Generalized Barrier Functions: Integral Conditions & Recurrent Relaxations

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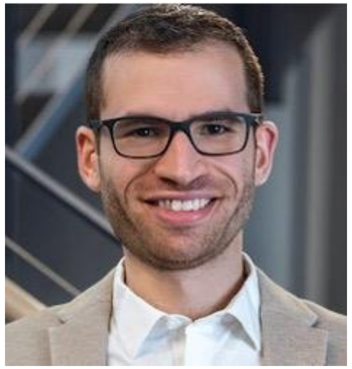
Acknowledgements

Yue Shen

Apple



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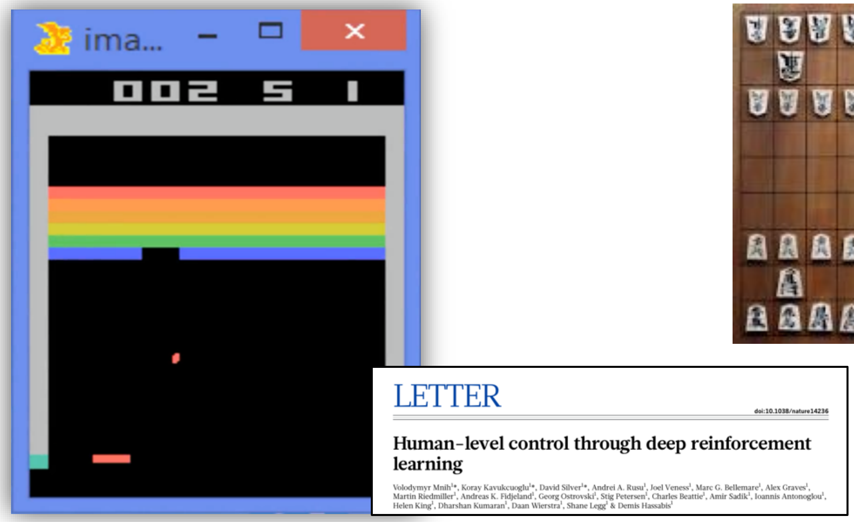
NSF

JOHNS HOPKINS INSTITUTE *for* ASSURED AUTONOMY

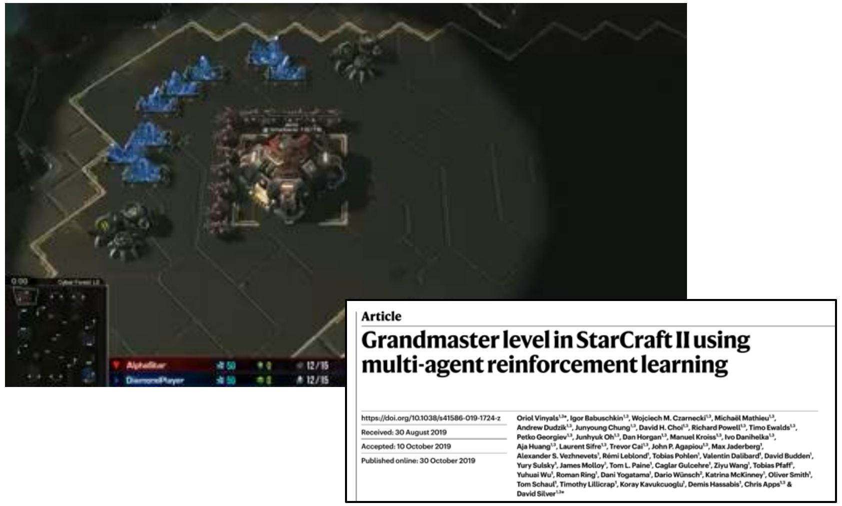
A World of Success Stories

2017 Google DeepMind's DQN



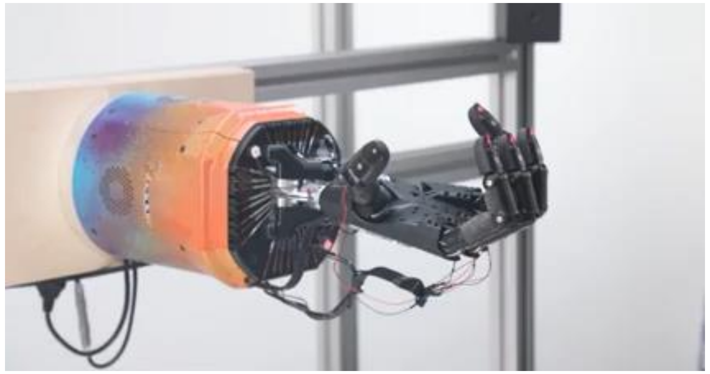
2017 AlphaZero – Chess, Shogi, Go

2019 AlphaStar – Starcraft II

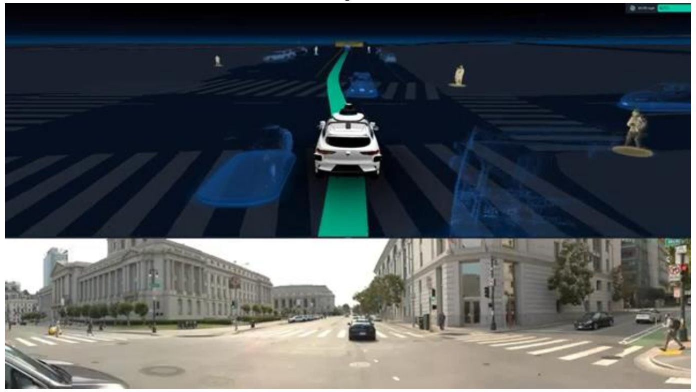


Boston Dynamics

OpenAI – Rubik's Cube



Waymo



The Need for Safety Guarantees

Angry Residents, Abrupt Stops: Waymo Vehicles Are Still Causing Problems in Arizona

RAY STERN | MARCH 31, 2021 | 8:26AM

GARY MARCUS BUSINESS 08.14.2019 09:00 AM

DeepMind's Losses and the Future of Artificial Intelligence

Alphabet's DeepMind unit, conqueror of Go and other games, is losing lots of money. Continued deficits could imperil investments in AI.

AARIAN MARSHALL BUSINESS 12.07.2020 04:06 PM

Uber Gives Up on the Self-Driving Dream

The ride-hail giant invested more than $1 billion in autonomous vehicles. Now it's selling the unit to Aurora, which makes self-driving tech.

Tesla Recalls Nearly All Vehicles Due to Autopilot Failures

Tesla disagrees with feds' analysis of glitches

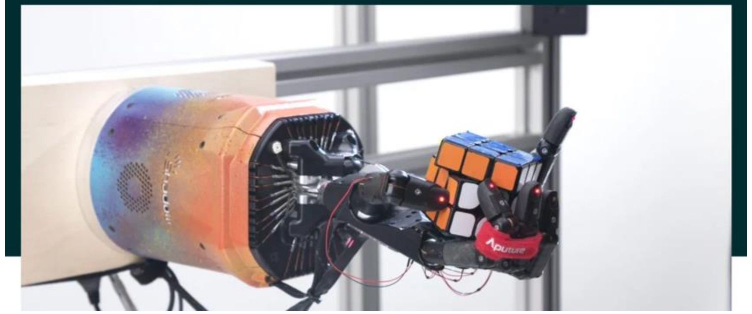
BY LINA FISHER, 2:54PM, WED. DEC. 13, 2023

CRUISE KNEW ITS SELF-DRIVING CARS HAD PROBLEMS RECOGNIZING CHILDREN — AND KEPT THEM ON THE STREETS

According to internal materials reviewed by The Intercept, Cruise cars were also in danger of driving into holes in the road.

OpenAI disbands its robotics research team

Kyle Wiggers @Kyle_L_Wiggers July 16, 2021 11:24 AM



Self-driving Uber car that hit and killed woman did not recognize that pedestrians jaywalk

The automated car lacked "the capability to classify an object as a pedestrian unless that object was near a crosswalk," an NTSB report said.

