^*\}$, then:*

- $\dot{V} \leq 0 \to x^*$ *stable*
- $\dot{V} < 0 \to x^*$ *as. stable*



Challenge: Couples shape of V and vector field f

- Towards decoupling the $V - f$ geometry
  - Controlling regions where $\dot{V} \geq 0$ **[Karafyllis '09, Liu et al '20]**
  - Higher order conditions: $g(V^{(q)}, \ldots, \dot{V}, V) \leq 0$ **[Butz '69, Gunderson '71, Ahmadi '06, Meigoli '12]**
  - Discretization approach: $V\big(x(T)\big) \leq V(x(0))$ **[Coron et al '94, Aeyels et. al '98, Karafyllis '12]**
  - Multiple Lyapunov Functions: $\{V_j : j \in [k]\}$ **[Ahmadi et al '14]**

A Butz. Higher order derivatives of Lyapunov functions. IEEE Transactions on automatic control, 1969

Gunderson. A comparison lemma for higher order trajectory derivatives. Proceedings of the American Mathematical Society, 1971

Coron, Lionel Rosier. A relation between continuous time-varying and discontinuous feedback stabilization. J. Math. Syst., Estimation, Control, 1994

Aeyels, Peuteman. A new asymptotic stability criterion for nonlinear time-variant differential equations. IEEE Transactions on automatic control, 1998

Ahmadi. Non-monotonic Lyapunov functions for stability of nonlinear and switched systems: theory and computation, 2008

Karafyllis, Kravaris, Kalogerakis. Relaxed Lyapunov criteria for robust global stabilisation of non-linear systems. International Journal of Control, 2009

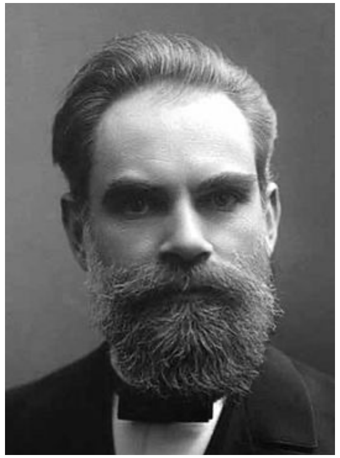
Meigoli, Nikravesh. Stability analysis of nonlinear systems using higher order derivatives of Lyapunov function candidates. Systems & Control Letters, 2012

Karafyllis. Can we prove stability by using a positive definite function with non sign-definite derivative? IMA Journal of Mathematical Control and Information, 2012

Ahmadi, Jungers, Parrilo, Roozbehani. Joint spectral radius and path-complete graph Lyapunov functions. SIAM Journal on Control and Optimization, 2014

Liu, Liberzon, Zharnitsky. Almost Lyapunov functions for nonlinear systems. Automatica, 2020

Lyapunov's Direct Method



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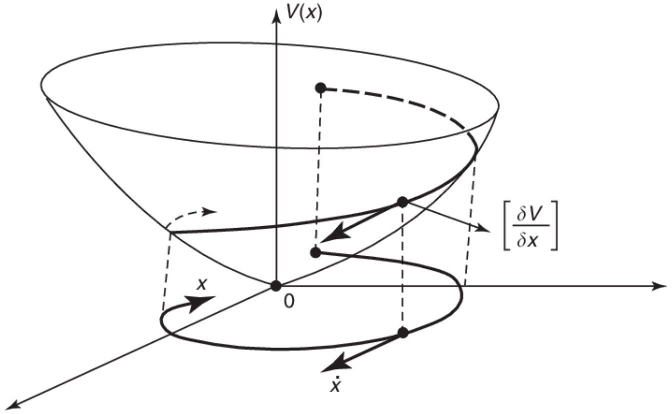

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Question: Can we provide stability conditions based on recurrence?

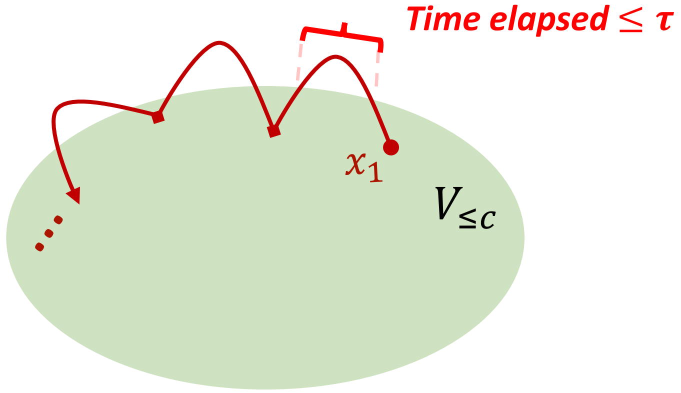
Recurrent Lyapunov Functions

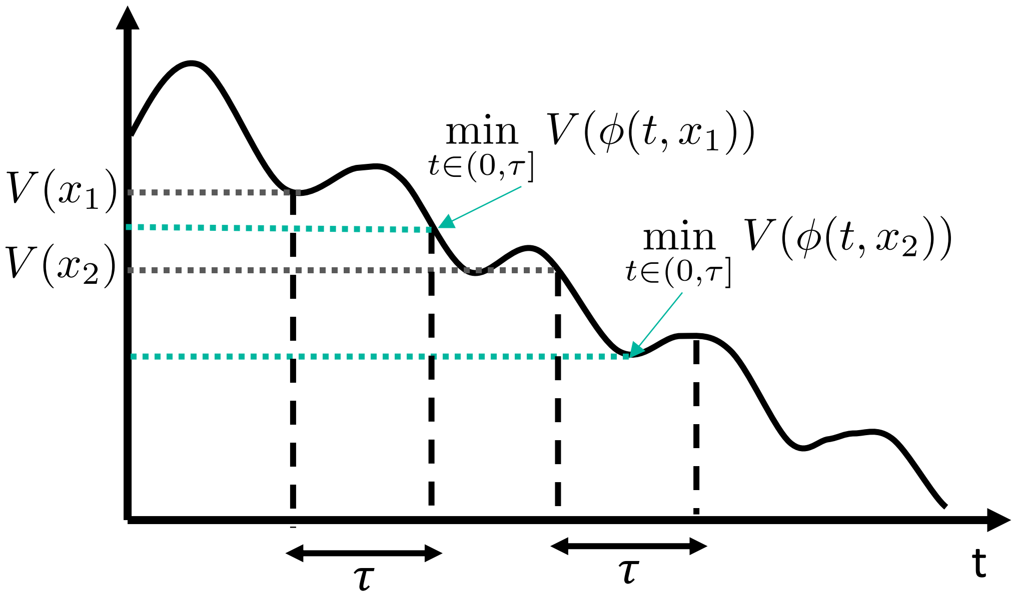
A continuous function $V: \mathbb{R}^d \to \mathbb{R}_+$ is a **recurrent Lyapunov function** if

$$\mathcal{L}_f^{(0,\tau]} V(x) := \min_{t \in (0,\tau]} V(\phi(t,x)) - V(x) \leq 0 \quad \forall x \in \mathbb{R}^d$$

Preliminaries:

- Sub-level sets $\{V(x) \leq c\}$ are $\boldsymbol{\tau}$-recurrent sets.





Definition: A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **τ-recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \in (t, t+\tau]$ s.t. $\phi(t', x_0) \in \mathcal{R}$.

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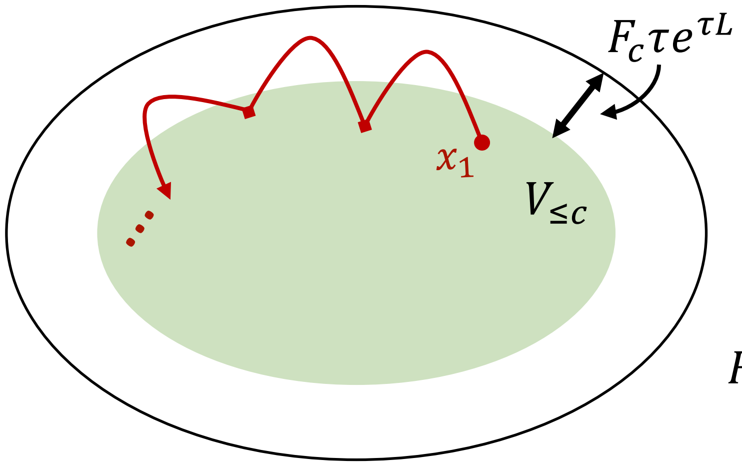
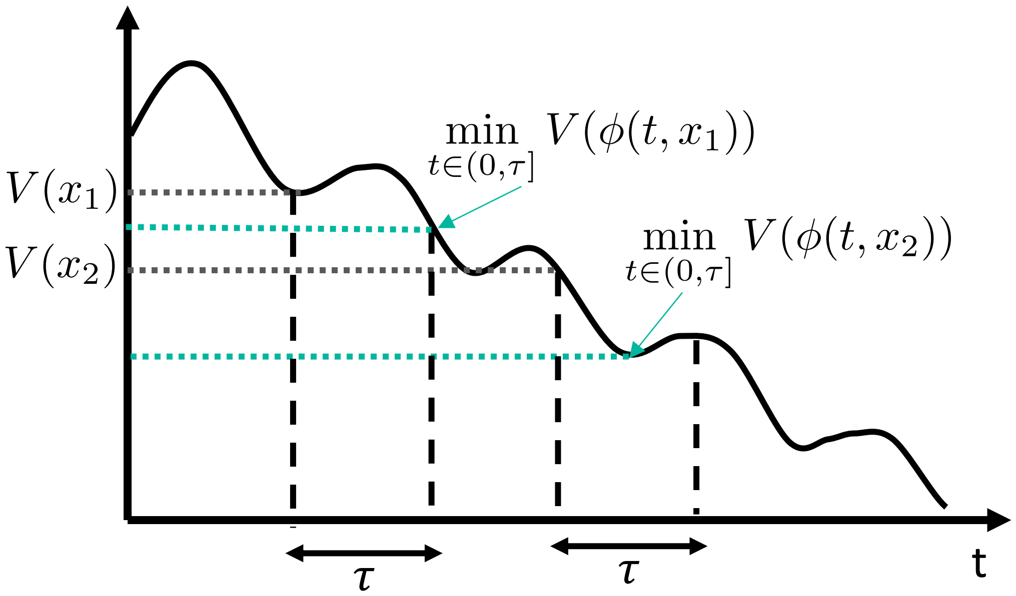
Preliminaries:

- Sub-level sets $\{V(x) \leq c\}$ are **τ**-recurrent sets.
- When f is L-Lipschitz, one can trap trajectories.



$$F_c = \max_{x \in V_{\leq c}} ||f(x)||$$



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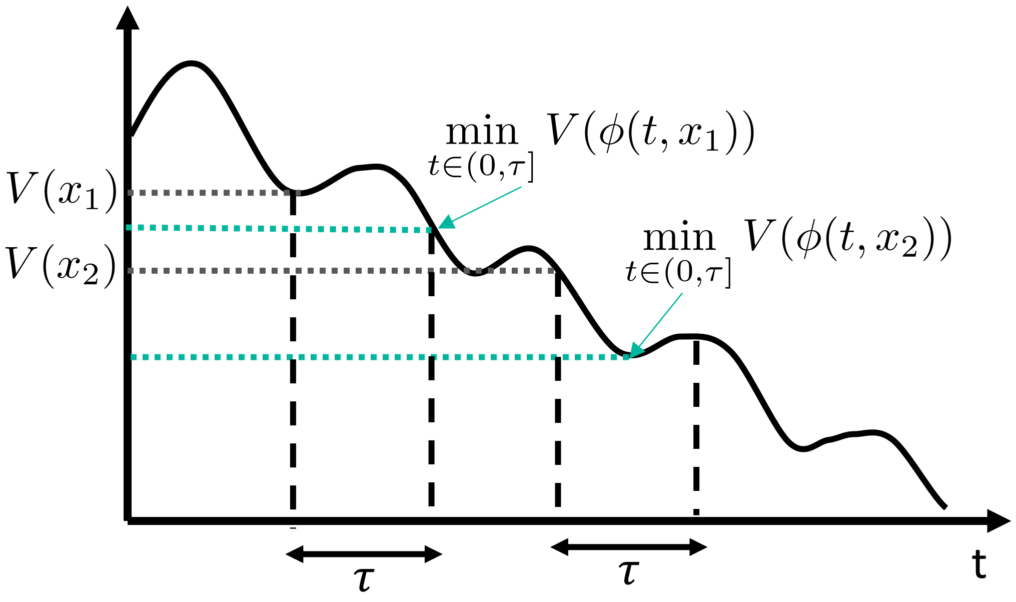
(*)

Theorem [CDC 23]: *Let* $V: \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ *be a recurrent Lyapunov function and let* f *be L-Lipschitz*

- *Then, the equilibrium* x^* *is stable.*
- *Further, if the* ***inequality is strict****, then* x^* *is asymptotically stable!*



Siegelmann, Shen, Paganini, M, *A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions*, **CDC 2023**

Exponential Stability Analysis

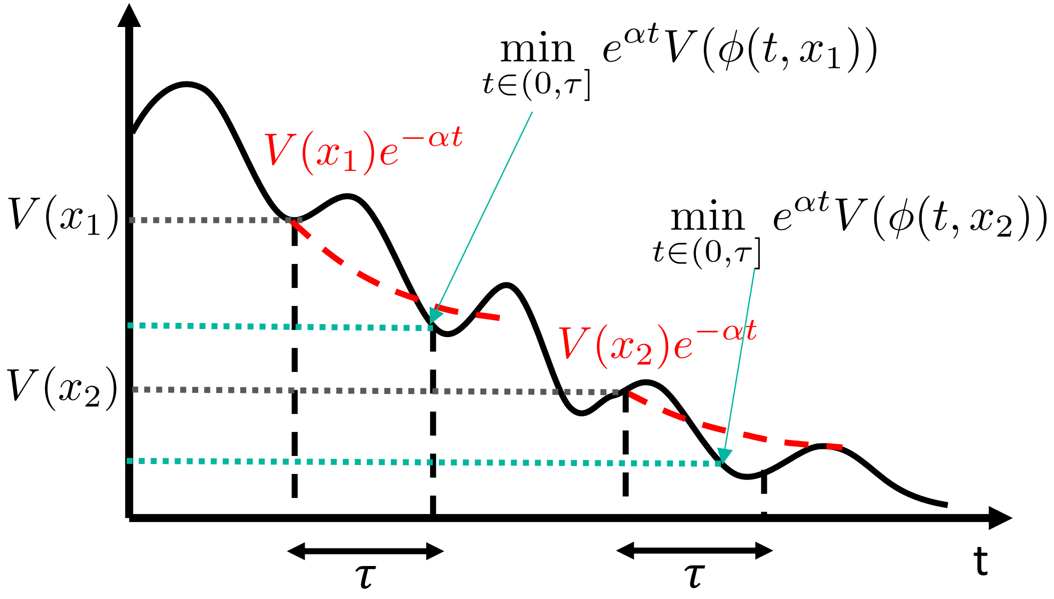
The function $V\colon \mathbb{R}^d \to \mathbb{R}_+$ is **α-exponential recurrent Lyapunov function** if

$$\mathcal{L}_{f,\alpha}^{(0,\tau]} V(x) := \min_{t\in(0,\tau]} e^{\alpha t} V(\phi(t,x)) - V(x) \leq 0 \quad \forall x \in \mathbb{R}^d$$

Theorem [CDC 23]: *Let $V\colon \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ satisfy*

$$\alpha_1 ||x - x^*|| \leq V(x) \leq \alpha_2 ||x - x^*||.$$

Then, if V is α-exponential recurrent Lyapunov function, x^* is α-exponentially stable.



Siegelmann, Shen, Paganini, M, *A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions*, **CDC 2023**

A (Sub-optimal) Converse Theorem

Theorem: Assume x^* is λ-exponentially stable: $\exists\ K, \lambda > 0$ such that:

$$\big|\big|\phi(t,x) - x^*\big|\big| \leq K e^{-\lambda t}\big|\big|x_0 - x^*\big|\big|, \qquad \forall x \in \mathbb{R}^d.$$

Then, $V(x) = \big|\big|x - x^*\big|\big|$ *is* *α-exponential recurrent Lyapunov function , i.e.,*

$$\min_{t\in(0,\tau]} e^{\alpha t}\big|\big|\phi(t,x) - x^*\big|\big| - \big|\big|x - x^*\big|\big| \leq 0, \qquad \forall x \in \mathbb{R}^d,$$

whenever $\alpha < \lambda$ *and* $\tau \geq \frac{1}{\lambda - \alpha} \ln K.$

Remarks:

- The rate α must be strictly smaller than the rate of convergence λ (giving up optimality).
- *Any norm is a Lyapunov function!*

Question: Is the struggle for its search over?