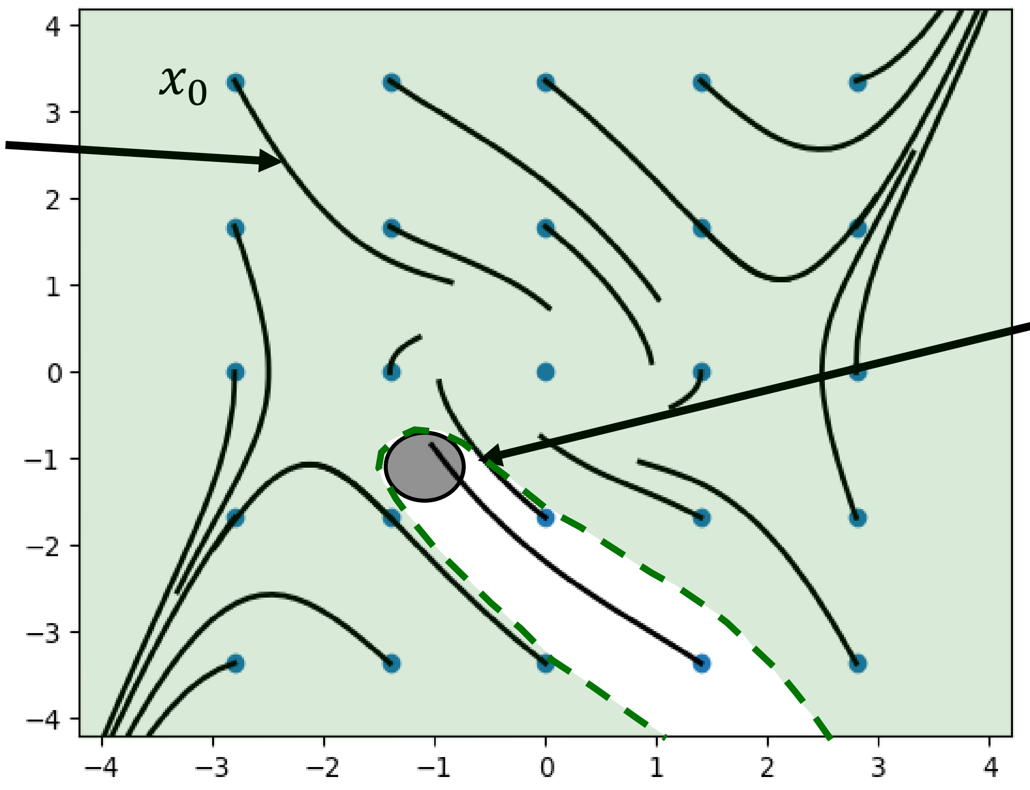
Safety in Dynamical Systems

Consider the continuous-time dynamical system: $\dot{x} = f(x)$

- $\phi(t, x_0)$: solution at time t starting from x_0
- X_u: set of unsafe states

Goal: Find the safe set $\mathcal{X}_s := \{x_0 \in \mathbb{R}^d | \phi(t, x_0) \notin \mathcal{X}_u, \forall t \geq 0\}$

$\phi(t, x_0)$



X_u: unsafe set
(e.g., obstacles)

X_s :

Safety in Dynamical Systems via Invariant Sets

Consider the continuous-time dynamical system: $\dot{x} = f(x)$

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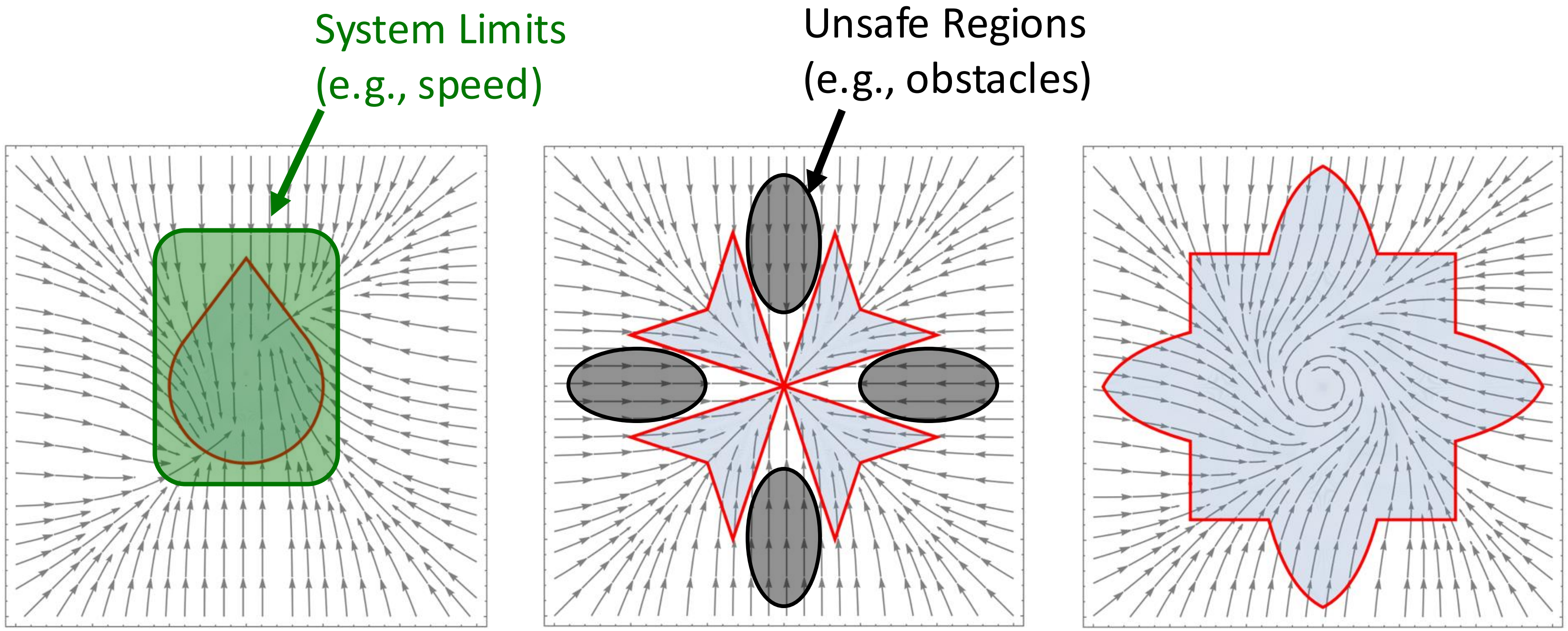
- X_u: set of unsafe states

Goal: Find the safe set $\mathcal{X}_s := \{x_0 \in \mathbb{R}^d | \phi(t, x_0) \notin \mathcal{X}_u, \forall t \geq 0\}$

General Approach: Use invariant sets!

A set $\mathcal{S} \subseteq \mathbb{R}^d$ is **invariant** if and only if: $x_0 \in \mathcal{S} \rightarrow \phi(t, x_0) \in \mathcal{S}, \ \forall t \geq 0$



Certifying Safety using Barrier Functions

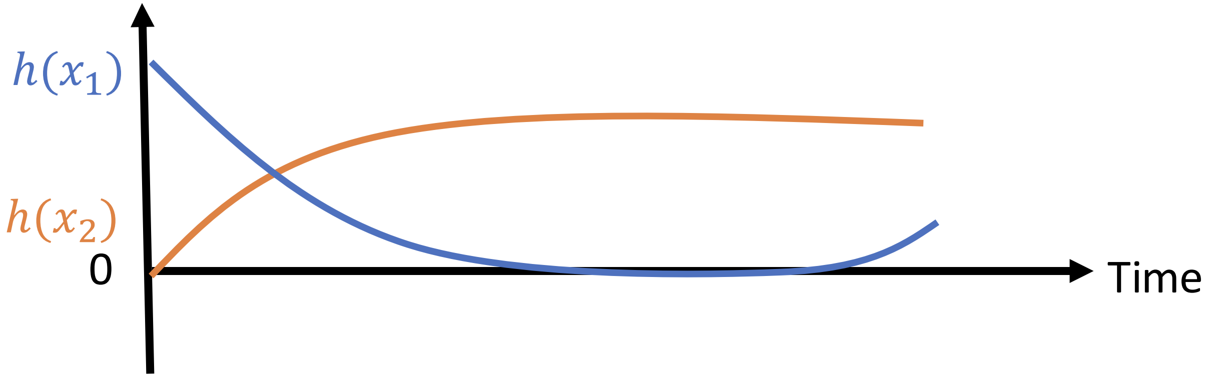
Theorem - Nagumo's Barrier Functions [Nagumo '42] **:**