Nonparametric Verification of Exponential Stability

Proposition [CDC 23*]: *Let* $\|\cdot\|$ *be any norm and* $x^* = 0$. *Then, whenever*

$$\min_{t\in(0,\tau]} e^{\alpha t}\left(\|\phi(x,t)\| + re^{Lt}\right) \leq \|x\| - r$$

for all y *with* $\|y - x\| \leq r$

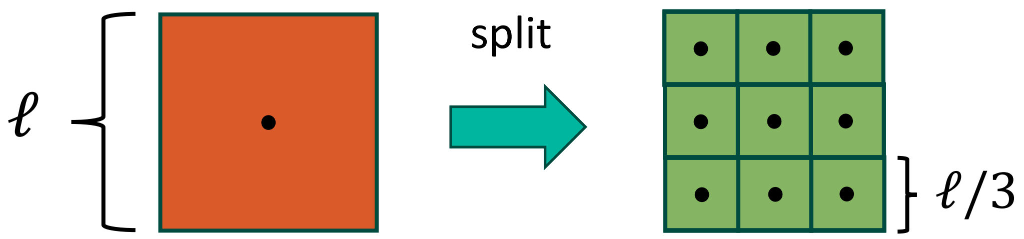
$$\min_{t\in(0,\tau]} e^{\alpha t}\|\phi(y,t)\| \leq \|y\|$$

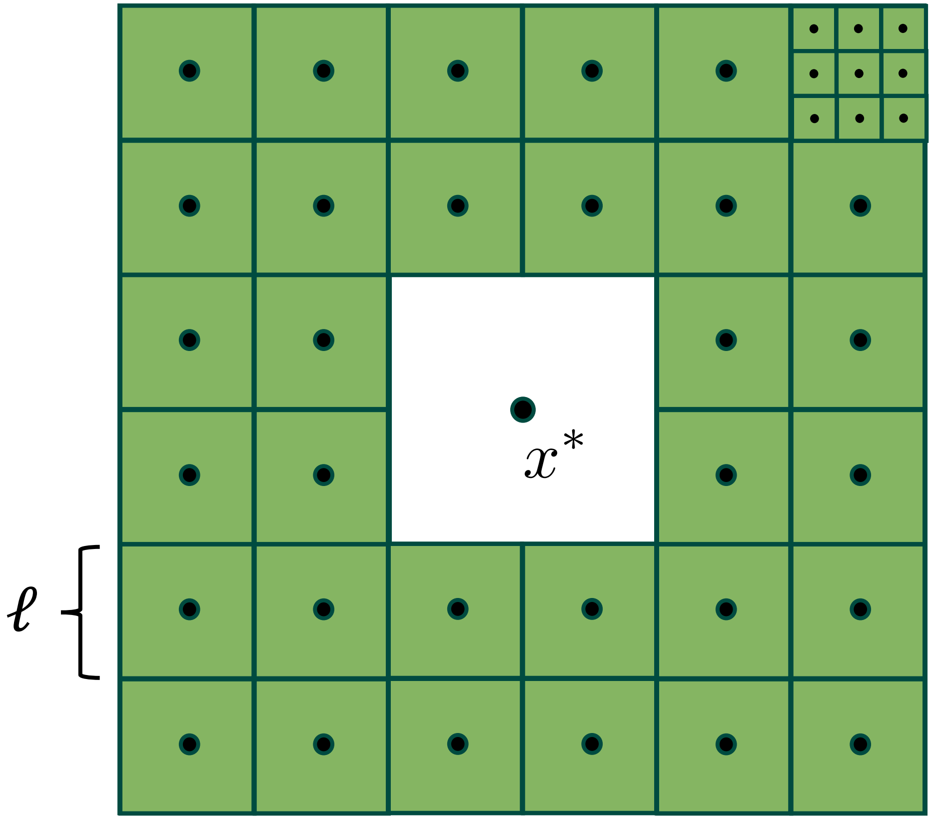
Remarks:

- Only requires a trajectory of length τ
- Trades off between **radius** $\boldsymbol{r}$ and verified **performance** $\boldsymbol{\alpha}$
- Amenable for parallel computations **using GPUs**

Nonparametric Stability Verification via GPUs

- **Basic Algorithm:**
  - Consider $V(x) = ||x - x^*||_\infty$
  - Build a grid of hypercubes surrounding x^*
  - Test grid center points:
    - Simulate trajectories of length τ
    - Find α s.t. the verified radius is $r \geq \ell/2$
  - Hypercube **not verified**, **split in 3^d** parts
  - Repeat testing of new points





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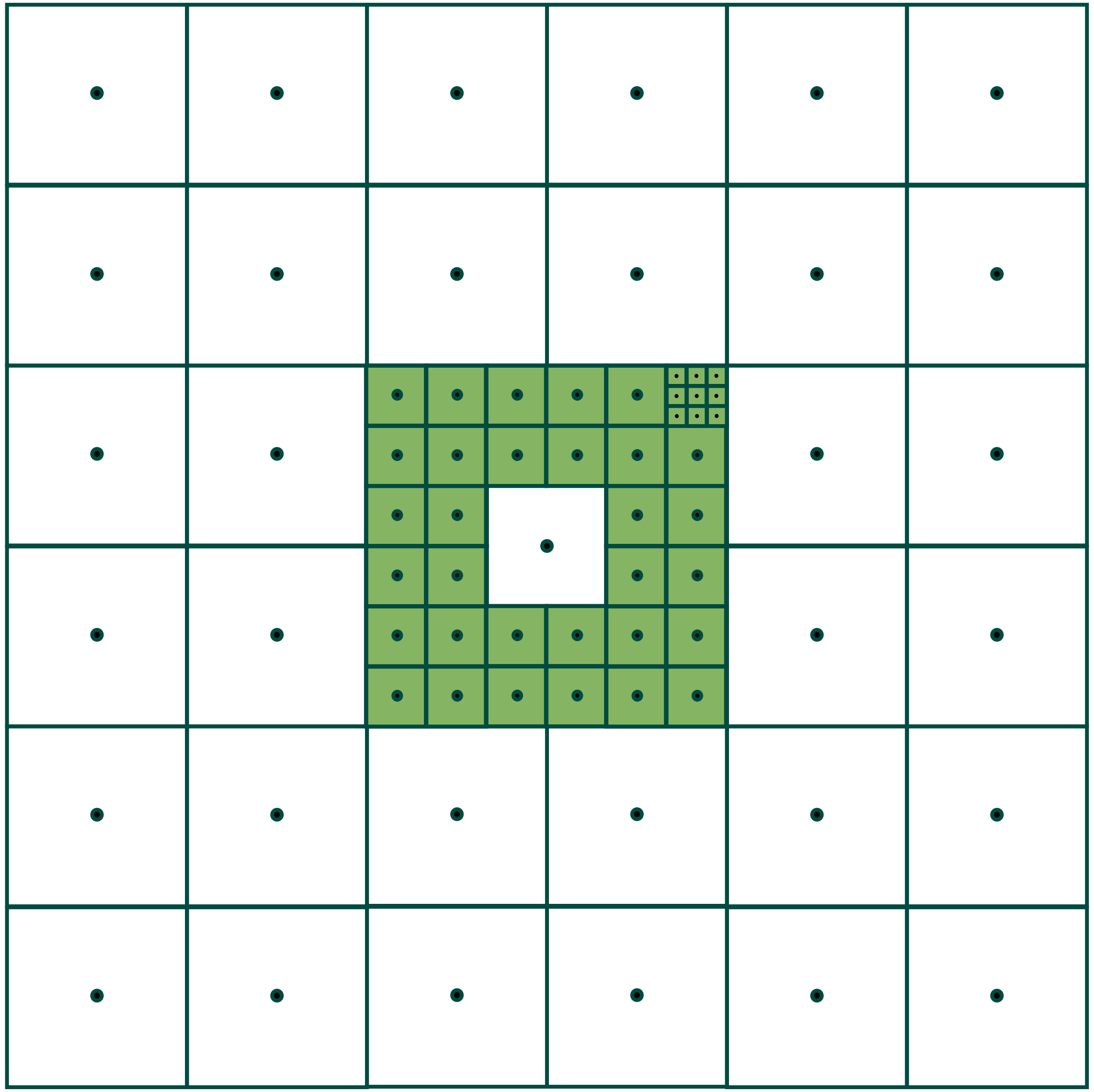
  - Hypercube **not verified**, **split in $\mathbf{3^d}$** parts
  - Repeat testing of new points
  - Exponentially expand to outer layer
  - Repeat testing in new layer



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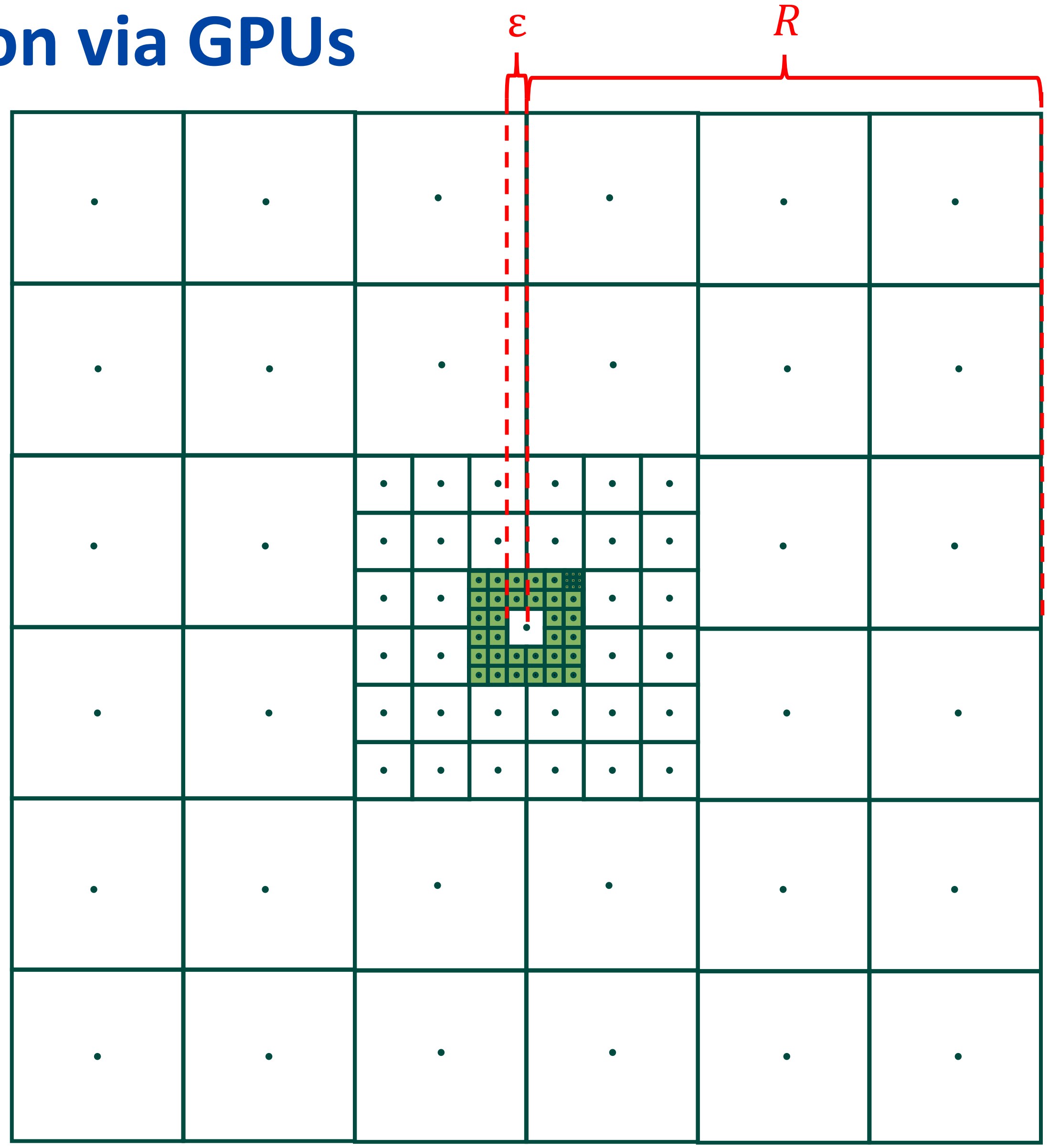
Q: How many samples are needed?
If x^* is λ-exp. stable

$$\mathcal{O}\left(q^{-d} \log\left(\frac{R}{\varepsilon}\right)\right)$$

with $q = \frac{1-K e^{(\alpha-\lambda)\tau}}{1+e^{(L+\alpha)\tau}} < 1.$



Numerical Illustration

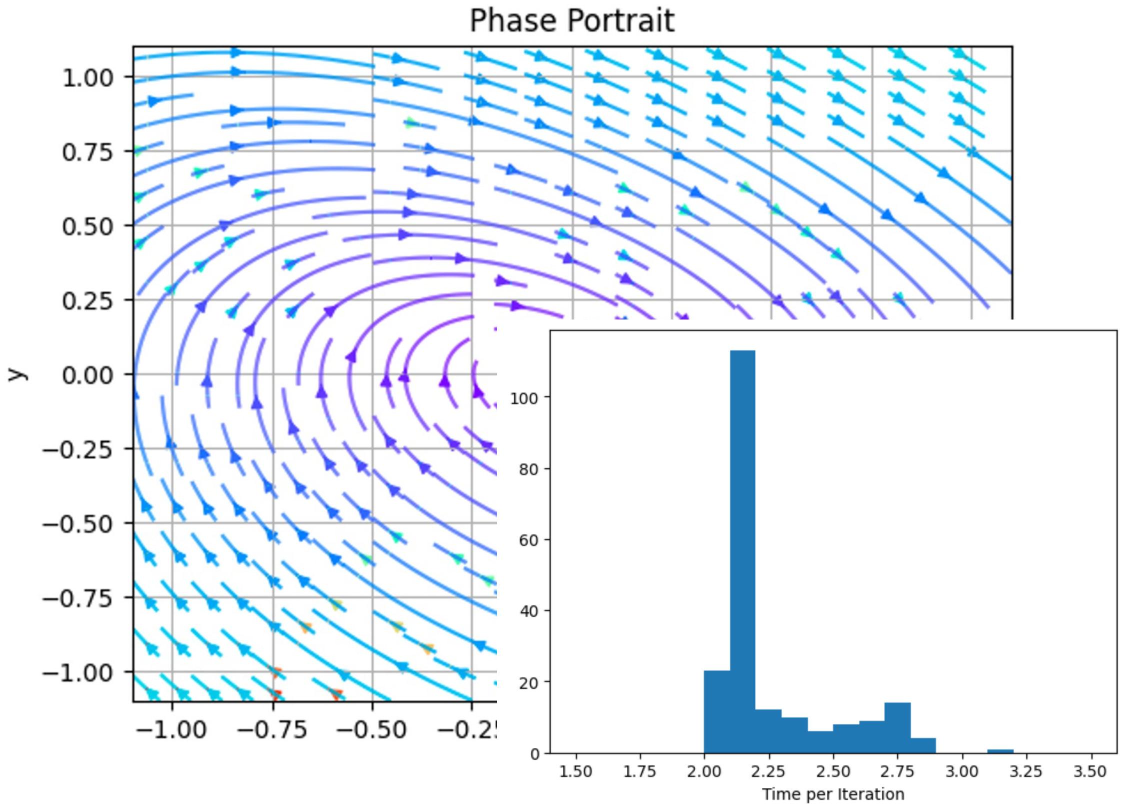
Consider the 2-d non-linear system:

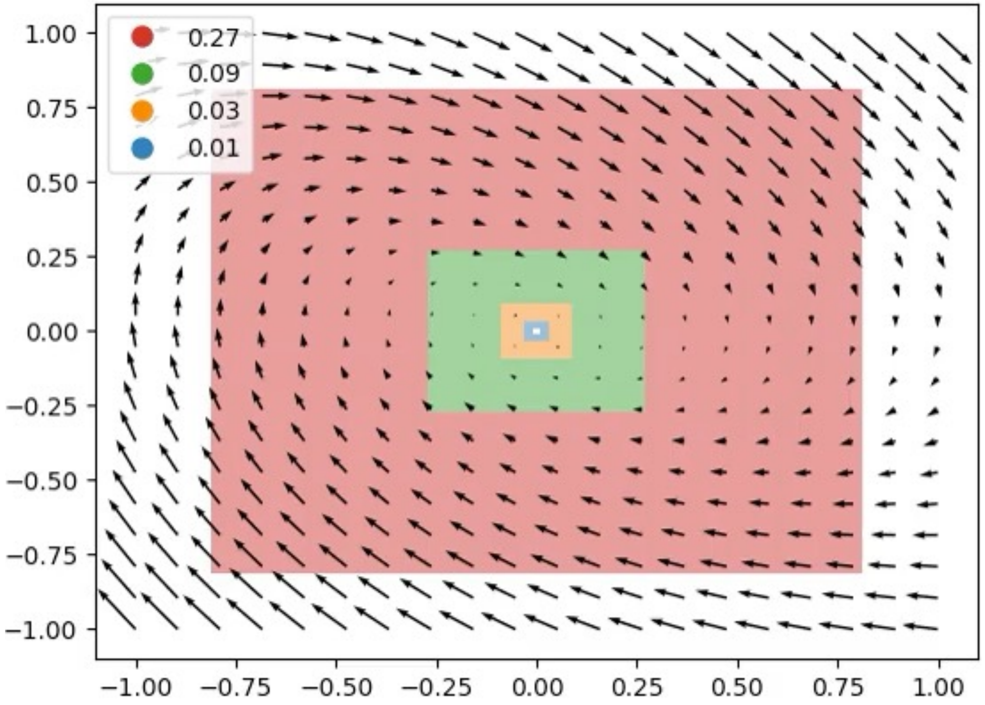
$$\dot{x} = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} x + B \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$$

with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

Parameter	Value
L	1.8
τ	1.5
ℓ	0.01

$\sigma = 0.2$





Numerical Illustration

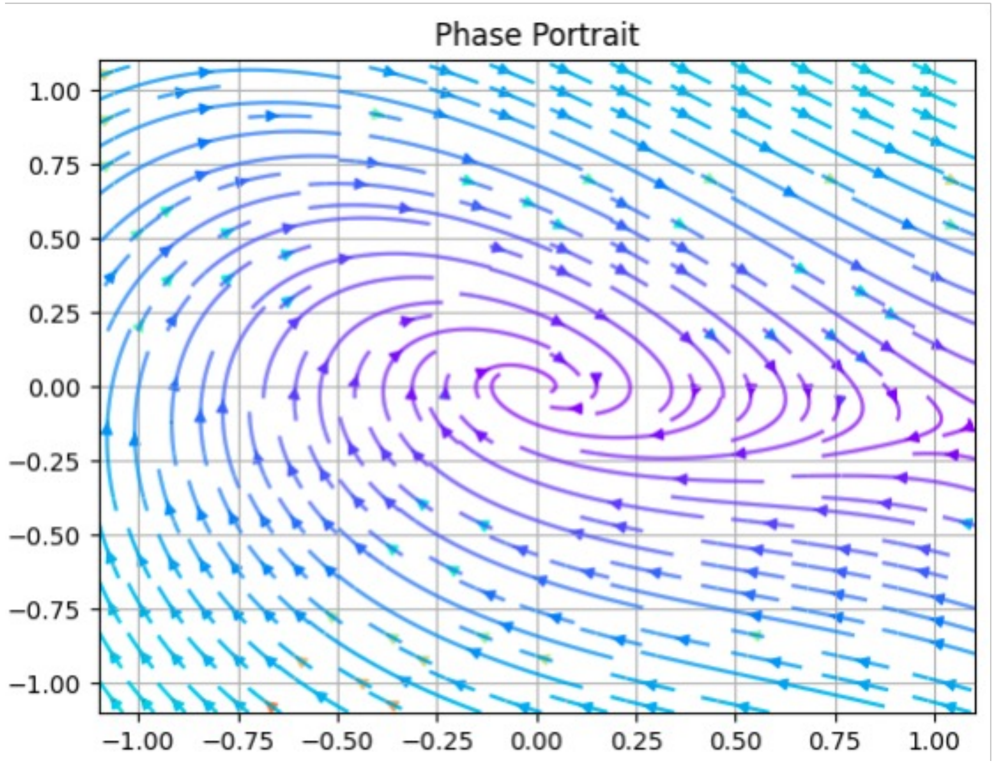
Consider the 2-d non-linear system:

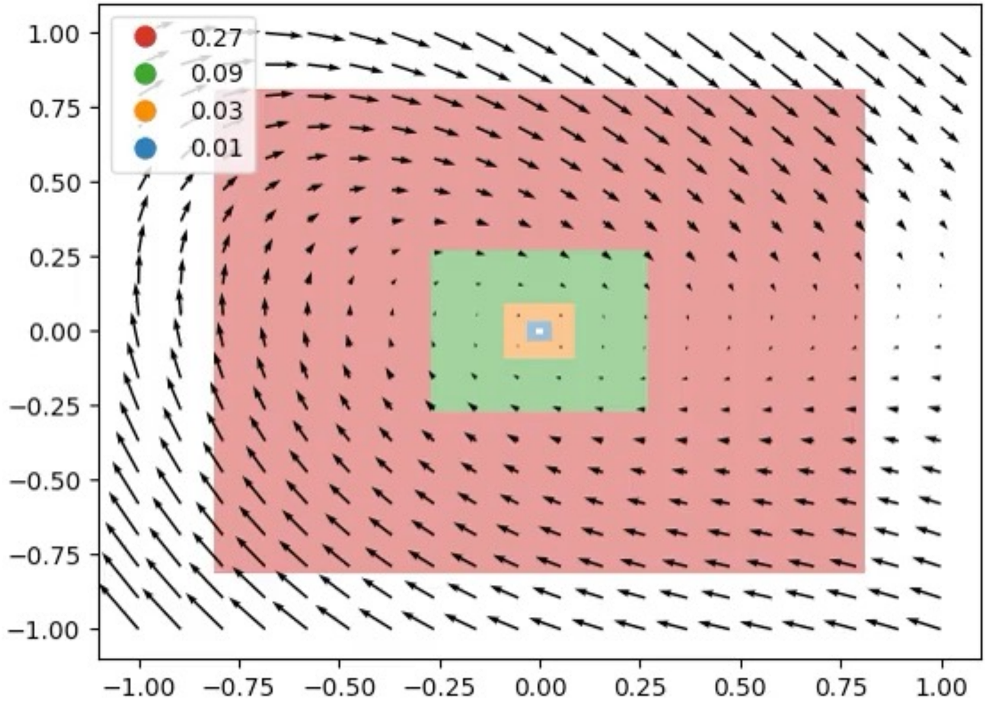
$$\dot{x} = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} x + B \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$$

with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

Parameter	Value
L	1.8
τ	1.5
ℓ	0.01

$\sigma = 0.5$





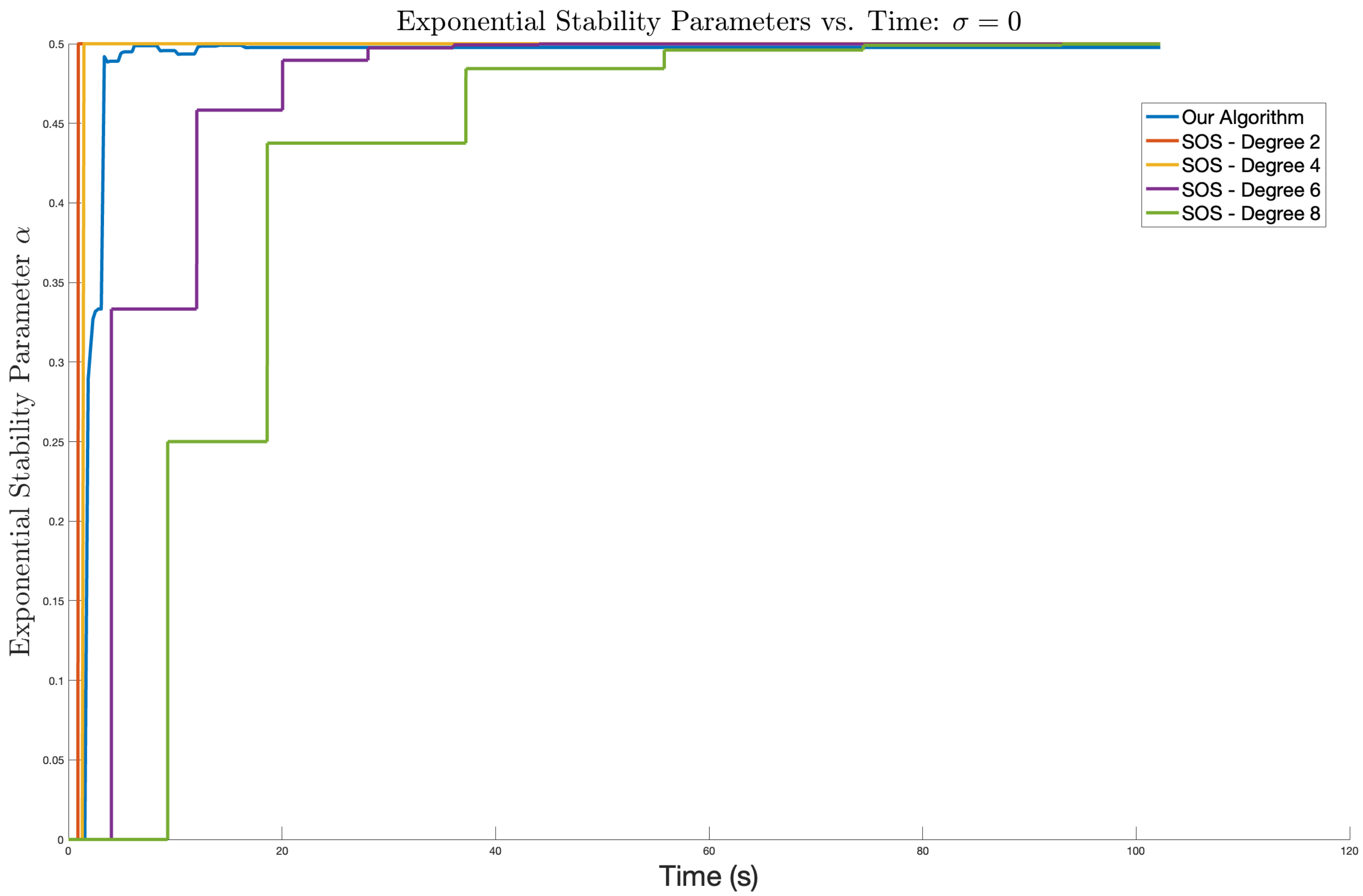
Comparison with SoS

Consider the 2-d non-linear system: $\dot{x} = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} x + B \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$

with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

Parameter	Value
L	1.8
τ	1.5
ℓ	0.01

$\sigma = 0.0$



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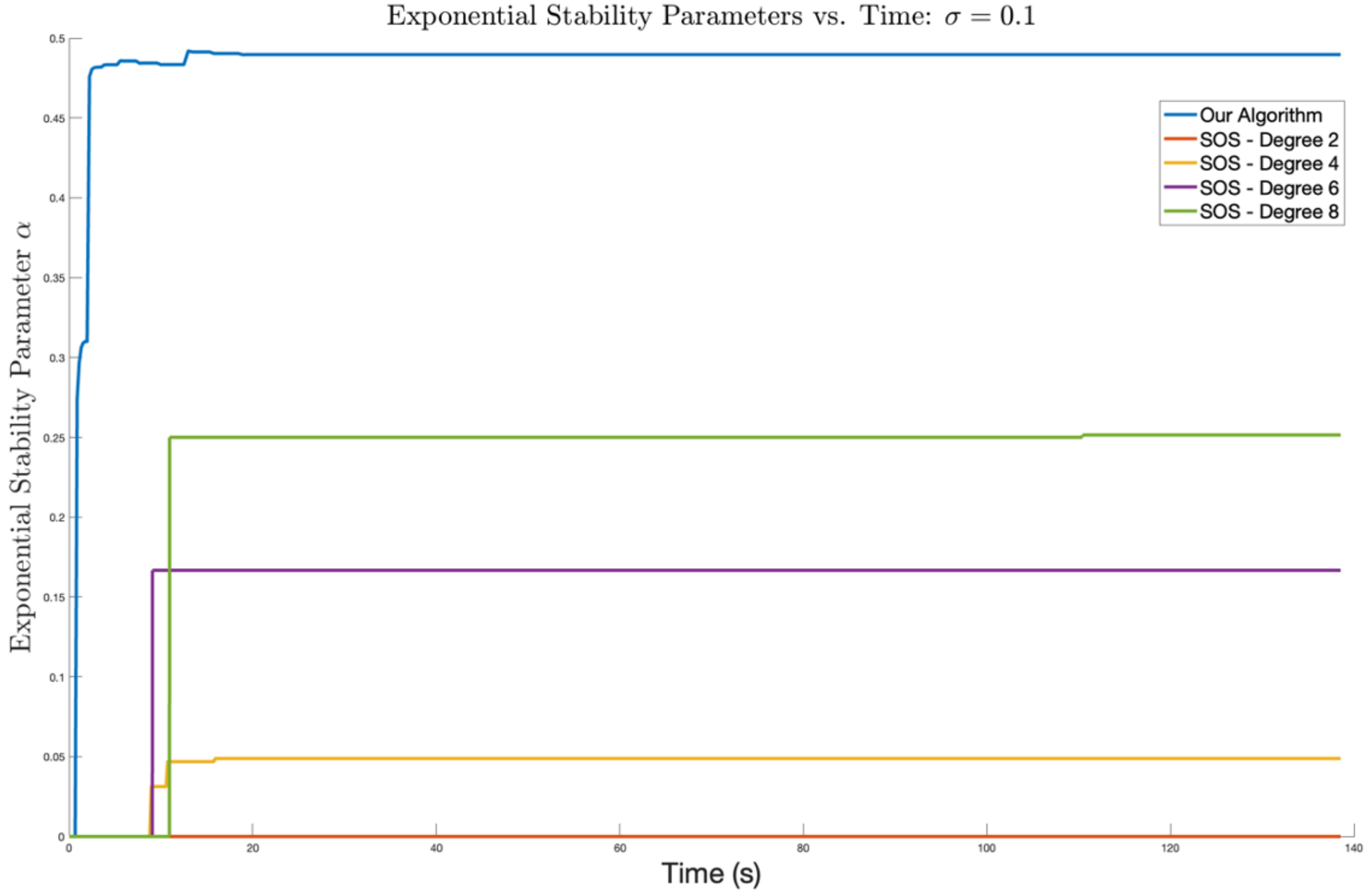
Consider the 2-d non-linear system: $$\dot{x} = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} x + B \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$$

with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

Parameter	Value
L	1.8
τ	1.5
ℓ	0.01

$\sigma = 0.1$



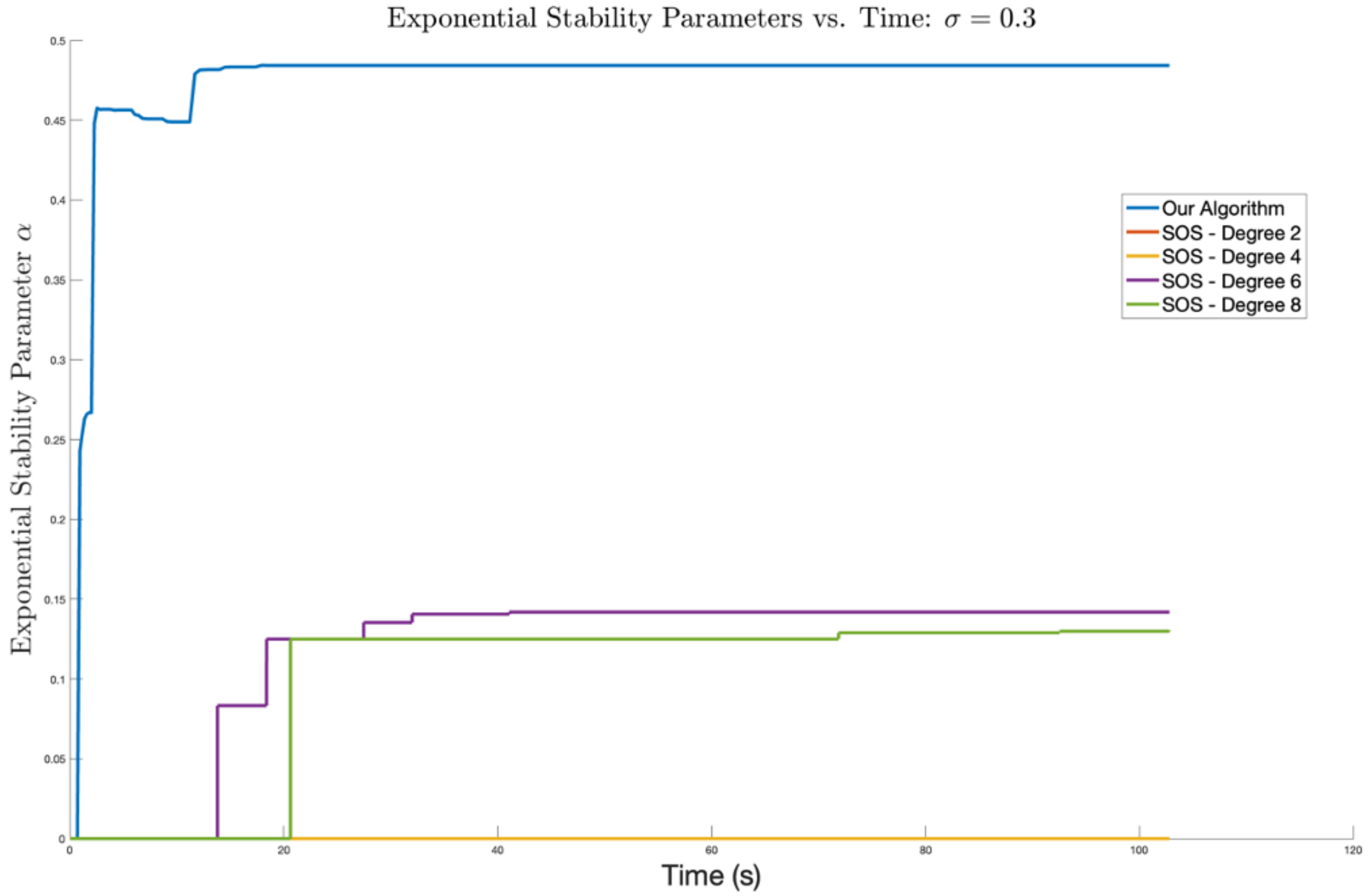
Comparison with SoS

Consider the 2-d non-linear system: $\dot{x} = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} x + B \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$

with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

Parameter	Value
L	1.8
τ	1.5
ℓ	0.01

$\sigma = 0.3$



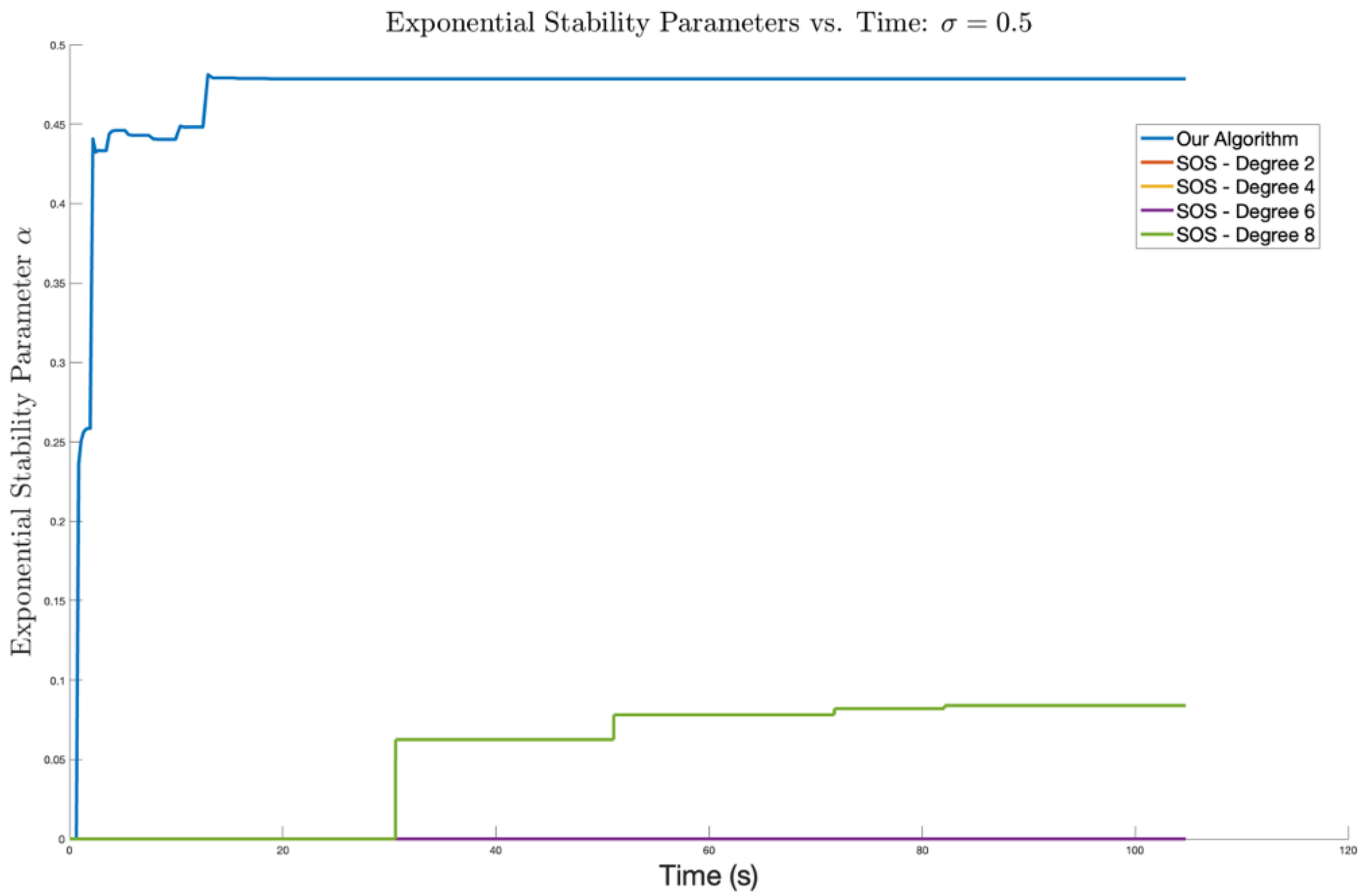
Comparison with SoS

Consider the 2-d non-linear system: $\dot{x} = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} x + B \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$