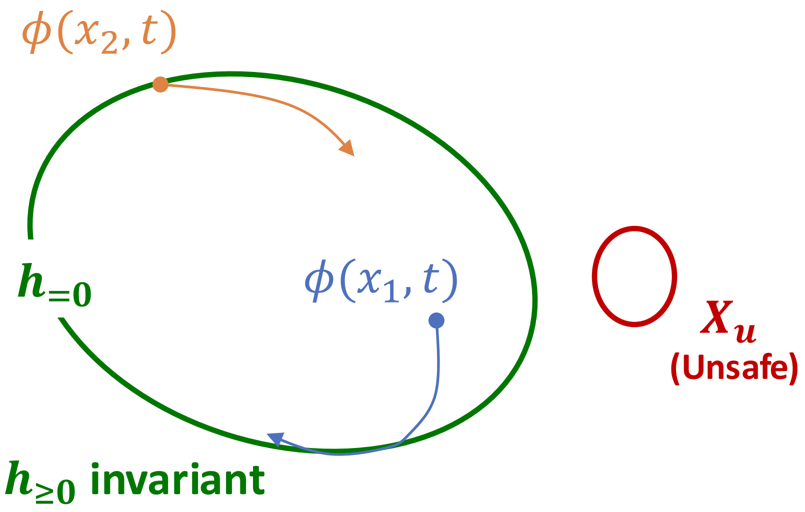
Let $h : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}$ be differentiable, with 0 being a *regular value.*
Then h is a Nagumo's Barrier Function (NBF) satisfying:

[redacted]

if and only if $h_{\geq 0} := \{x \in \mathbb{R}^d \,|\, h(x) \geq 0\}$ *is invariant.*

Mitio Nagumo





Then $h_{\geq 0}$ is a safe set whenever $h_{\geq 0} \cap X_u = \emptyset$

M. Nagumo, "Über die lage der integralkurven gewöhnlicher differentialgleichungen," Proceedings of the Physico-Mathematical Society of Japan 1942

Shaping Safe Behavior using Barrier Functions (BFs)

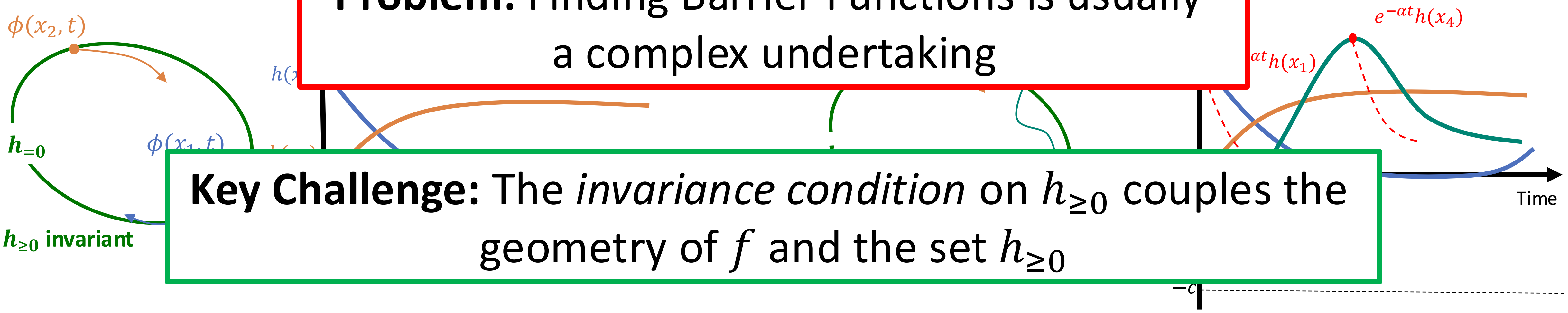
Barrier functions provide a flexible framework to shape the behavior of trajectories

Nagumo's (NBF)

$$L_f h(x) \geq 0\,, \quad \forall x \in h_{=0}$$

Exponential Barrier Functions (EBF)

$$L_f h(x) \geq -\alpha h(x)\,, \quad \forall x \in h_{\geq -c}$$



Problem: Finding Barrier Functions is usually a complex undertaking

Key Challenge: The *invariance condition* on $h_{\geq 0}$ couples the geometry of f and the set $h_{\geq 0}$

Other: Zeroing BFs (ZBFs), Minimal BFs (MBFs), Control BFs (CBFs), High Order CBFs (HOCBFs), …

S. Prajna, A. Jadbabaie. *Safety Verification of Hybrid Systems Using Barrier Certificates*. HSCC 2004
P. Wieland, F. Allgöwer. *Constructive safety using control barrier functions*. IFAC Proceedings Volumes 2007
A. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, P. Tabuada. *Control barrier functions: Theory and applications*. IEEE ECC 2019
R. Konda, A. Ames, S. Coogan. *Characterizing safety: Minimal control barrier functions from scalar comparison systems*. IEEE L-CSS 2020
W. Xiao, C. Belta. *High-order control barrier functions*. IEEE TAC 2021

Outline

- Letting things go and come back: *Recurrent sets*

- Generalized barriers: Integral forms and recurrent relaxations

- Safety verification via Recurrent Barrier Functions

Outline

- **Letting things go and come back: *Recurrent sets***
- Generalized barriers: Integral forms and recurrent relaxations
- Safety verification via Recurrent Barrier Functions

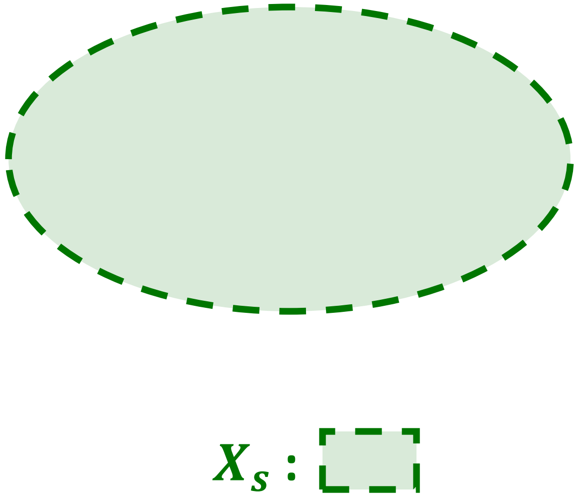
Core challenge: Invariant safe sets are hard to find

A set $\mathcal{S} \subseteq \mathbb{R}^d$ is **positively invariant** if and only if: $x_0 \in \mathcal{S} \rightarrow \phi(t, x_0) \in \mathcal{S},\ \forall t \geq 0$

Any trajectory starting in the set remains in inside it for all times

Goal: Given an unknown safe set X_s, find an invariant set $\mathcal{S}$ such that $\mathcal{S} \subset X_s$.

Example:



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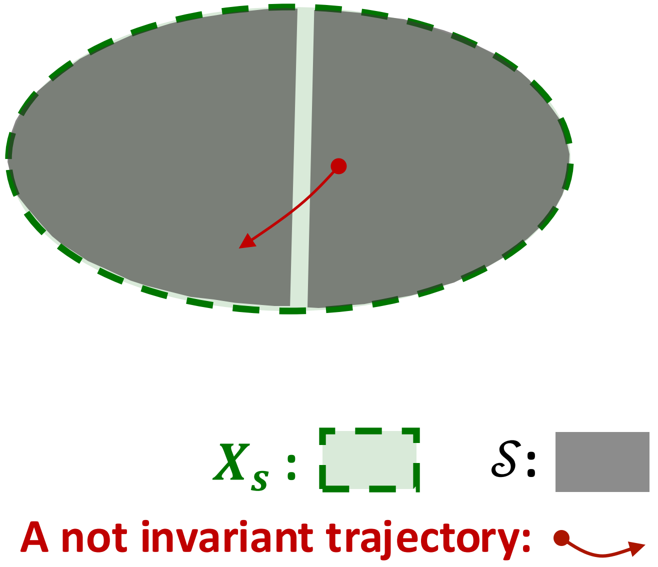
Challenges:

- $\mathcal{S}$ is topologically constrained
  - Trajectory cannot cross disconnected parts of $\mathcal{S}$

Example:
$\mathcal{S} \subseteq X_s$ is not invariant!



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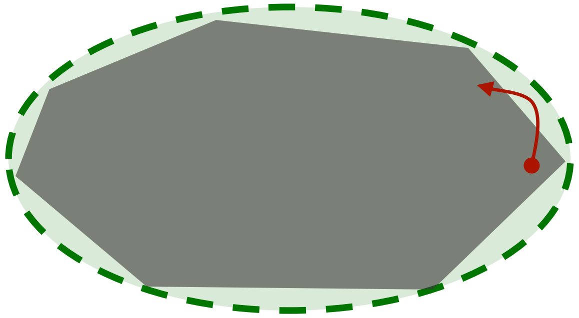
- $\mathcal{S}$ is geometrically constrained
  - f should point inwards for $x \in \partial\mathcal{S}$

Example:
$\boldsymbol{\mathcal{S}} \subseteq \boldsymbol{X_s}$ is not invariant!



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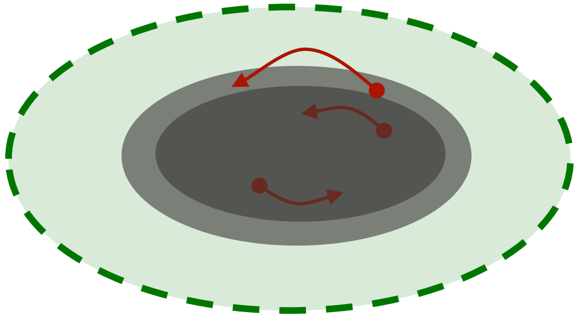
Challenges:

- $\mathcal{S}$ is topologically constrained
  - Trajectory cannot cross disconnected parts of $\mathcal{S}$
- $\mathcal{S}$ is geometrically constrained
  - f should point inwards for $x \in \partial\mathcal{S}$
- $\mathcal{S}$ is hard to grow
  - $\mathcal{S}$ should adapt to new trajectories

Invariance introduces strict constraints on the shape, topology, and the future extension of the set $\mathcal{S}$!

Example:
$\boldsymbol{\mathcal{S}} \subseteq \boldsymbol{X_s}$ is not invariant!

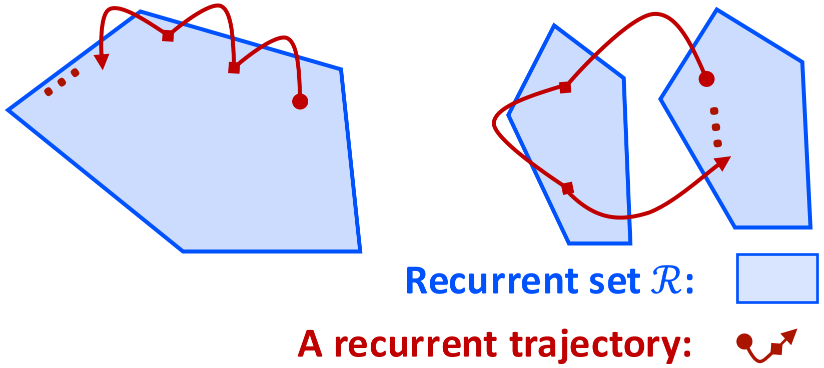


Recurrent sets: Letting things go, and come back

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \geq t$ s.t. $\phi(t', x_0) \in \mathcal{R}$.

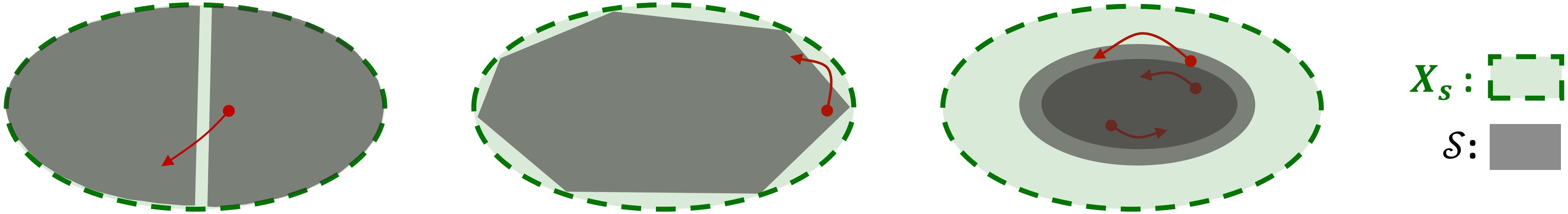
Property of Recurrent Sets

- $\mathcal{R}$ need **not** be **connected**
- $\mathcal{R}$ does **not** require f to **point inwards** on all $\partial\mathcal{R}$



> Recurrent sets, while not invariant, guarantee that solutions that start in this set, will come back **infinitely often, forever!**

Previous good inner approximations of $\mathbf{X}_s$ are recurrent sets



Outline

- Letting things go and come back: *Recurrent sets*
- **Generalized barriers: Integral forms and recurrent relaxations**
- Safety verification via Recurrent Barrier Functions

Integral Nagumo's Barrier Function (INBF)

Nagumo's Barrier Functions [Nagumo '42]**:**
Let h *be differentiable, regular at zero, and*

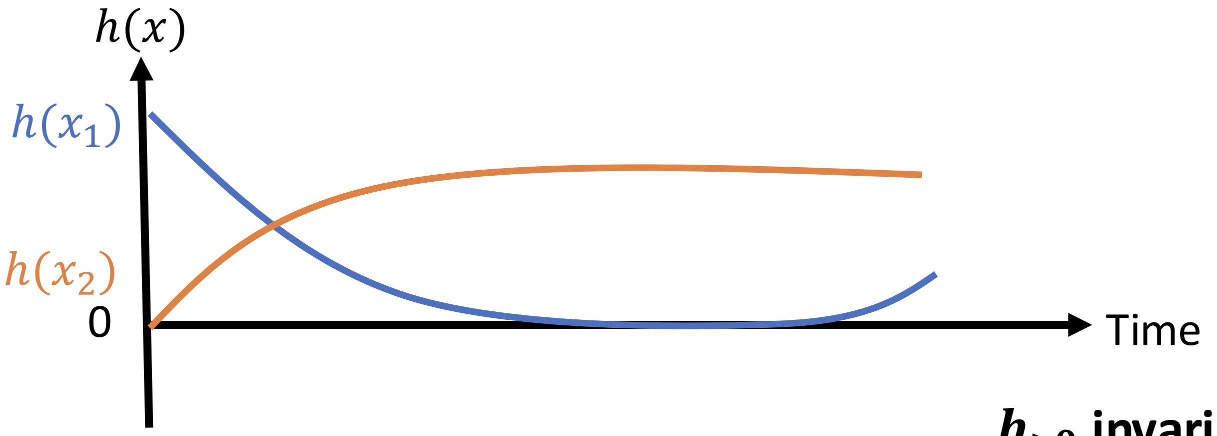
$$L_f h(x) \geq 0\,,\ \forall x \in h_{=0}$$

Integral Nagumo's Barrier Function:
Let h *be continuous, and*

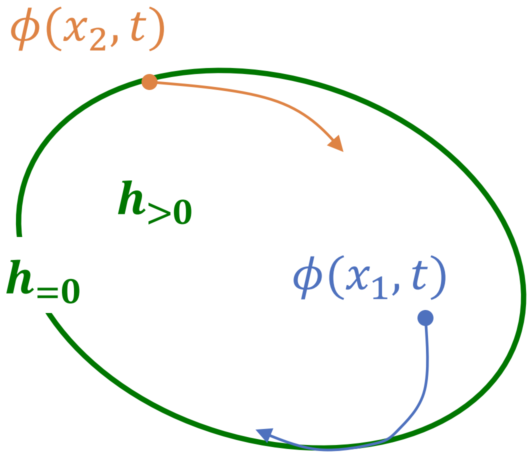
$$h(\phi(t,x)) \geq 0\,,\ \forall x \in h_{=0}\,, t \geq 0$$

The super-level set $h_{\geq 0}$ is invariant.