with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

Parameter	Value
L	1.8
τ	1.5
ℓ	0.01

$\sigma = 0.5$



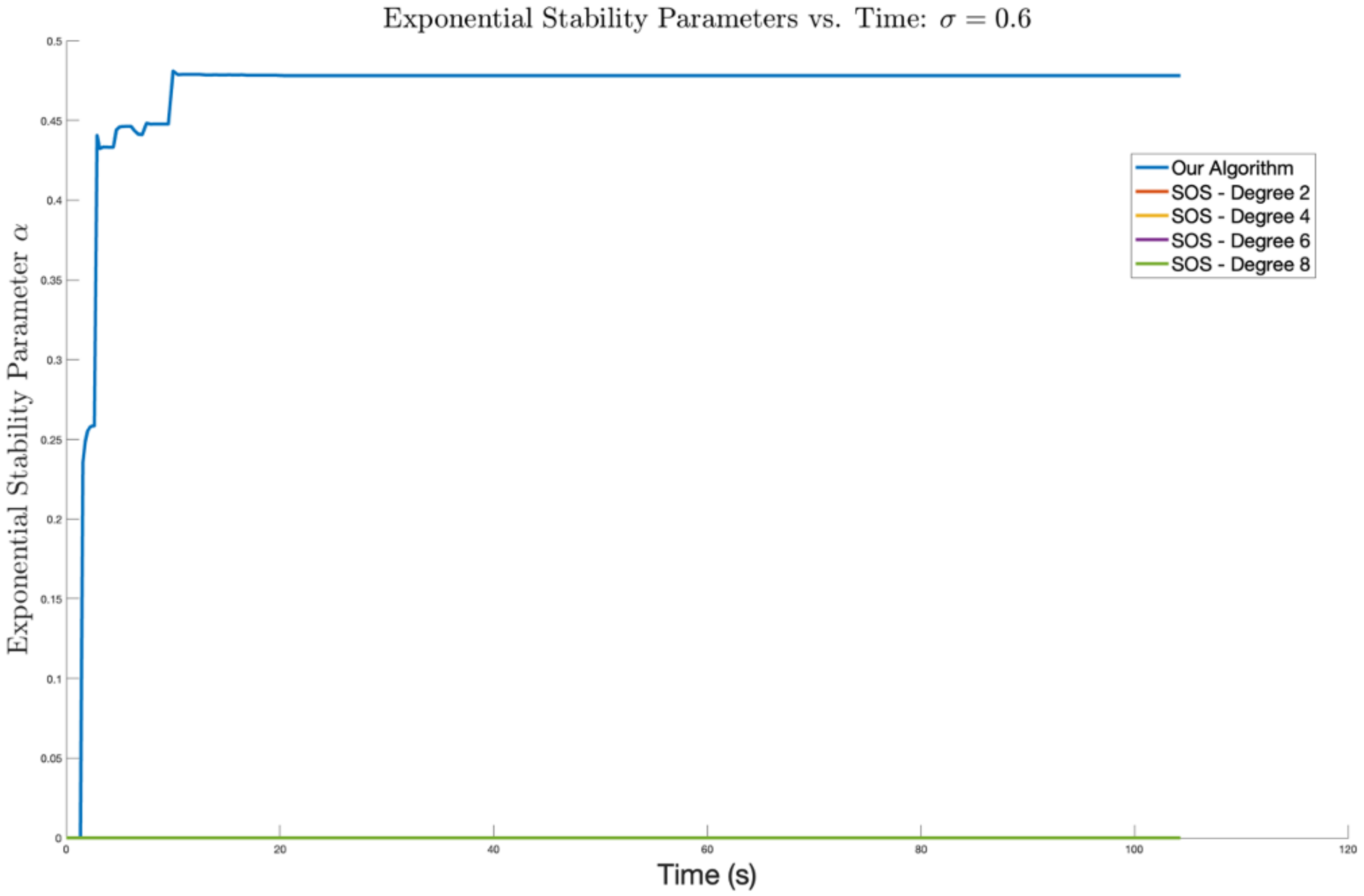
Comparison with SoS

Consider the 2-d non-linear system: $\dot{x} = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} x + B \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$

with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

Parameter	Value
L	1.8
τ	1.5
ℓ	0.01

$\sigma = 0.6$



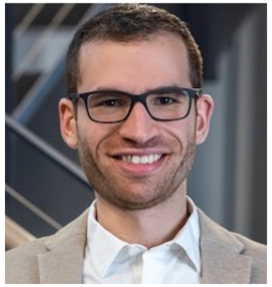
Outline

- Invariance: Merits and trade-offs
- Letting things go and come back: Recurrent sets
  - Approximating regions of attractions via recurrent sets
- **Non-parametric analysis of dynamical systems**
  - **Stability analysis via non-monotonic Lyapunov conditions**
  - Safety verification via generalized Barrier functions

Outline

- Invariance: Merits and trade-offs
- Letting things go and come back: Recurrent sets
  - Approximating regions of attractions via recurrent sets
- **Non-parametric analysis of dynamical systems**
  - Stability analysis via non-monotonic Lyapunov conditions
  - **Safety verification via generalized Barrier functions**

Yue Shen



Hussein Sibai

Nonparametric Safety Verification using Recurrence

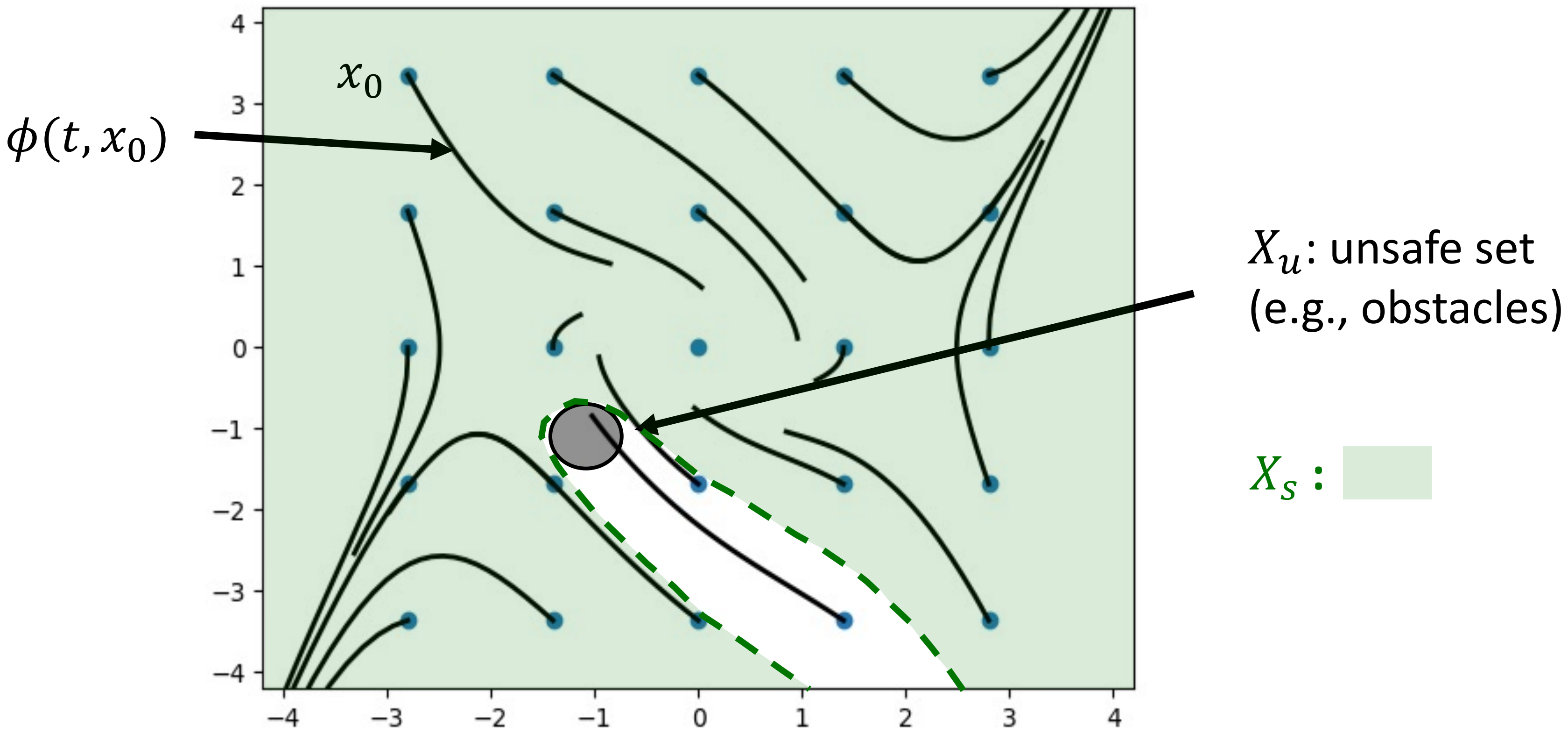
Y. Shen, H. Sibai, E. Mallada, "Generalized Barrier Functions: Integral Conditions and Recurrent Relaxations", in 60th Allerton Conference on Communication, Control, and Computing 2024

Safety in Dynamical Systems

Consider the continuous-time dynamical system: $\dot{x} = f(x)$

- $\phi(t, x_0)$: solution at time t starting from x_0
- X_u: set of unsafe states

Goal: Find the safe set $\mathcal{X}_s := \{x_0 \in \mathbb{R}^d | \phi(t, x_0) \notin \mathcal{X}_u, \forall t \geq 0\}$



Safety in Dynamical Systems via Invariant Sets

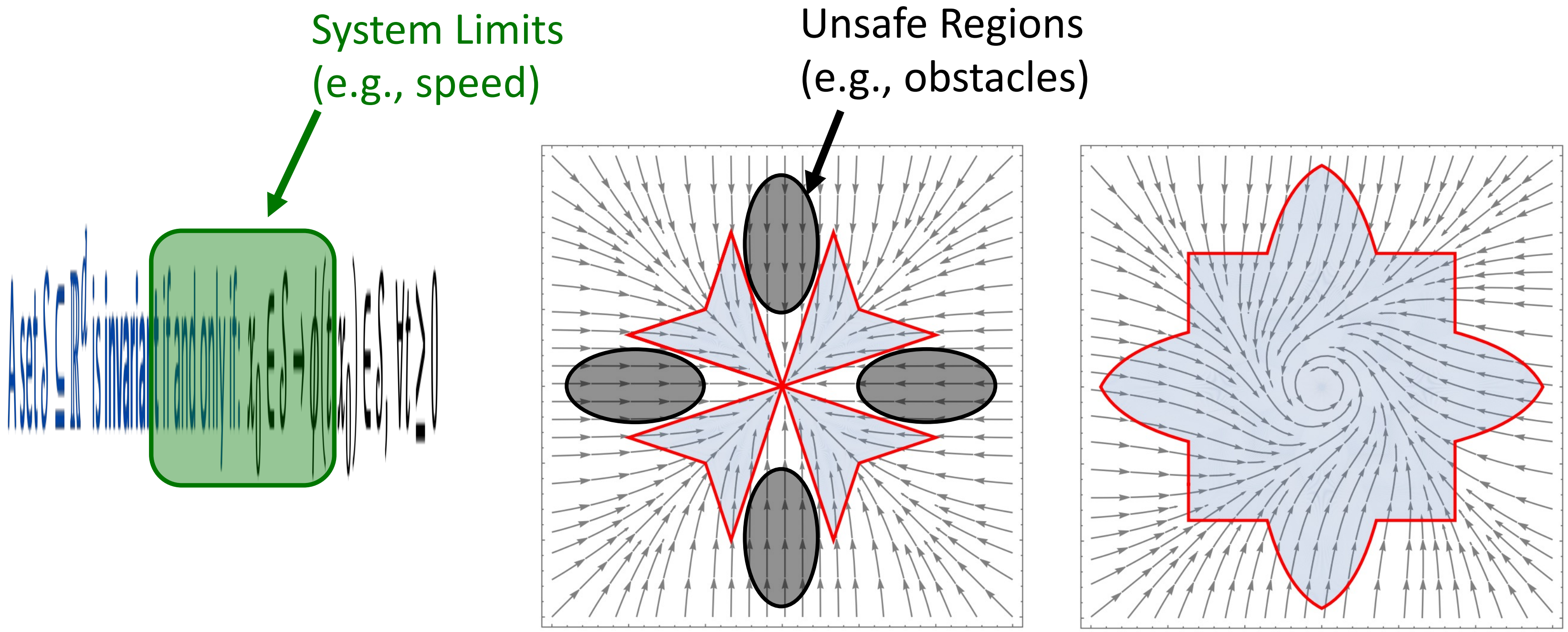
Consider the continuous-time dynamical system: $\dot{x} = f(x)$

- $\phi(t, x_0)$: solution at time t starting from x_0
- X_u: set of unsafe states

Goal: Find the safe set $\mathcal{X}_s := \{x_0 \in \mathbb{R}^d | \phi(t, x_0) \notin \mathcal{X}_u, \forall t \geq 0\}$

General Approach: Use invariant sets!

A set $\mathcal{S} \subseteq \mathbb{R}^d$ is **invariant** if and only if: $x_0 \in \mathcal{S} \rightarrow \phi(t, x_0) \in \mathcal{S}, \ \forall t \geq 0$



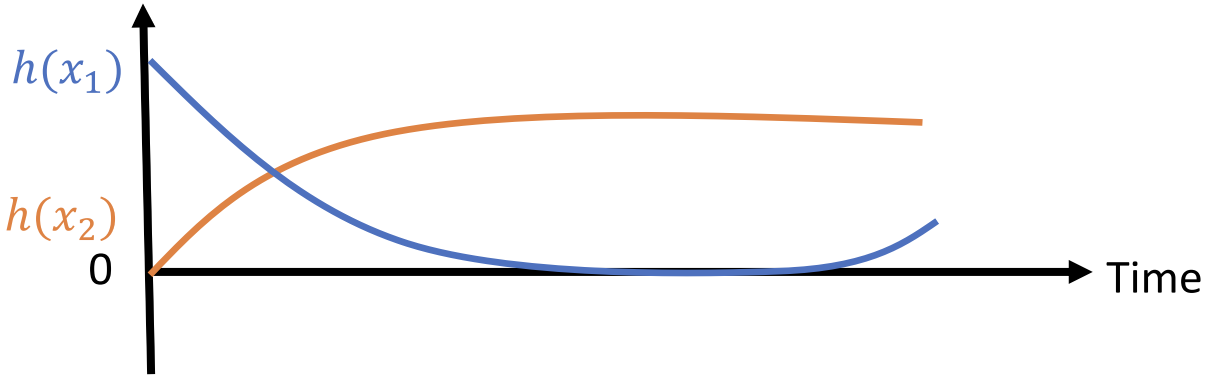
Certifying Safety using Barrier Functions

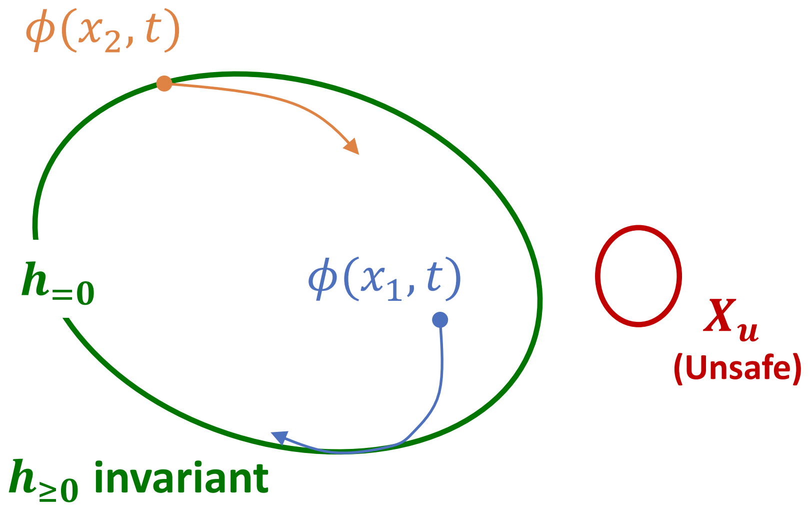
Theorem - Nagumo's Barrier Functions [Nagumo '42] **:**
Let $h\colon \mathbb{R}^{\mathrm{d}} \to \mathbb{R}$ be differentiable, with 0 being a *regular value.*
Then h is a Nagumo's Barrier Function (NBF) satisfying:

$$L_f h(x) := \lim_{t \to 0} \frac{h(\phi(t,x)) - h(x)}{t} \geq 0, \quad \forall x \in h_{=0},$$

if and only if $h_{\geq 0} \coloneqq \{x \in \mathbb{R}^d \,|\, h(x) \geq 0\}$ *is invariant.*


Mitio Nagumo





Then $h_{\geq 0}$ is a safe set whenever $h_{\geq 0} \cap X_u = \emptyset$

M. Nagumo, "Über die lage der integralkurven gewöhnlicher differentialgleichungen," Proceedings of the Physico-Mathematical Society of Japan 1942

Shaping Safe Behavior using Barrier Functions (BFs)

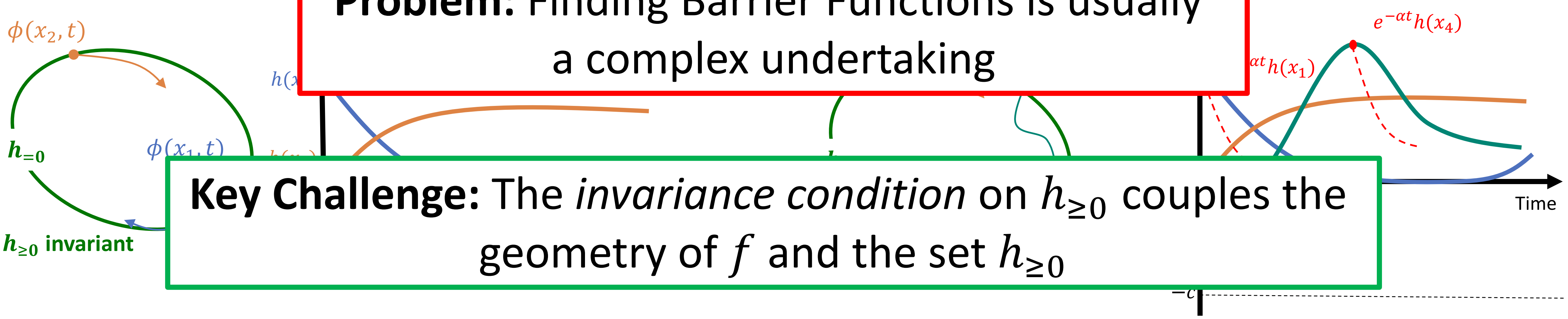
Barrier functions provide a flexible framework to shape the behavior of trajectories

Nagumo's (NBF)

$$L_f h(x) \geq 0\,, \quad \forall x \in h_{=0}$$

Exponential Barrier Functions (EBF)

$$L_f h(x) \geq -\alpha h(x)\,, \quad \forall x \in h_{\geq -c}$$



Problem: Finding Barrier Functions is usually a complex undertaking

Key Challenge: The *invariance condition* on $h_{\geq 0}$ couples the geometry of f and the set $h_{\geq 0}$

Other: Zeroing BFs (ZBFs), Minimal BFs (MBFs), Control BFs (CBFs), High Order CBFs (HOCBFs), …

S. Prajna, A. Jadbabaie. *Safety Verification of Hybrid Systems Using Barrier Certificates.* HSCC 2004

P. Wieland, F. Allgöwer. *Constructive safety using control barrier functions*. IFAC Proceedings Volumes 2007

A. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, P. Tabuada. *Control barrier functions: Theory and applications*. IEEE ECC 2019