* requires more conditions on h



$h_{\geq 0}$ invariant



Recurrent Nagumo's Barrier Function (RNBF)

Thm: Integral Nagumo's Barrier Function:
Let h *be continuous. Then:*

$$h(\phi(t,x)) \geq 0\,,\ \forall x \in h_{=0}\,, t \geq 0$$

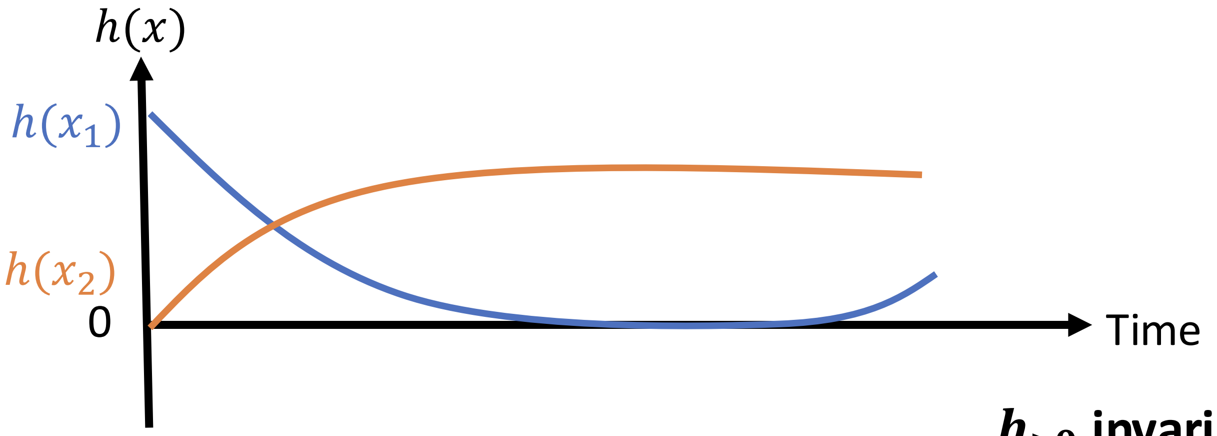
if and only if $h_{\geq 0}$ is **invariant**

Thm: Recurrent Nagumo's Barrier Function:
Let h *be continuous. Then:*

$$\max_{t\in(0,\tau]} h(\phi(t,x)) \geq 0\,,\ \forall x \in h_{=0}$$

if and only if $h_{\geq 0}$ is **τ-recurrent**

Definition: A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **τ-recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \in (t, t+\tau]$ s.t. $\phi(t', x_0) \in \mathcal{R}$.



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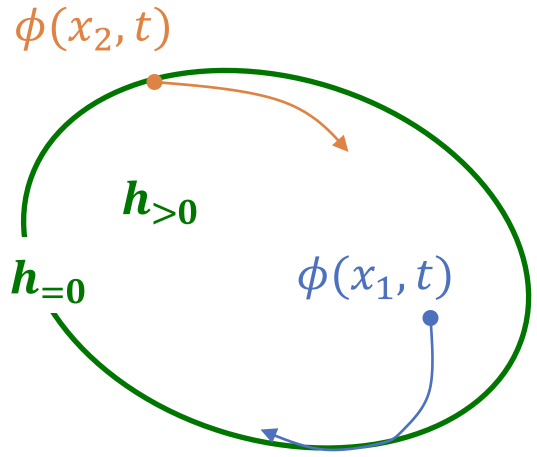

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As $\tau \to 0$

By definition

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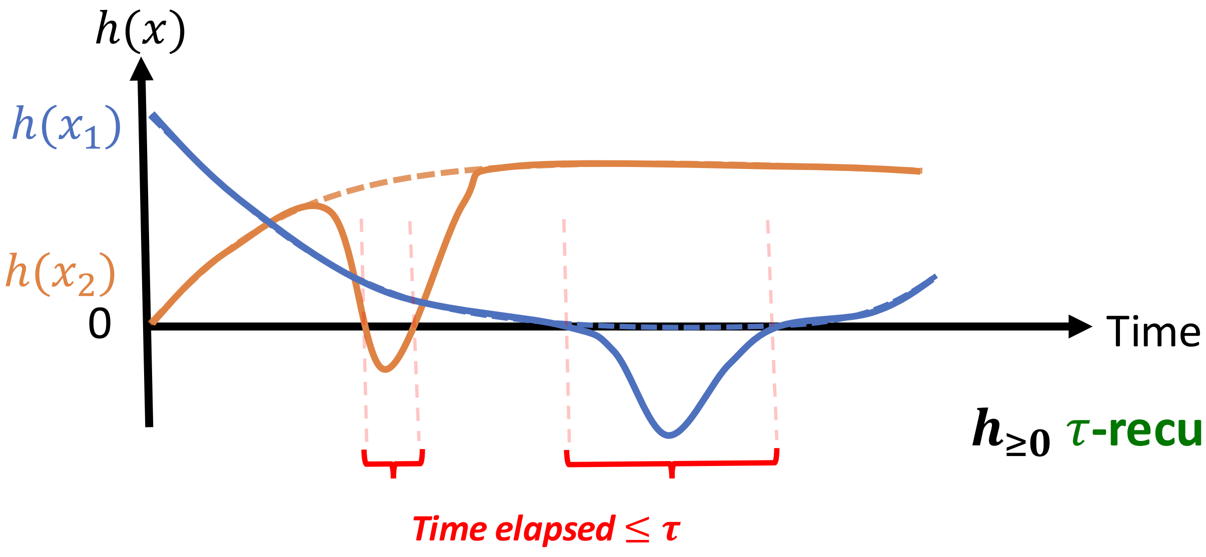
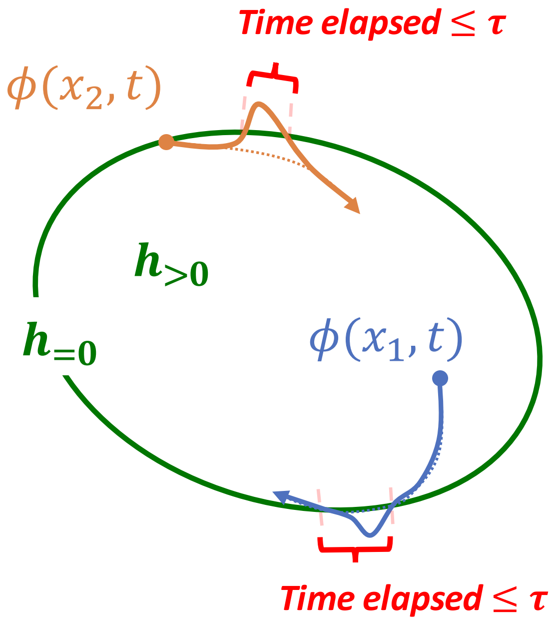


$h_{\geq 0}$ τ-recurrent



Integral Exponential Barrier Function (IEBF)

Exponential Barrier Functions:
Let h *be differentiable, and*

$$L_f h(x) \geq -\alpha h(x)\,,\ \forall x \in h_{\geq -c}$$

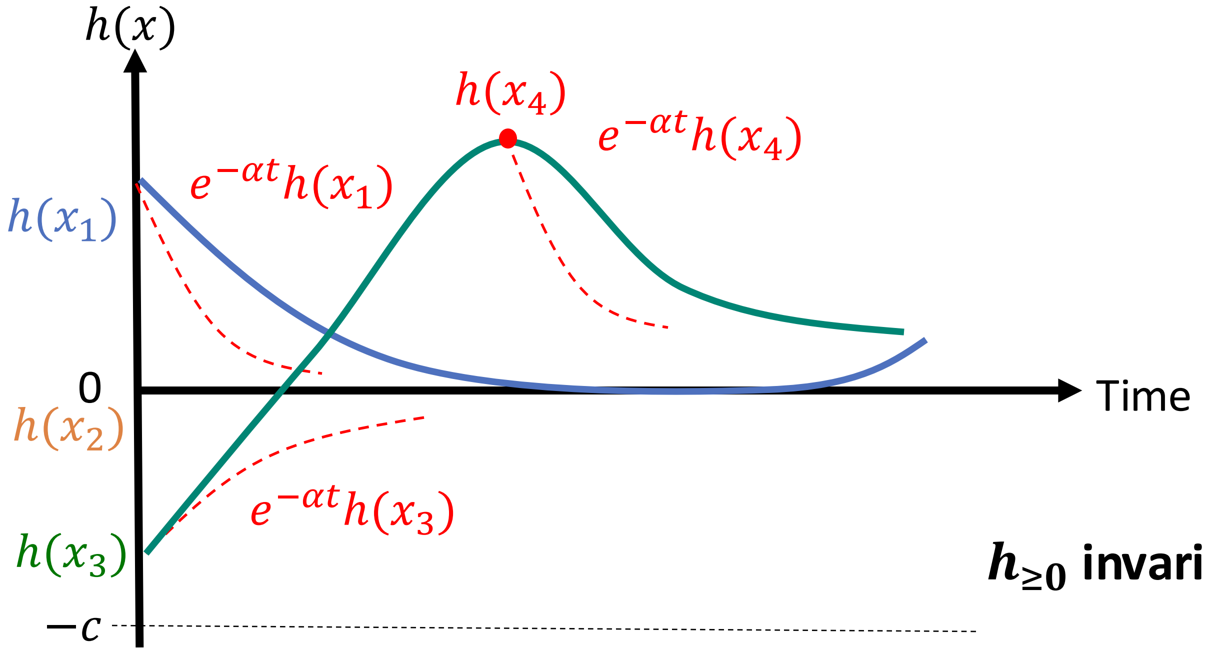
Integral Exponential Barrier Function:
Let h *be continuous, and*

$$h(\phi(t,x)) \geq e^{-\alpha t} h(x)\,,\ \forall x \in h_{\geq -c}$$

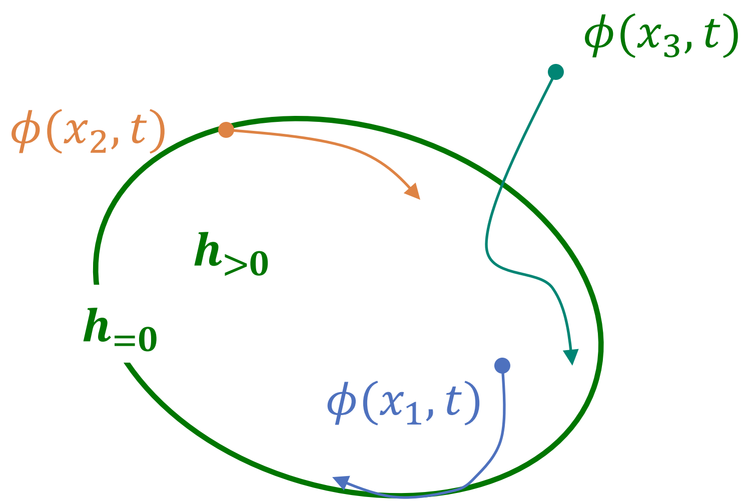
for all $t \geq 0$

The super-level set $h_{\geq 0}$ is invariant.

* requires more conditions on h



$h_{\geq 0}$ invariant



Recurrent Exponential Barrier Function (REBF)

Thm: Integral Exponential Barrier Function:
Let h *be continuous. If:*

$$h(\phi(t,x)) \geq e^{-\alpha t} h(x)\,,\ \forall x \in h_{\geq -c}$$

for all $t \geq 0$, then, $h_{\geq 0}$ is **invariant**

Thm: Recurrent Exponential Barrier Function:
Let h *be continuous. If:*

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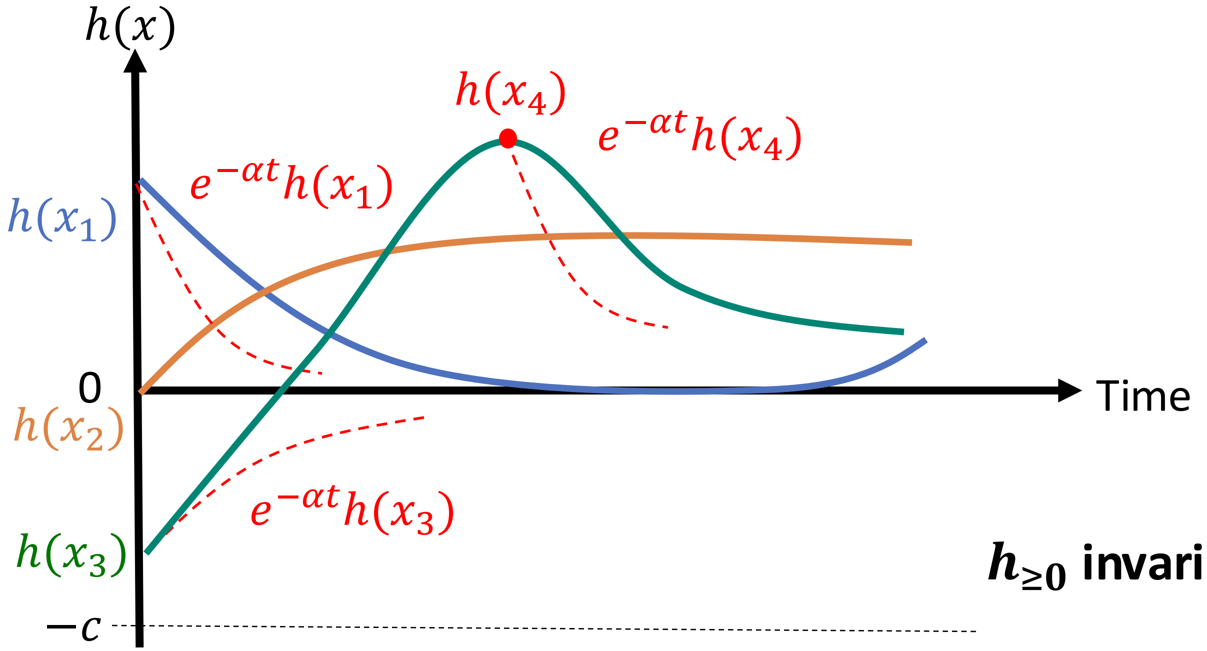
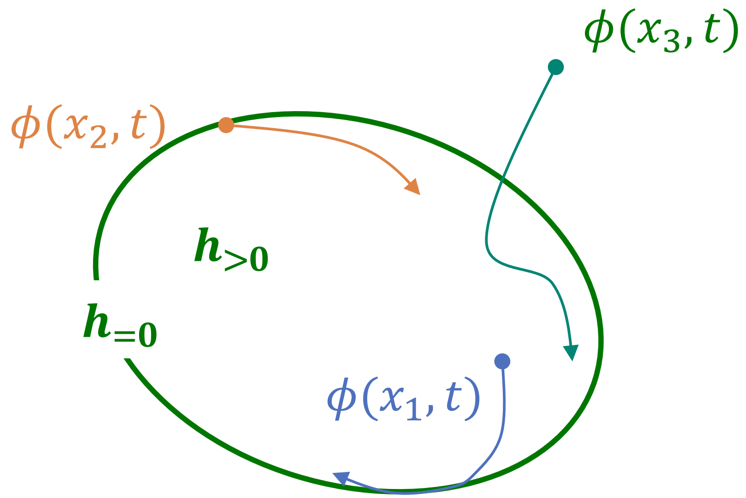


$h_{\geq 0}$ invariant



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As $\tau \to 0$ ⟵
⟶ By definition