R. Konda, A. Ames, S. Coogan. *Characterizing safety: Minimal control barrier functions from scalar comparison systems.* IEEE L-CSS 2020

W. Xiao, C. Belta. *High-order control barrier functions*. IEEE TAC 2021

Integral Nagumo's Barrier Function (INBF)

Nagumo's Barrier Functions [Nagumo '42]**:**
Let h *be differentiable, regular at zero, and*

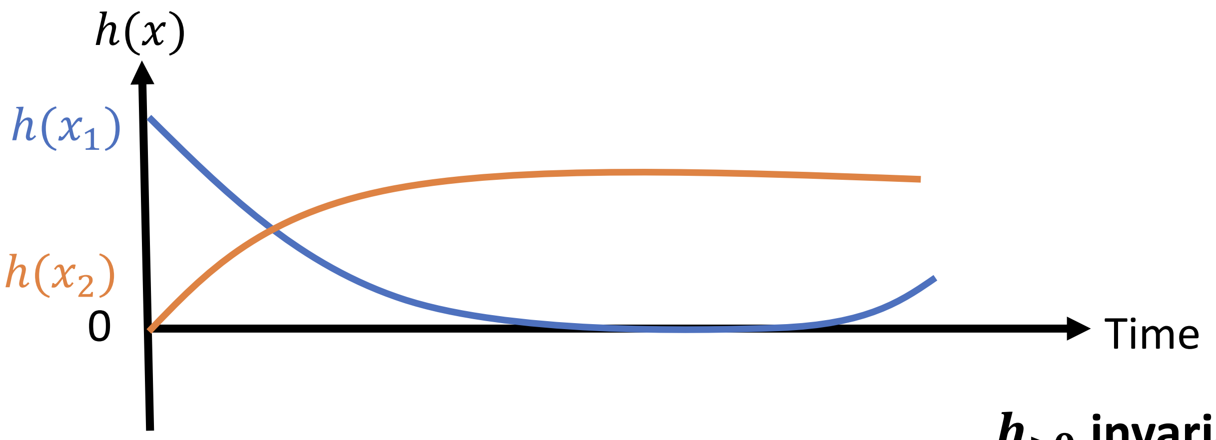
$$L_f h(x) \geq 0\,,\ \forall x \in h_{=0}$$

Integral Nagumo's Barrier Function:
Let h *be continuous, and*

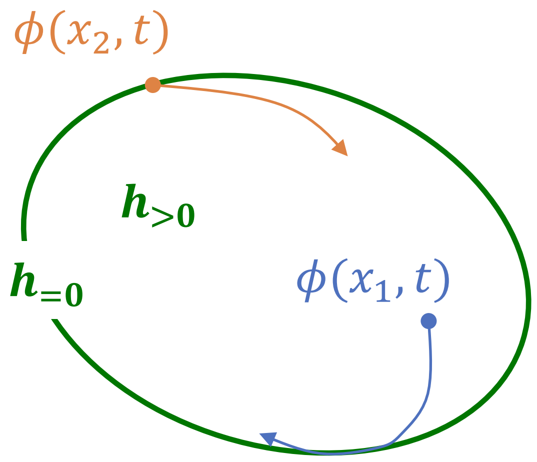
$$h(\phi(t,x)) \geq 0\,,\ \forall x \in h_{=0}\,, t \geq 0$$

The super-level set $h_{\geq 0}$ is invariant.

* requires more conditions on h



$h_{\geq 0}$ invariant



Recurrent Nagumo's Barrier Function (RNBF)

Thm: Integral Nagumo's Barrier Function:
Let h *be continuous. Then:*

$$h(\phi(t,x)) \geq 0\,,\ \forall x \in h_{=0}\,, t \geq 0$$

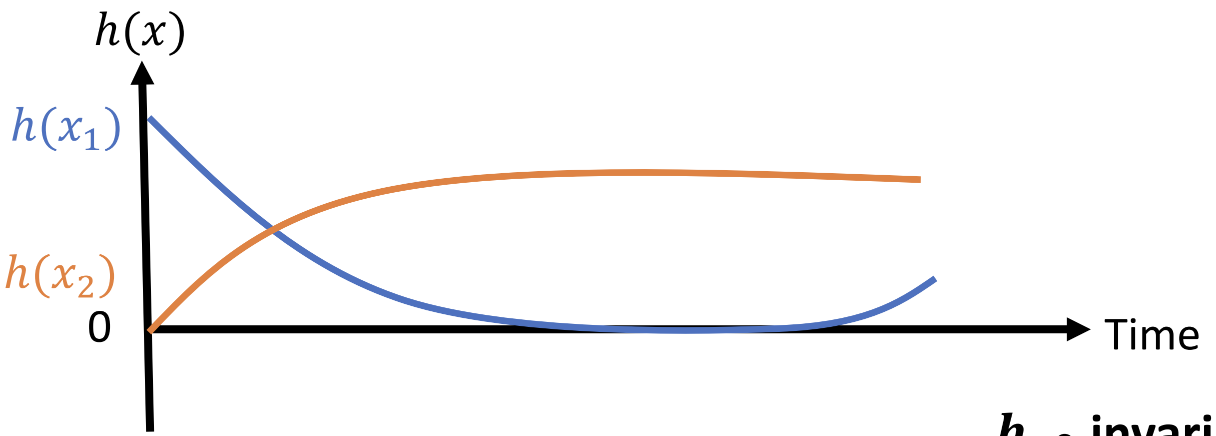
if and only if $h_{\geq 0}$ is **invariant**

Thm: Recurrent Nagumo's Barrier Function:
Let h *be continuous. Then:*

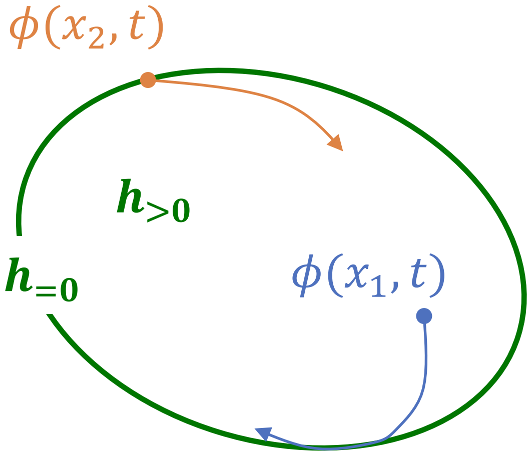
$$\max_{t\in(0,\tau]} h(\phi(t,x)) \geq 0\,,\ \forall x \in h_{=0}$$

if and only if $h_{\geq 0}$ is $\boldsymbol{\tau}$**-recurrent**

Recall: A set $\mathcal{R} \subseteq \mathbb{R}^d$ is $\boldsymbol{\tau}$**-recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \in (t, t+\tau]$ s.t. $\phi(t', x_0) \in \mathcal{R}$.



$h_{\geq 0}$ invariant



Recurrent Nagumo's Barrier Function (RNBF)

Thm: Integral Nagumo's Barrier Function:
Let h *be continuous. Then:*

$$h(\phi(t,x)) \geq 0\,,\ \forall x \in h_{=0}\,, t \geq 0$$

if and only if $h_{\geq 0}$ is **invariant**

As $\tau \to 0$

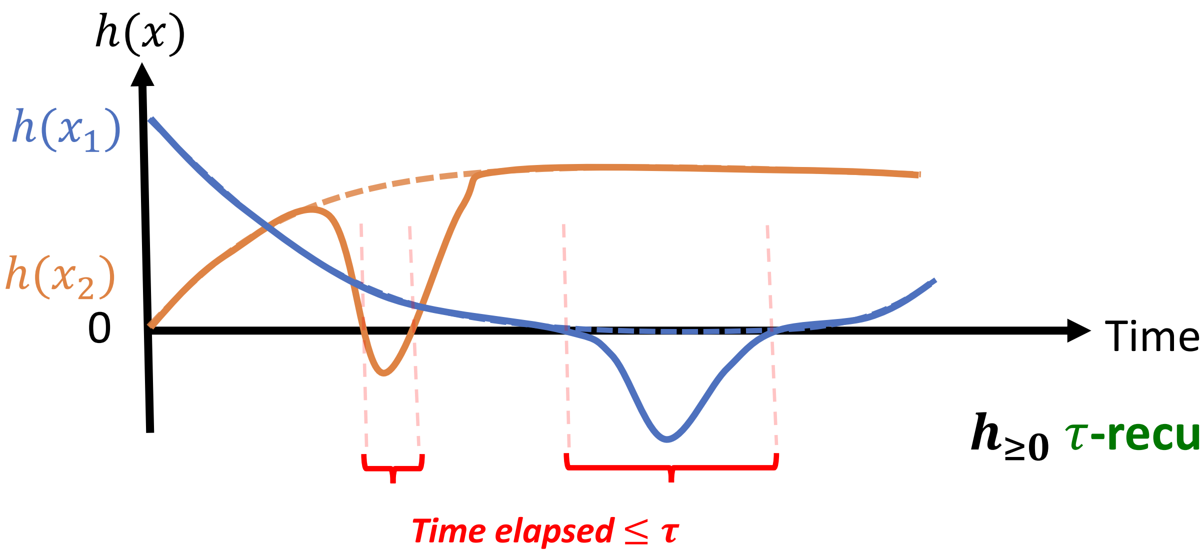
By definition

Thm: Recurrent Nagumo's Barrier Function:
Let h *be continuous. Then:*

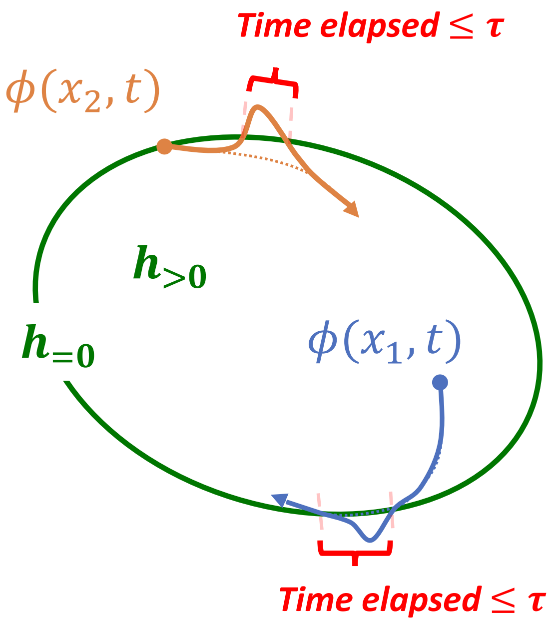
$$\max_{t \in (0,\tau]} h(\phi(t,x)) \geq 0\,,\ \forall x \in h_{=0}$$

if and only if $h_{\geq 0}$ is **τ-recurrent**

Recall: A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **τ-recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \in (t, t+\tau]$ s.t. $\phi(t', x_0) \in \mathcal{R}$.



$h_{\geq 0}$ τ-recurrent



Integral Exponential Barrier Function (IEBF)

Exponential Barrier Functions:
Let h *be differentiable, and*

$$L_f h(x) \geq -\alpha h(x)\,,\ \forall x \in h_{\geq -c}$$

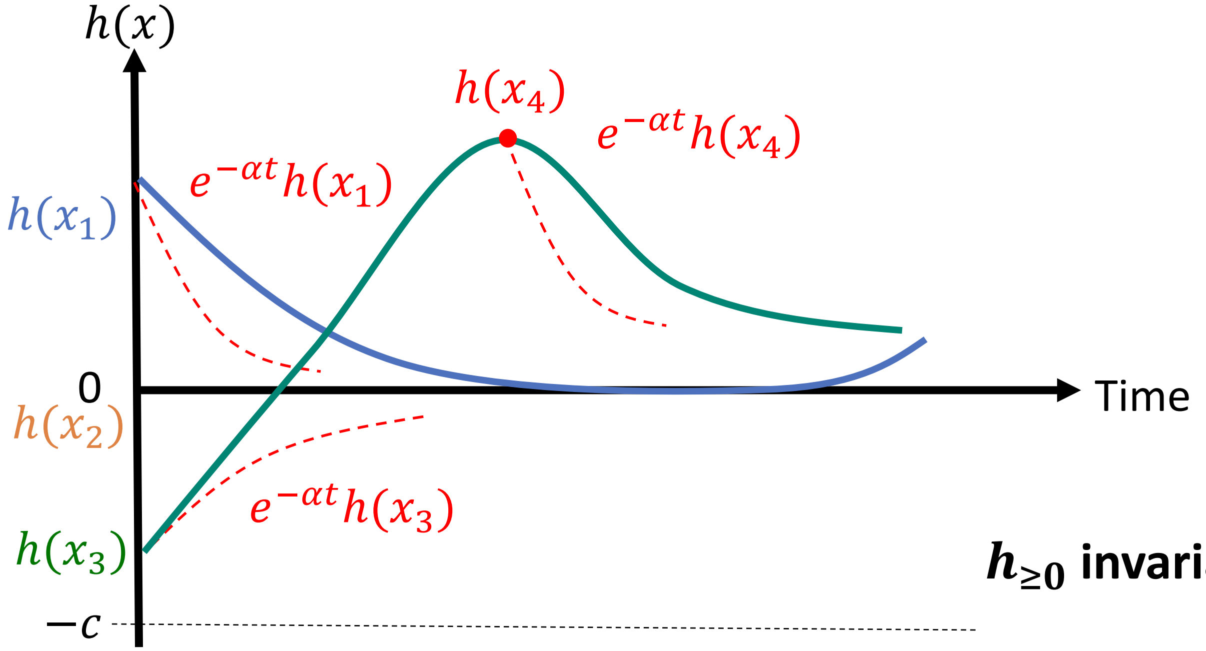
Integral Exponential Barrier Function:
Let h *be continuous, and*

$$h(\phi(t,x)) \geq e^{-\alpha t} h(x)\,,\ \forall x \in h_{\geq -c}$$

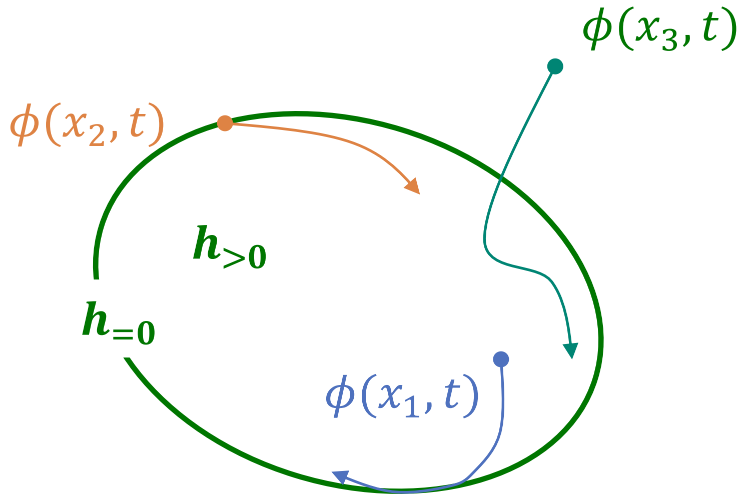
for all $t \geq 0$

The super-level set $h_{\geq 0}$ is invariant.

* requires more conditions on h



$h_{\geq 0}$ invariant



Recurrent Exponential Barrier Function (REBF)

Thm: Integral Exponential Barrier Function:
Let h *be continuous. If:*

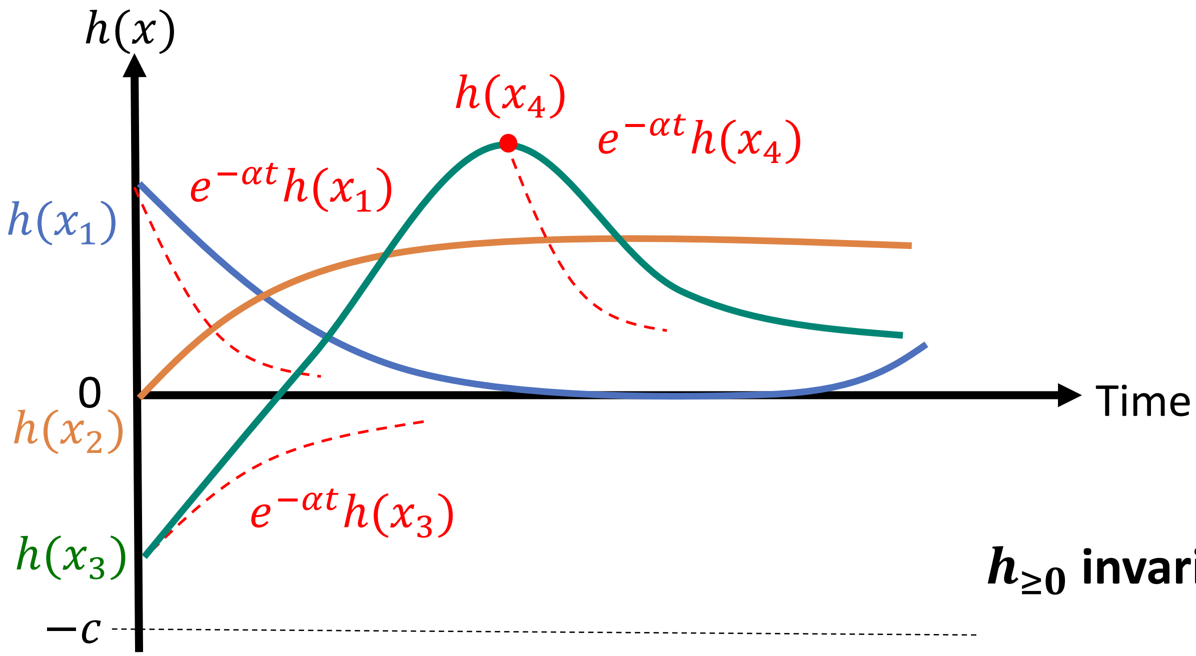
$$h(\phi(t,x)) \geq e^{-\alpha t} h(x)\,,\ \forall x \in h_{\geq -c}$$

for all $t \geq 0$, then, $h_{\geq 0}$ is **invariant**

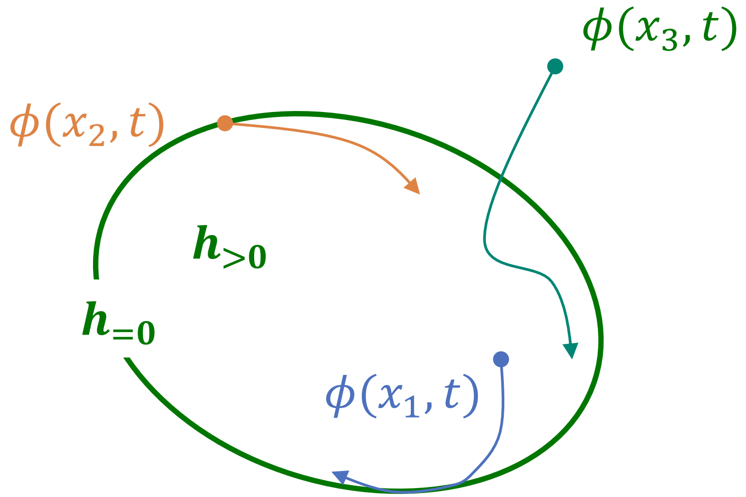
Thm: Recurrent Exponential Barrier Function:
Let h *be continuous. If:*

$$\max_{t\in(0,\tau]} e^{\alpha t} h(\phi(t,x)) \geq h(x)\,,\ \forall x \in h_{\geq -c}$$

then, $h_{\geq 0}$ is **τ-recurrent**



$h_{\geq 0}$ invariant



Recurrent Exponential Barrier Function (REBF)