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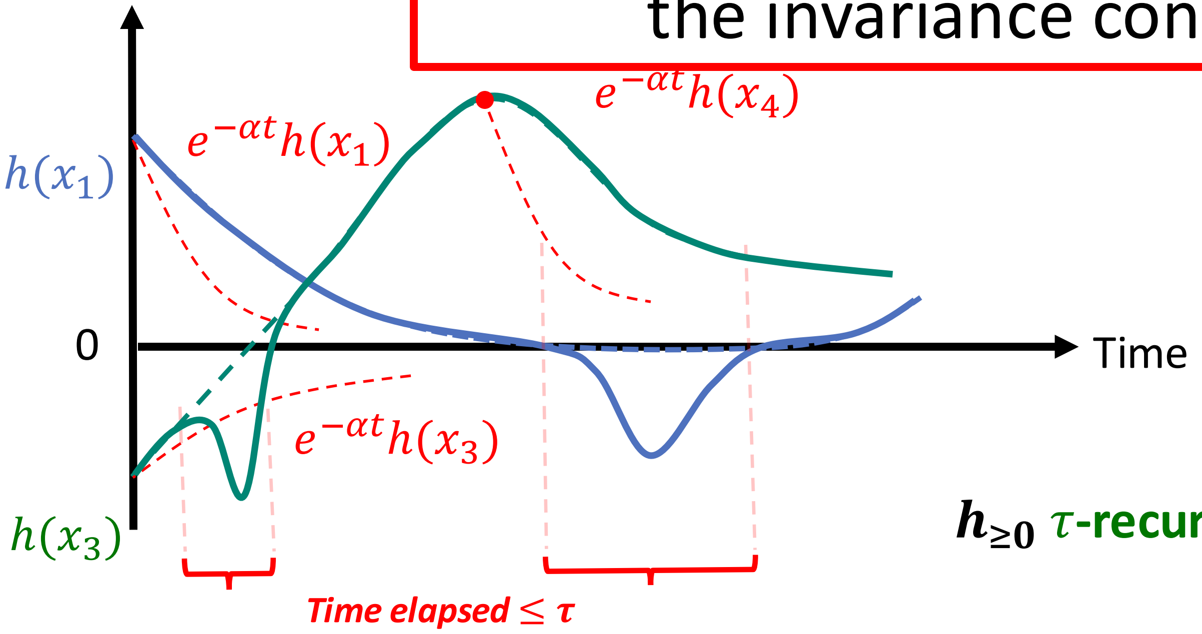
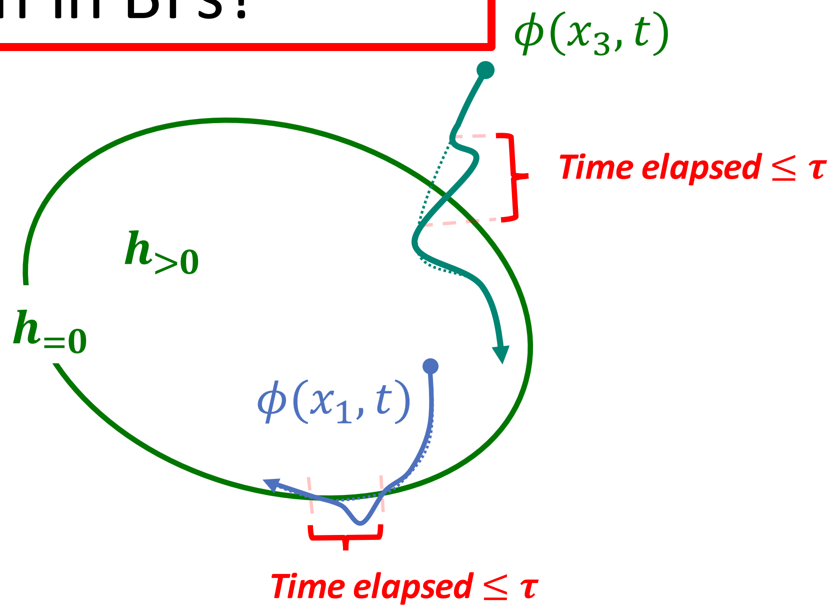
then, $h_{\geq 0}$ is **τ-recurrent**

Question: Do we gain anything from relaxing the invariance condition in BFs?



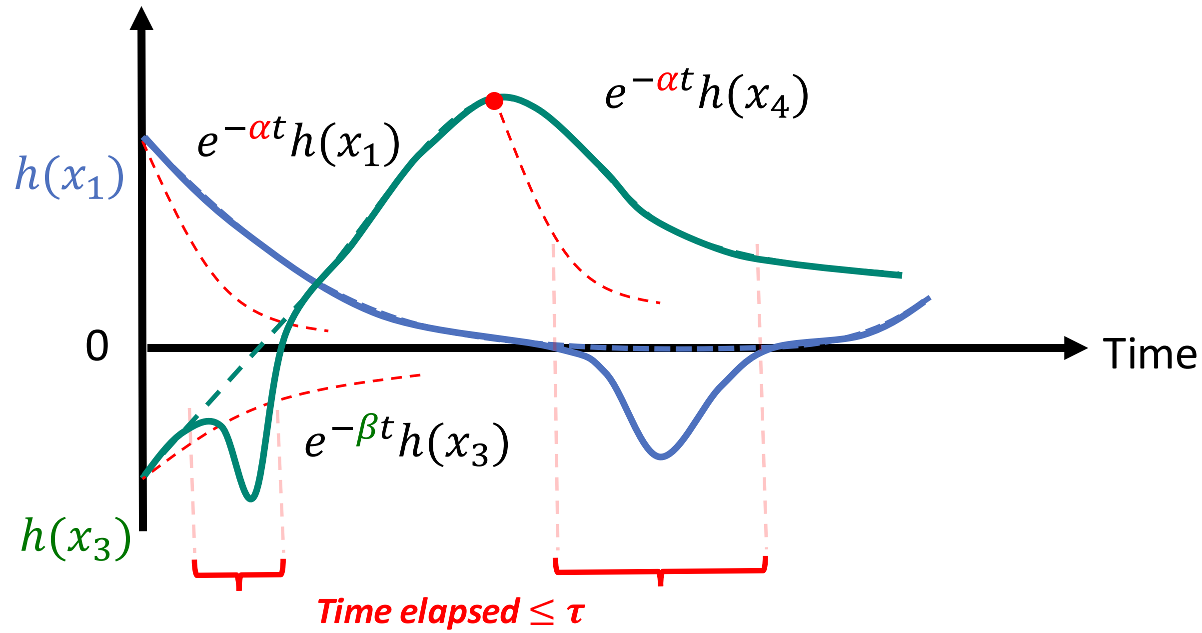
$h_{\geq 0}$ τ-recurrent

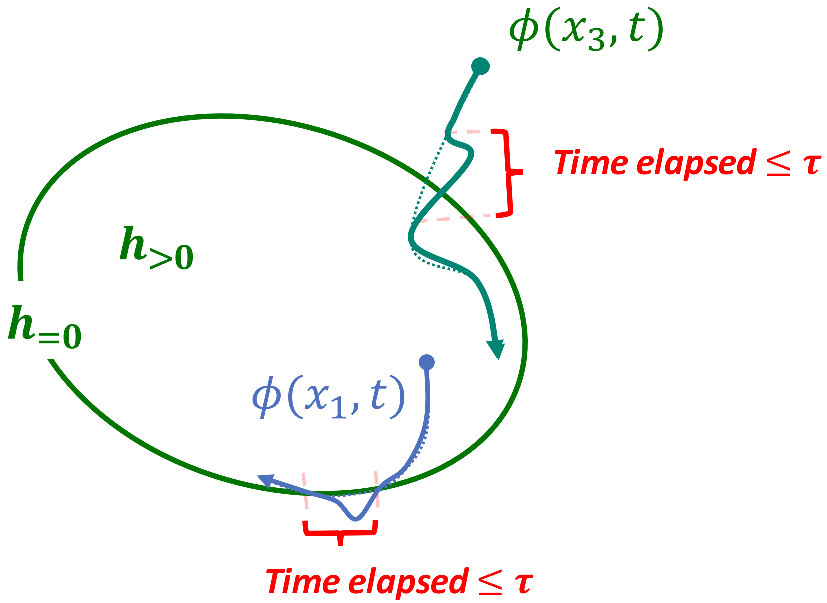


Signed Norms are Recurrent Barrier Functions!