Thm: Integral Exponential Barrier Function:
Let h *be continuous. If:*

$$h(\phi(t,x)) \geq e^{-\alpha t} h(x)\,,\ \forall x \in h_{\geq -c}$$

for all $t \geq 0$, then, $h_{\geq 0}$ is **invariant**

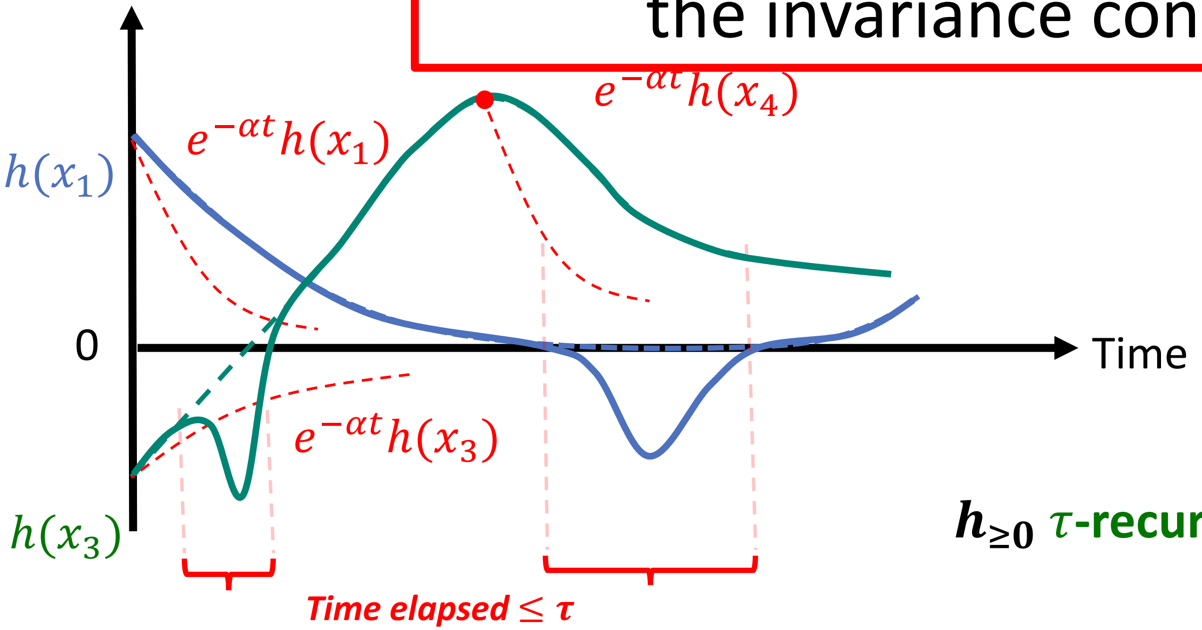
As $\tau \to 0$ ⇄ By definition

Thm: Recurrent Exponential Barrier Function:
Let h *be continuous. If:*

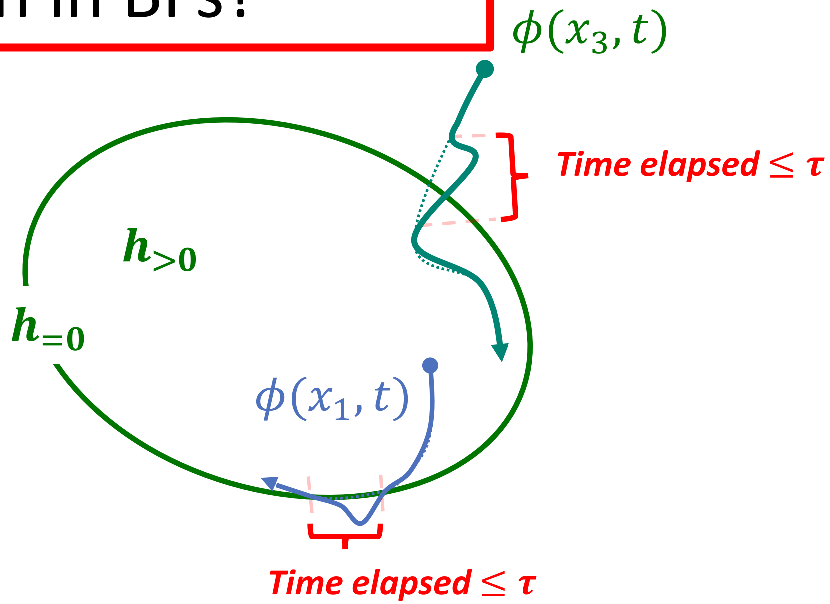
$$\max_{t \in (0,\tau]} e^{\alpha t} h(\phi(t,x)) \geq h(x)\,,\ \forall x \in h_{\geq -c}$$

then, $h_{\geq 0}$ is **τ-recurrent**

Question: Do we gain anything from relaxing the invariance condition in BFs?



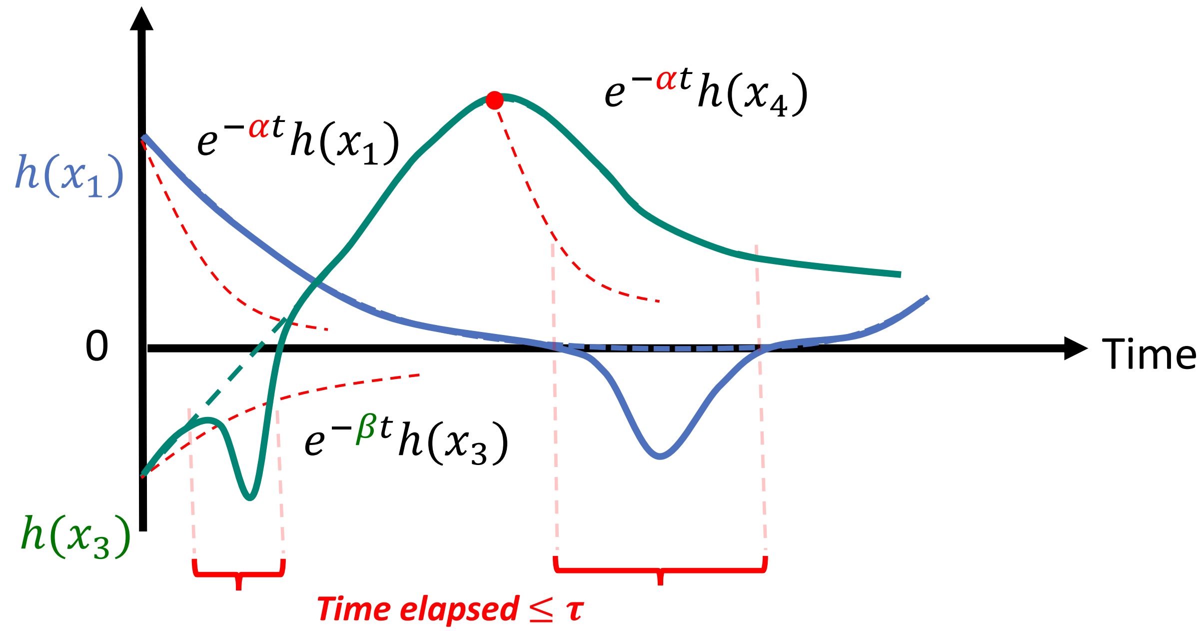
$h_{\geq 0}$ τ-recurrent

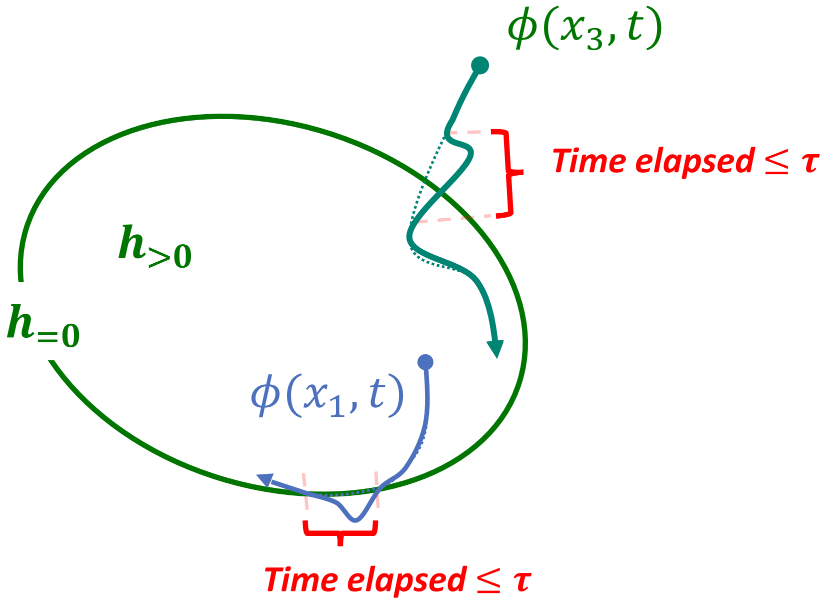


Bi-Exponential Recurrent Barrier Functions

We first generalize REBF using different exponential rates $\alpha, \beta > 0$:

$$\max_{t\in(0,\tau]} e^{\alpha t}[h(\phi(t,x))]_{+} + e^{\beta t}[h(\phi(t,x))]_{-} \geq h(x)\,,\ \forall x \in h_{\geq -c}$$





All Signed Norms are Recurrent Barrier Functions!

Theorem: Assume there exists an Integral Exponential BF (IEBF), h , defined over $D_0 \coloneqq h_{\geq -c}$ for some $c > 0$. Then $\exists\ \alpha > 0$ such that:

$$e^{\alpha t}\, h\big(\phi(t,x)\big) \geq h(x), \qquad \forall x \in h_{\geq -c}$$

for all $t \geq 0$.

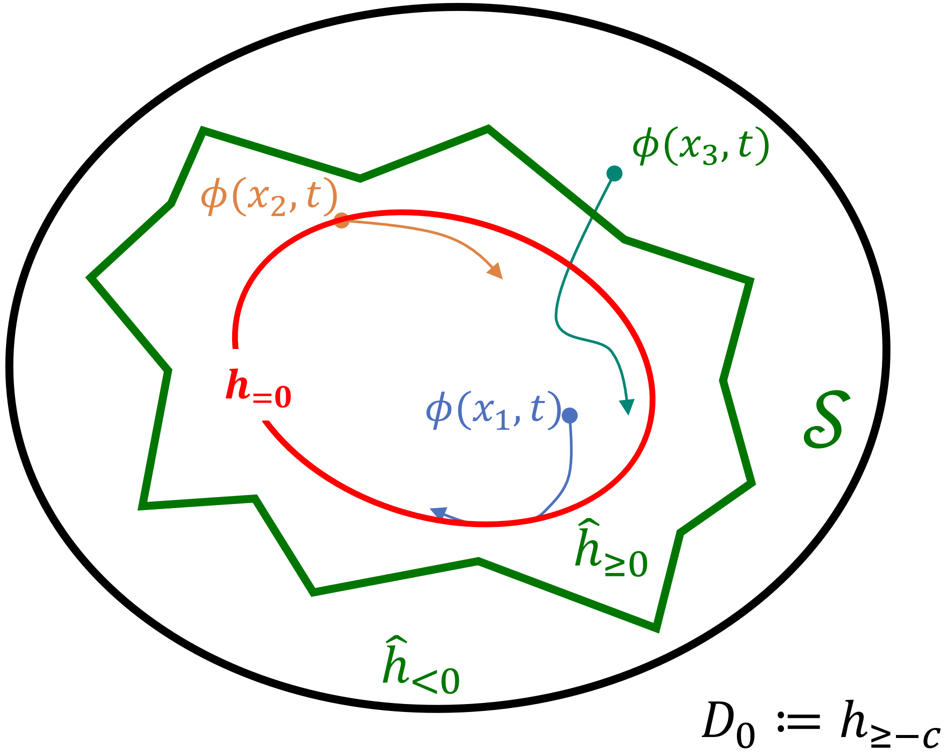
Then for any set $\mathcal{S}$ with $h_{\geq 0} \subseteq \mathcal{S} \subseteq h_{\leq -c}$, the function

$$\hat{h}(x) \coloneqq -\mathrm{sd}(x, \mathcal{S})$$

is a Recurrent Exponential Barrier Function (REBF):

$$\max_{t \in (0,\tau]} e^{\bar{\alpha}\, t}\left[h\big(\phi(t,x)\big)\right]_+ + e^{\underline{\alpha} t}\left[h\big(\phi(t,x)\big)\right]_- \geq h(x), \qquad \forall x \in h_{\geq -c}$$

with any parameters $\underline{\alpha} < \alpha < \bar{\alpha}$ whenever $\tau \geq \bar{\tau}(\bar{\alpha} - \alpha, \underline{\alpha} - \alpha)$



Remarks:

- The rates $\underline{\alpha} < \bar{\alpha}$ must be strictly smaller/bigger than α (giving up optimality).
- *Any signed norm of most sets is a Recurrent Barrier function!*

Question: How to use Recurrent Barriers for safety?

Certifying Safety using Recurrent Sets

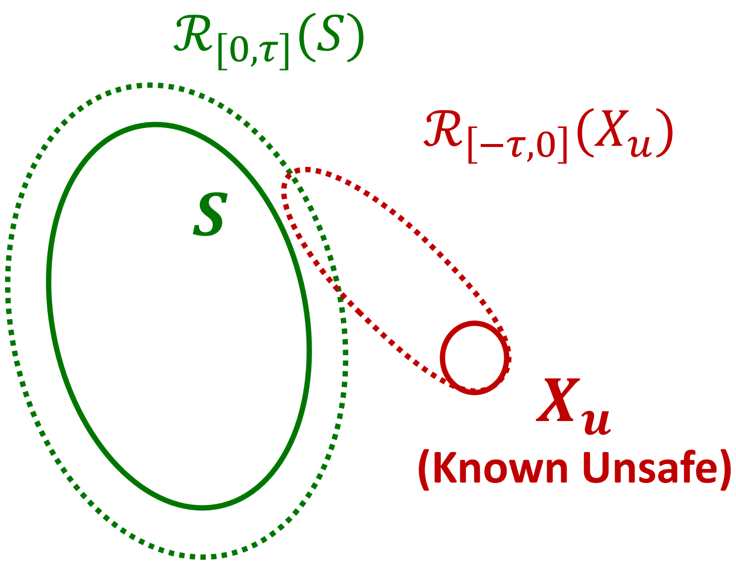
***Theorem* -** Consider a closed set S that is **τ-recurrent**.
Then its τ-reachable set:

$$\mathcal{R}_{[0,\tau]}(S) := \bigcup_{\substack{x \in S \\ t \in [0,\tau]}} \phi(t,x)$$

is **invariant**.

Moreover, S **is safe whenever:**

1. $\mathcal{R}_{[0,\tau]}(S) \cap X_u = \emptyset$,
2. $S \cap \mathcal{R}_{[-\tau,0]}(X_u) = \emptyset$



Nonparametric Safety Verification

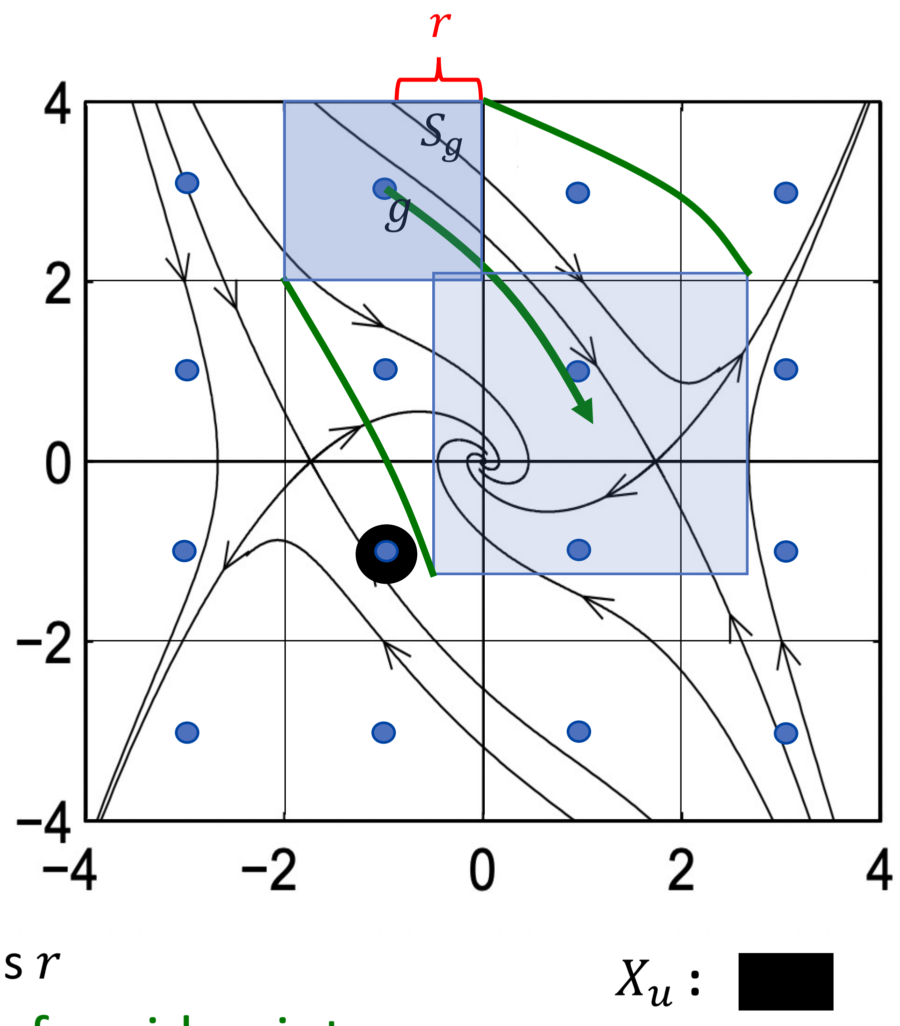
A set S is safe whenever:

Reachability Condition

- $\mathcal{R}_{[0,\tau]}(S) \cap X_u = \emptyset$

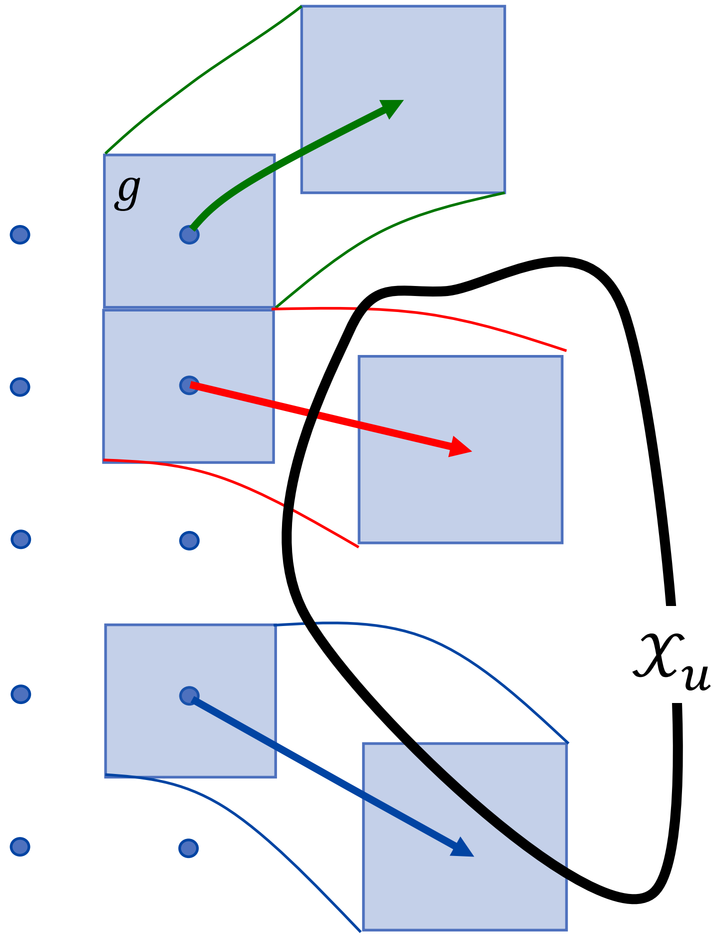
Recurrent Condition

- $\hat{h}(x) := -\,\mathrm{sd}\,(\,x\,, S\,)$ is RNBF or REBF



- Cover the region with a grid G
  - For each point $g \in G$, S_g represents its cell of radius r
- We build $S = \cup_{g \in G^s} S_g$, with G^s representing safe grid points
  - Initialize $G^s \leftarrow G$ (all grid points are initially safe)
- Check both **conditions** using only one trajectory for each cell!

Nonparametric Safety Verification: Reachability



Checking Reachability Condition

Starting from centers of grid cell g, simulate the trajectories for τ-seconds in parallel using a GPU

If $\phi(t,x) \notin \mathcal{X}_u, \forall x \in S_g, t \in [0,\tau]$
$\mathcal{R}_{[0,\tau]}(S_g) \cap X_u = \emptyset$

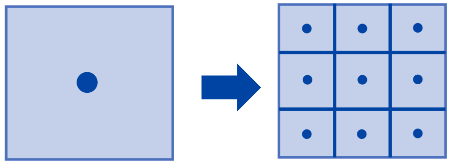
→ Keep g in G^s

If $\exists t \in [0,\tau]$ s.t. $\phi(t,x) \in \mathcal{X}_u, \forall x \in S_g$
$\mathcal{R}_t(S_g) \subseteq X_u$ for some $t \in [0,\tau]$

→ Remove g from G^s
Add g to G^u

Else
Undetermined

→ Split g



*Stop splitting g and mark it as unsafe whenever g is too small ($r \le r_{\min}$)

Nonparametric Safety Verification

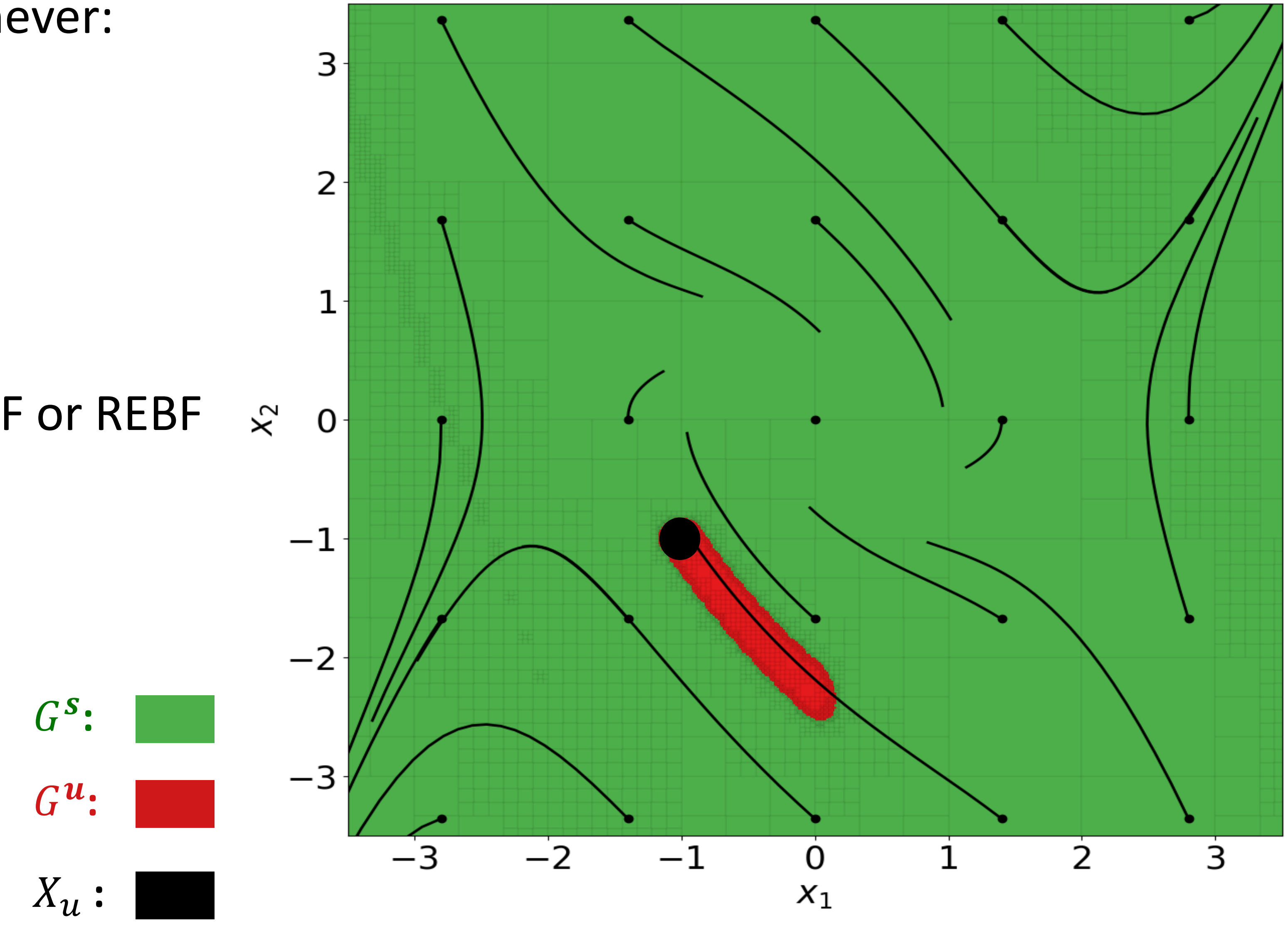
A set $S = \cup_{s \in G^s} S_g$ is safe whenever:

Reachability Condition

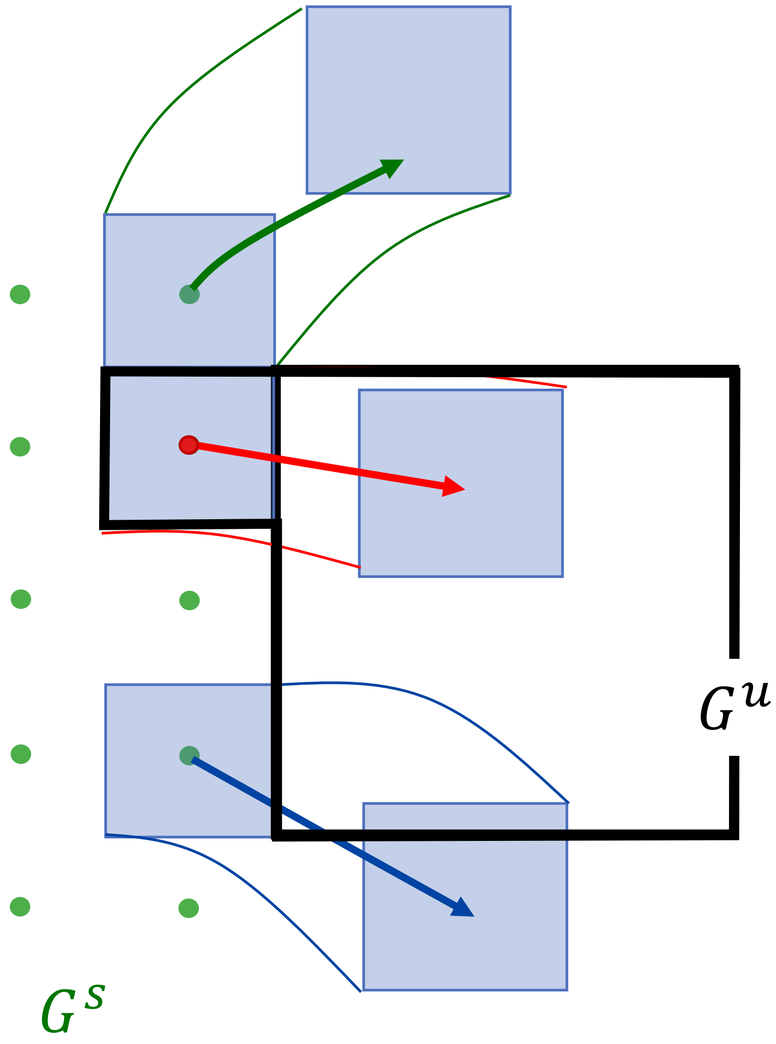
- $\mathcal{R}_{[0,\tau]}(S) \cap X_u = \emptyset$

Recurrent Condition

- $\hat{h}(x) \coloneqq -\,\text{sd}\,(\,x\,, S\,)$ is a RBF or REBF



Nonparametric Safety Verification: Recurrent Condition



Check the Recurrent condition

- Let $S = \cup_{g \in G^s} S_g$, $\hat{h}(x) := -\mathrm{sd}(x, S)$
- Starting from centers of grid cell $g \in G^s$, simulate the trajectories for τ-seconds in parallel using a GPU

If $\max_{t \in (0,\tau]} e^{\hat{\alpha} t} \hat{h}(\phi(t,x)) \geq \hat{h}(x), \forall x \in S_g$

REBF condition is satisfied within g

→ Keep g in G^s

If $\max_{t \in (0,\tau]} e^{\hat{\alpha} t} \hat{h}(\phi(t,x)) < \hat{h}(x), \forall x \in S_g$

REBF condition is NOT satisfied within S_g

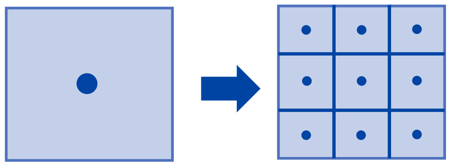
→ Remove g from G^s
Add g to G^u

Else

Undetermined

→ Split g

*Stop splitting g and mark it as unsafe whenever g is too small ($r \leq r_{\min}$)



A GPU based algorithm

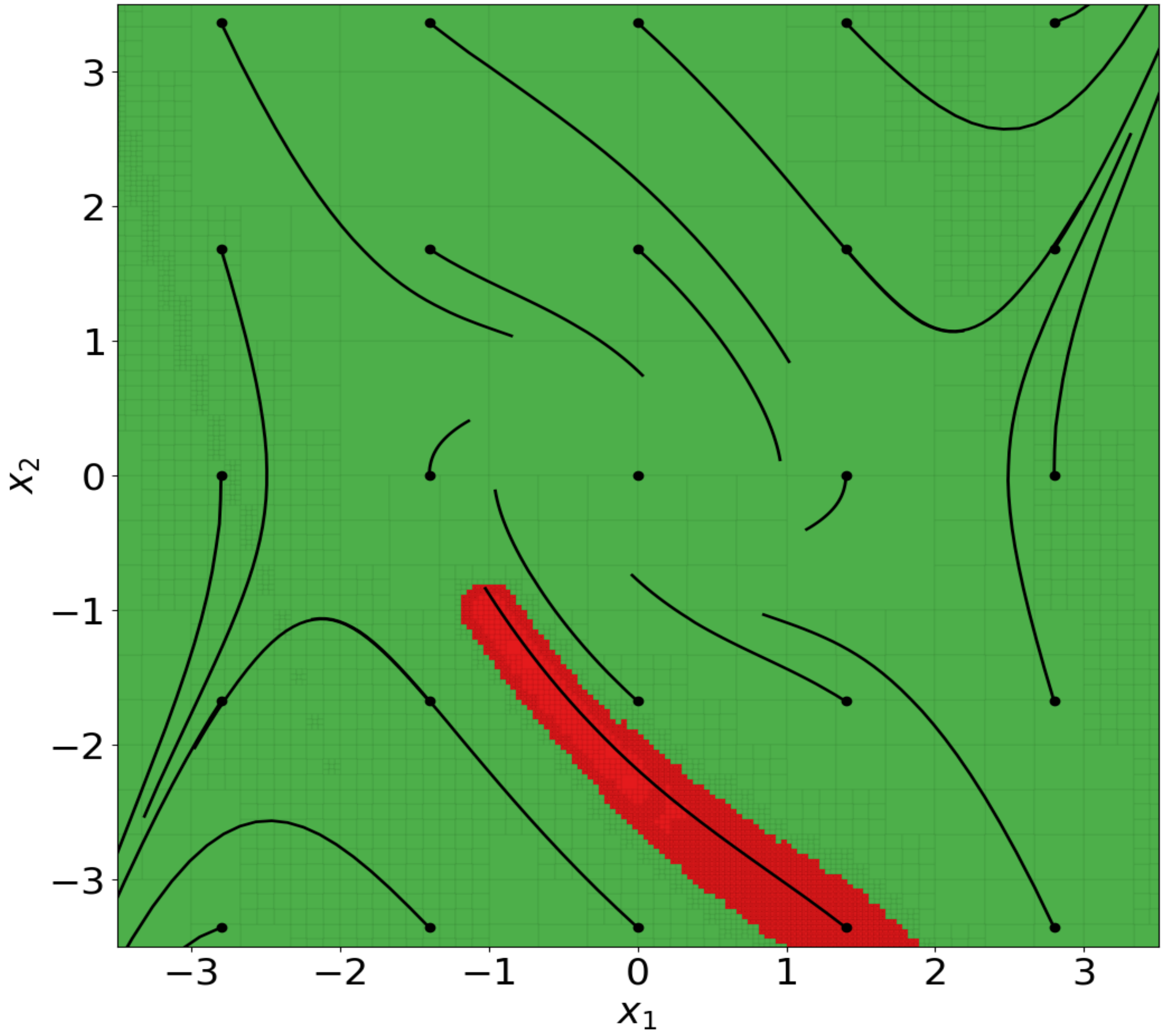
A set $S = \cup_{s \in G^s} S_g$ is safe whenever:

Reachability Condition

- $\mathcal{R}_{[0,\tau]}(S) \cap X_u = \emptyset$

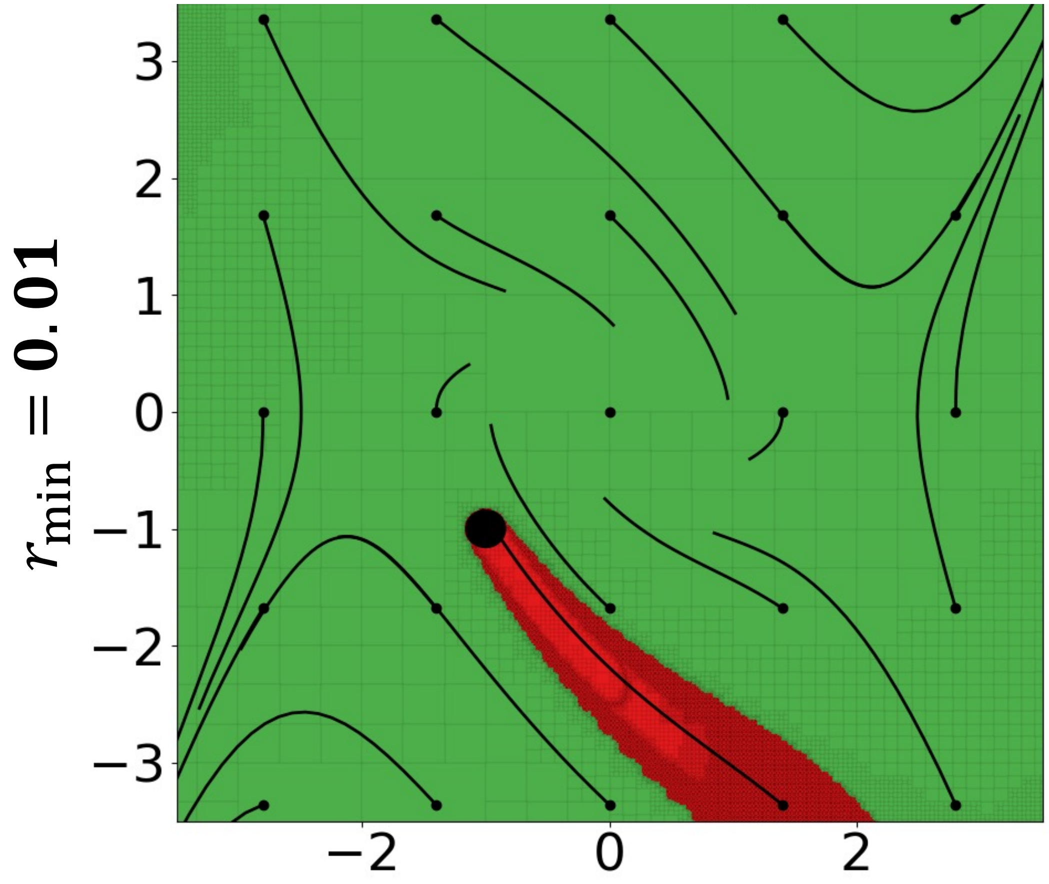
Recurrent Condition

- $\hat{h}(x) := -\,\mathrm{sd}\,(x, S)$ is a REBF or RNBF



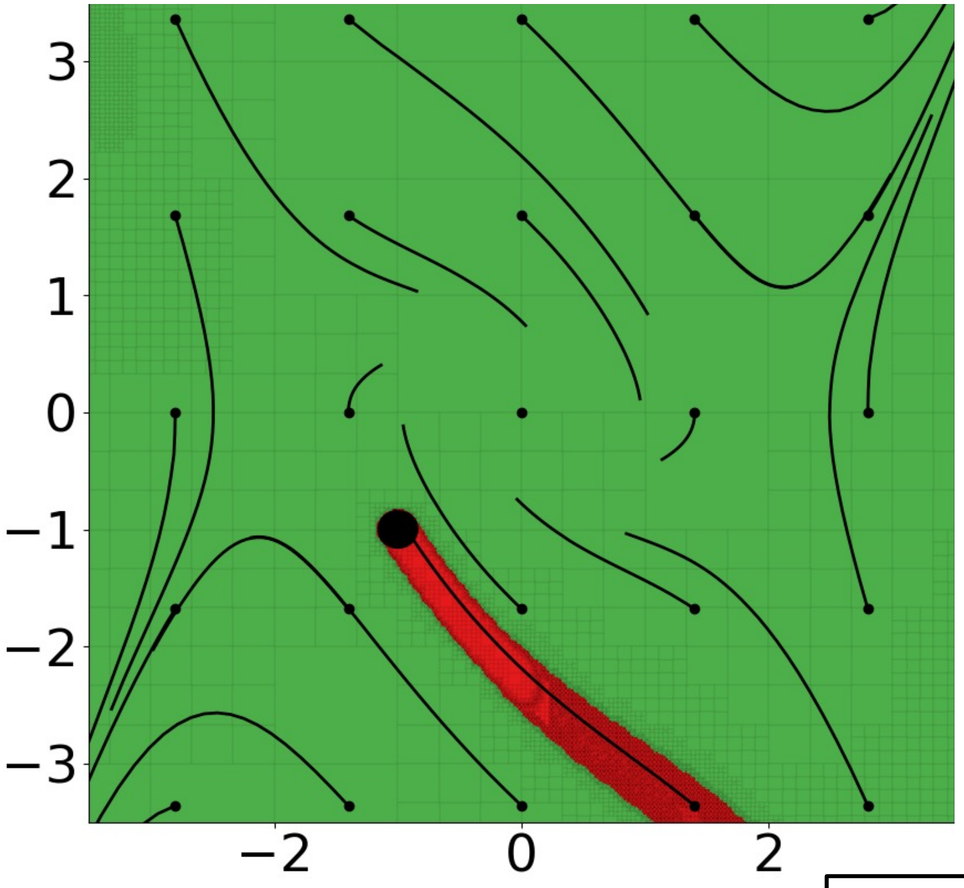
Numerical Validation: Reachability vs Recurrence

Reachability



running time: 354.12s
97.4% safe set covered

Recurrent (RNBF)

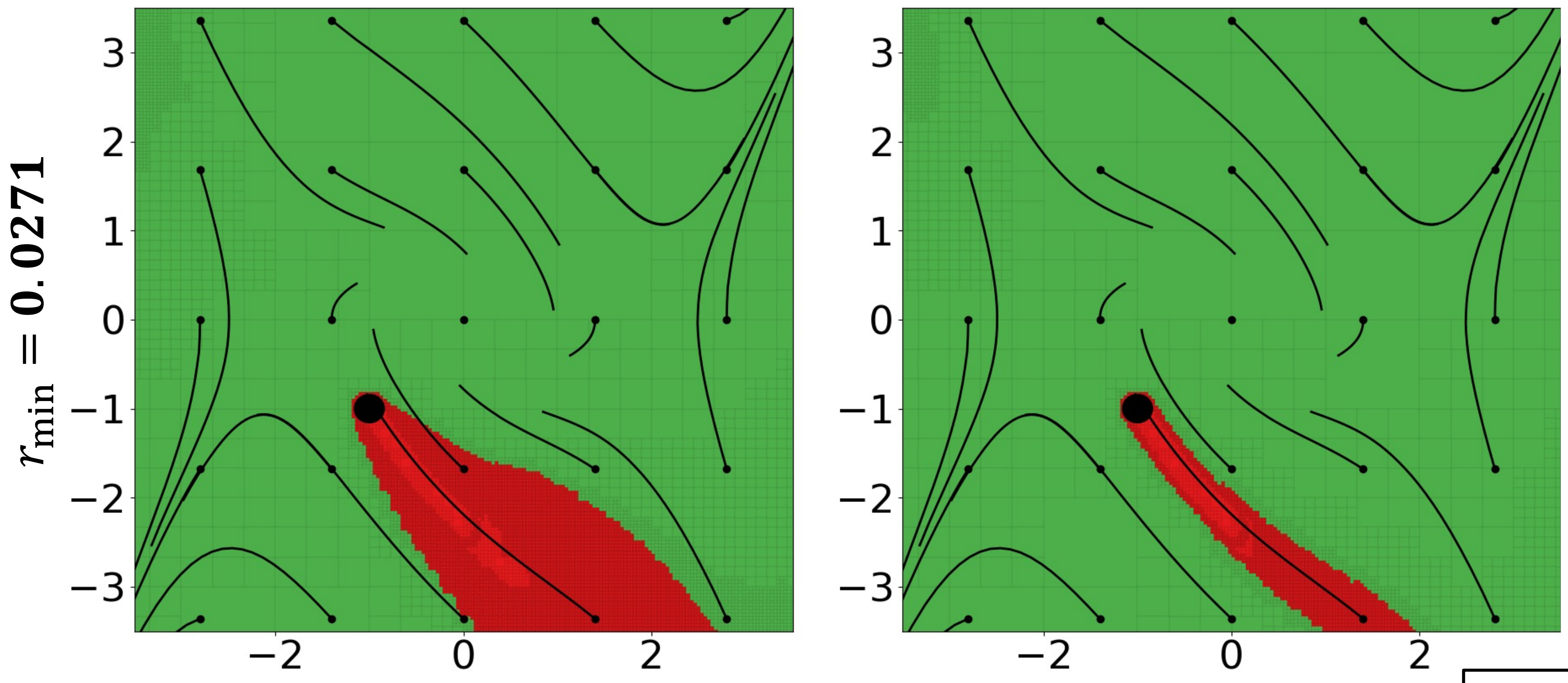


running time: 20.68s
99.6% safe set covered

~17x faster
+2.2% more area

Numerical Validation: Reachability vs Recurrence

Reachability | **Recurrent (RNBF)**

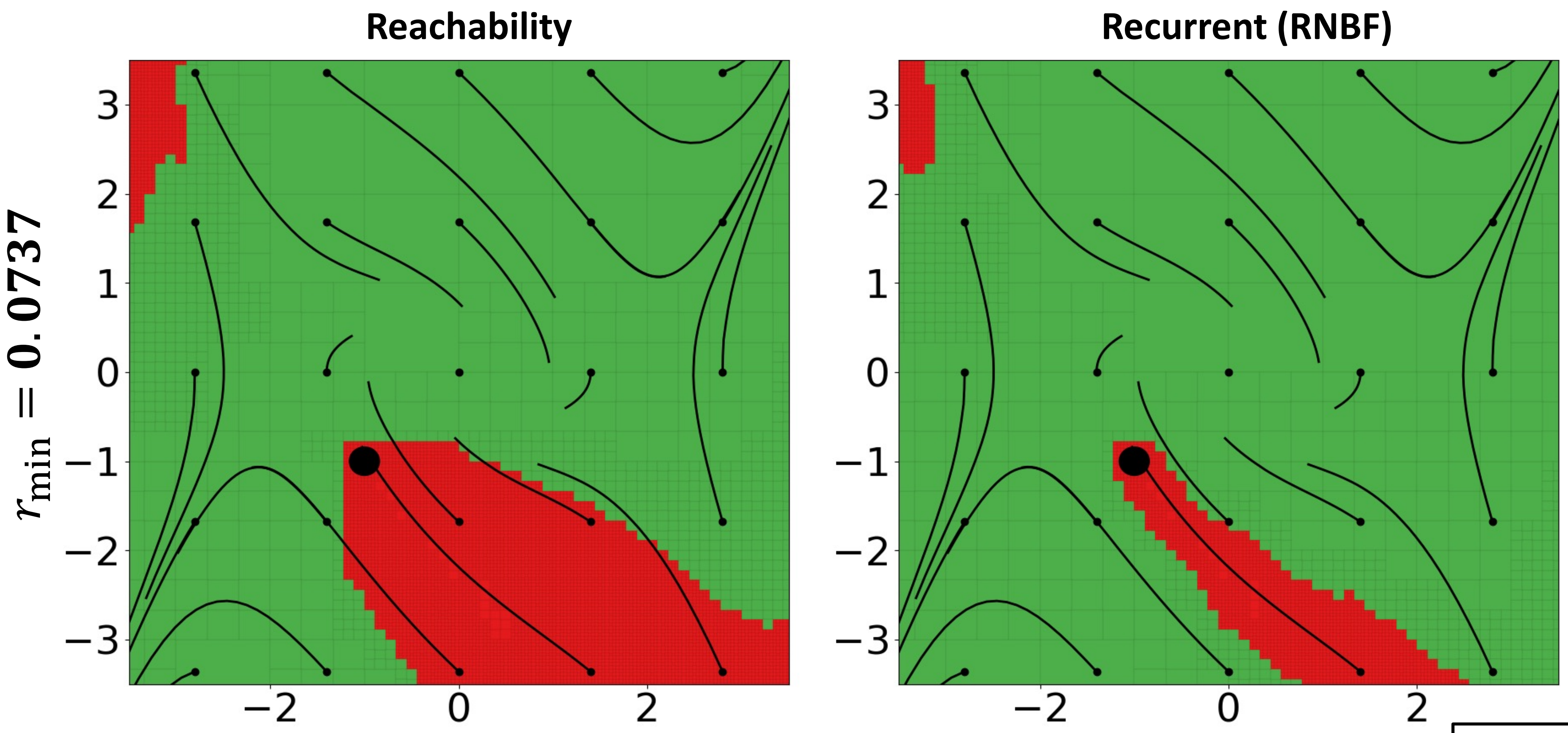


running time: 23.15s
92.2% safe set covered

running time: 3.81s
98.6% safe set covered

~6x faster
+6.4% more area

Numerical Validation: Reachability vs Recurrence

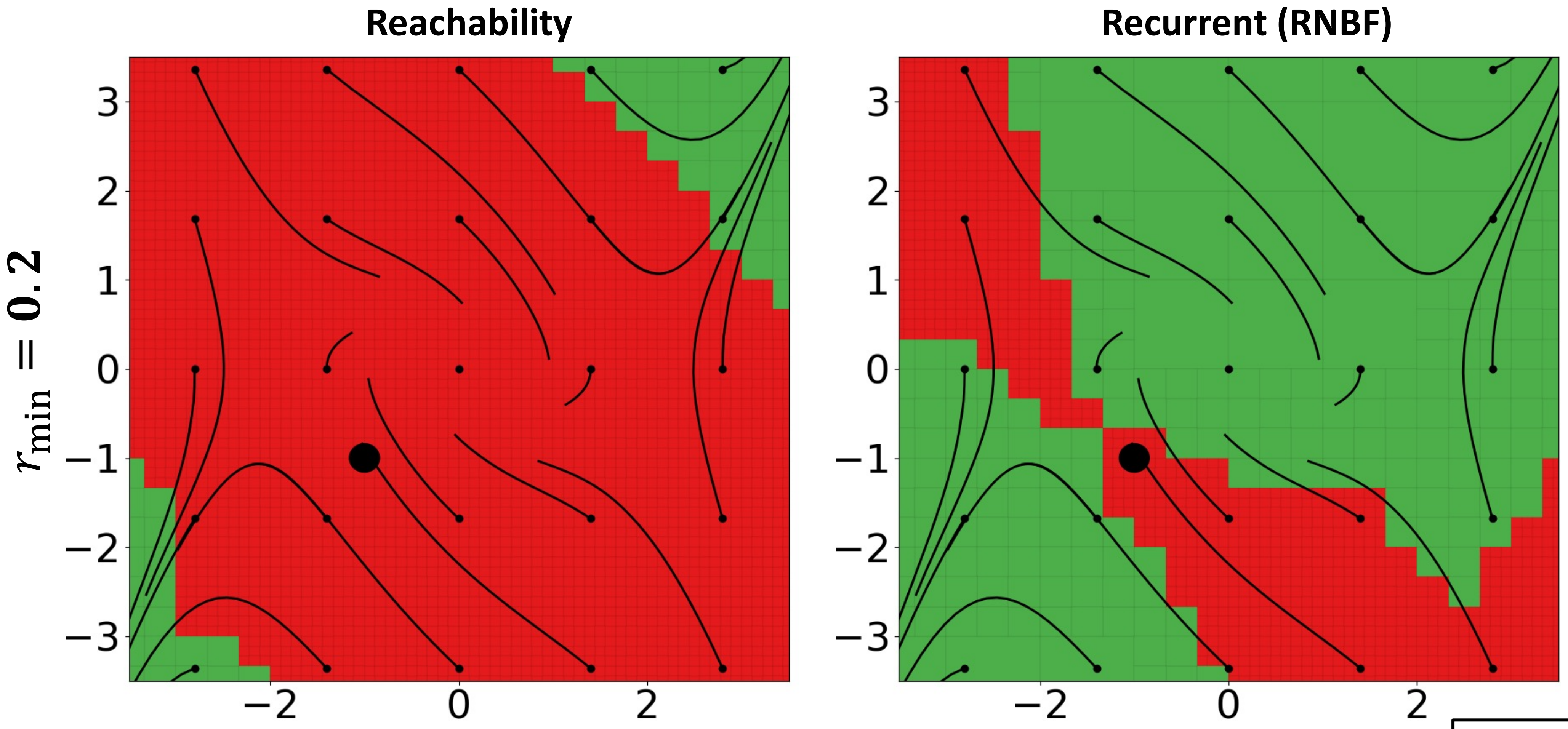


running time: 3.01s
83.3% safe set covered

running time: 1.56s
94.6% safe set covered

~2x faster
+11.2% more area

Numerical Validation: Reachability vs Recurrence



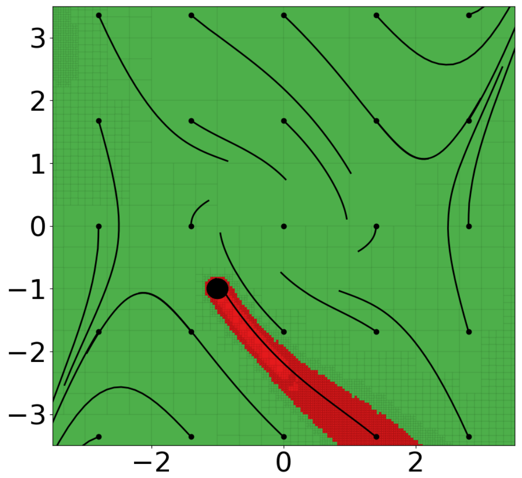
running time: 1.86s
9.3% safe set covered

running time: 0.31s
71.2% safe set covered

6x faster
+61.9% more area

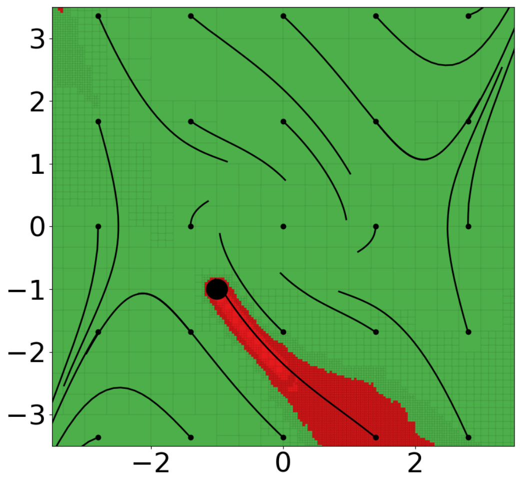
Numerical Validation: Recurrent Exponential Barrier Function

$\alpha = 4$



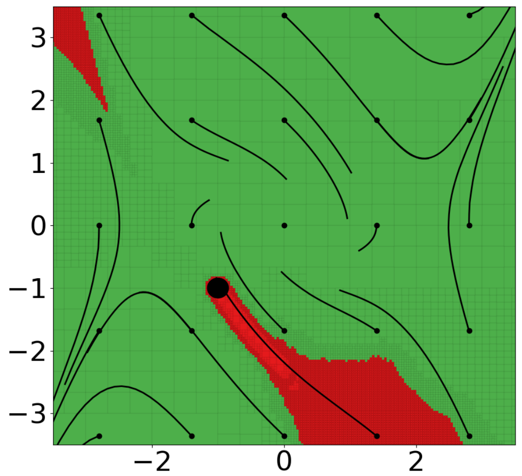
running time = 4.45s
98.3% Safe set covered

$\alpha = 3$



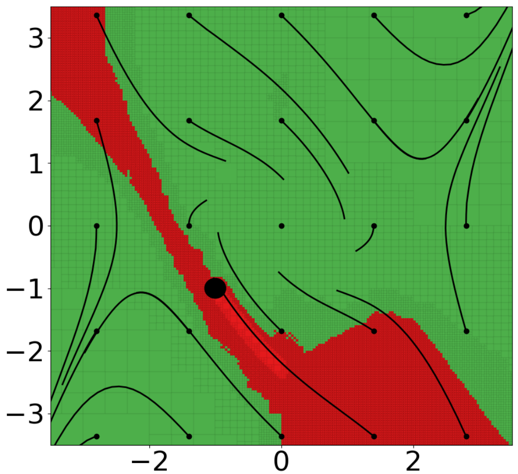
running time = 4.99s
97.0% Safe set covered

$\alpha = 2$



running time = 10.54s
93.6% Safe set covered

$\alpha = 1$



running time = 218s
82.75% Safe set covered

Conclusions and Future work

- **Takeaways**
  - Proposed a **relaxed notion of invariance: recurrence.**
  - Provide **necessary and sufficient conditions** for a recurrent set to be an **inner approximation** of the ROA.
  - **Nonparametric theory for dynamical systems analysis:**
    - Leading to general **Lyapunov** and **Barrier Function** conditions **satisfied by any norm!**
  - Our algorithms are **parallelizable and progressive/sequential.**

- **Ongoing work**
  - **Recurrent Sets:** Smart choice of multi-points, control recurrent sets, GPU implementation
  - **Lyapunov and Barrier Functions:** Generalize other Lyapunov notions, Control Lyapunov Functions, Control Barrier Functions, Contraction, etc.
  - **Recurrence Entropy:** Understanding the complexity of making a set recurrent compared to invariance.

Thanks!

Related Publications:

[arXiv 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* **CDC 2022**
[CDC 23] Siegelmann, Shen, Paganini, M, *A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions*, **CDC 2023**
[HSCC 24] Sibai, M, *Recurrence of nonlinear control systems: Entropy and bit rates,* **HSCC, 2024**
[Allerton 24] Shen, Sibai, M, *Generalized Barrier Functions: Integral conditions and recurrent relaxations,* **Allerton 2024**

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