We first generalize REBF using different exponential rates $\alpha, \beta > 0$:

$$\max_{t\in(0,\tau]} e^{\alpha t}[h(\phi(t,x))]_+ + e^{\beta t}[h(\phi(t,x))]_- \geq h(x)\,,\ \forall x \in h_{\geq -c}$$





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$$\max_{t\in(0,\tau]} e^{\alpha t}[h(\phi(t,x))]_+ + e^{\beta t}[h(\phi(t,x))]_- \geq h(x)\,,\ \forall x \in h_{\geq -c}$$

Theorem: Assume there exists an Integral Exponential BF (IEBF), h , defined over $D_0 := h_{\geq -c}$ for some $c > 0$. Then $\exists\ \alpha > 0$ such that:

$$e^{\alpha t}\, h\big(\phi(t,x)\big) \geq h(x), \qquad \forall x \in h_{\geq -c}$$

for all $t \geq 0$.

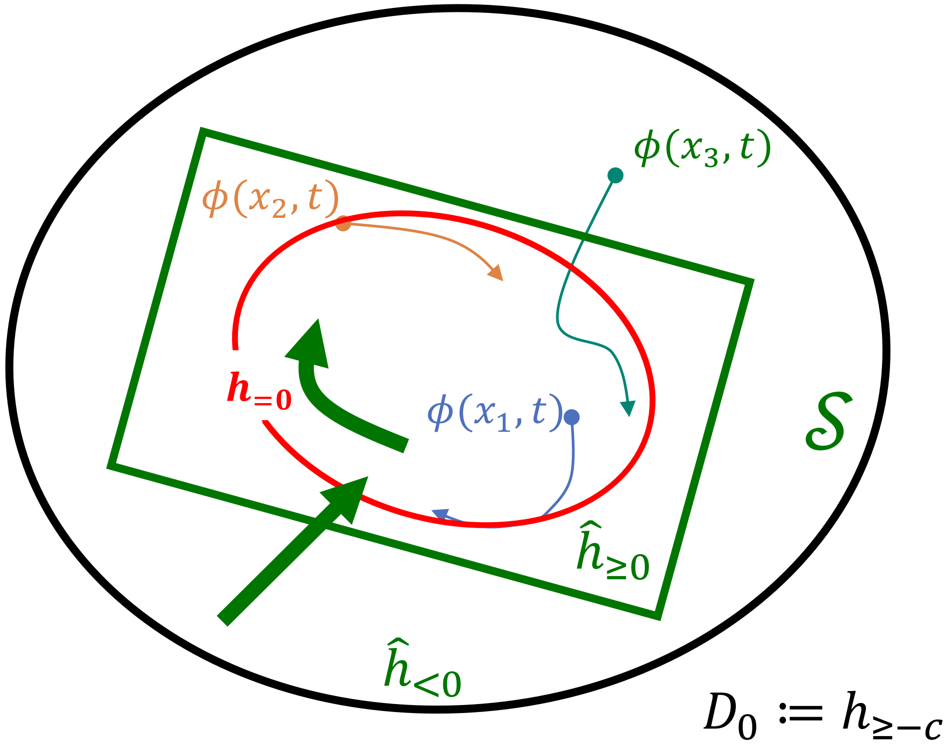
Then for any set $\mathcal{S}$ with $h_{\geq 0} \subseteq \mathcal{S} \subseteq h_{\leq -c}$, the function

$$\hat{h}(x) := -\mathrm{sd}(x, \mathcal{S})$$

is a Recurrent Exponential Barrier Function (REBF):

$$\max_{t\in(0,\tau]} e^{\hat{\alpha} t}\left[h\big(\phi(t,x)\big)\right]_+ + e^{\hat{\beta} t}\left[h\big(\phi(t,x)\big)\right]_- \geq h(x), \qquad \forall x \in h_{\geq -c}$$

with any parameters $\hat{\beta} < \alpha < \hat{\alpha}$ whenever $\tau \geq \bar{\tau}(\hat{\alpha} - \alpha, \hat{\beta} - \alpha)$



Outline

- Letting things go and come back: *Recurrent sets*
- Generalized barriers: Integral forms and recurrent relaxations
- **Safety verification via Recurrent Barrier Functions**

Safety Verification via Recurrent Sets

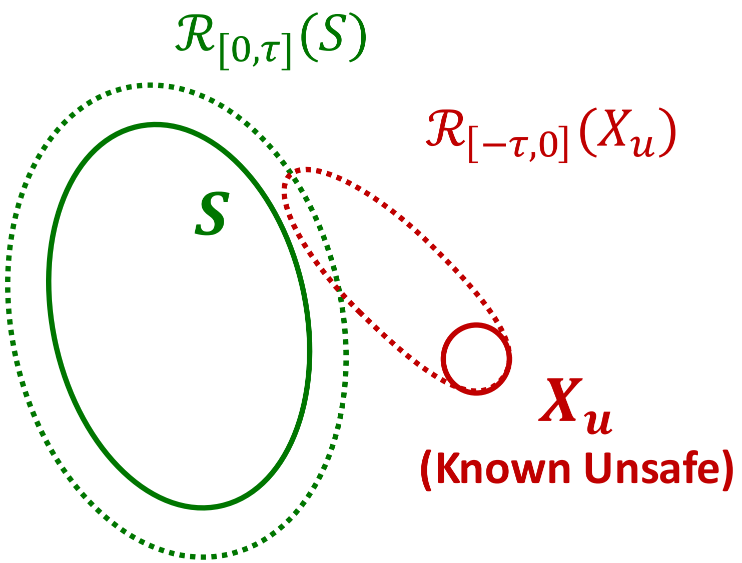
***Theorem* -** Consider a closed set S that is **τ-recurrent**. Then its τ-reachable set:

[redacted]

is **invariant**.

Moreover, S **is safe whenever:**

1. $\mathcal{R}_{[0,\tau]}(S) \cap X_u = \emptyset$,
2. $S \cap \mathcal{R}_{[-\tau,0]}(X_u) = \emptyset$



A GPU based algorithm

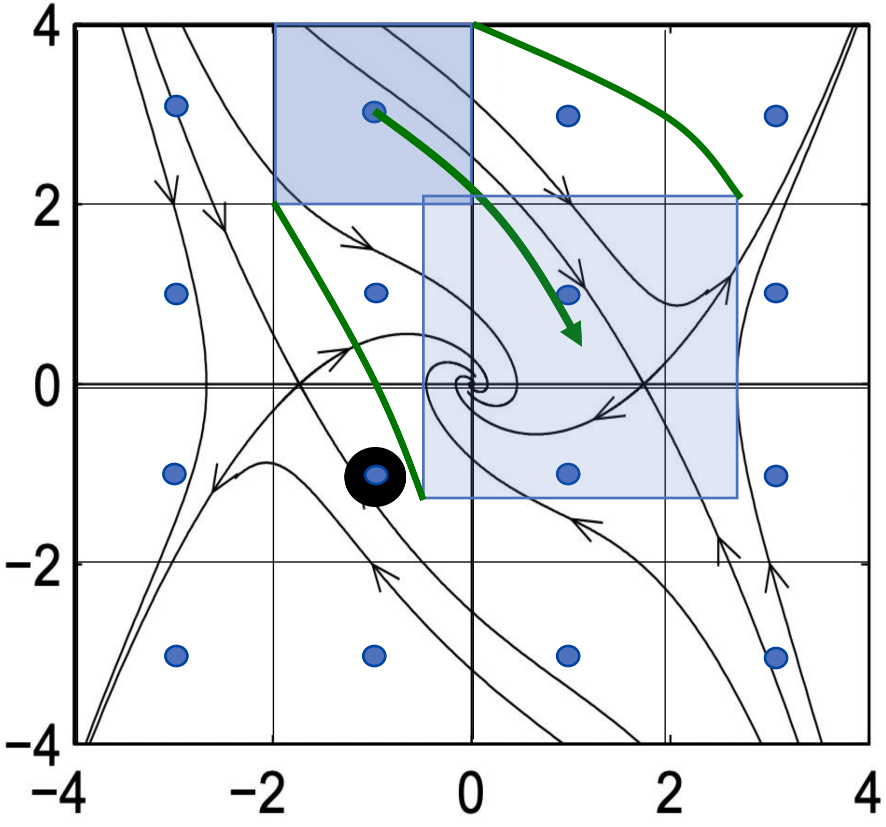
A set S is safe whenever:

Reachability Condition

- $\mathcal{R}_{[0,\tau]}(S) \cap X_u = \emptyset$

Recurrent Condition

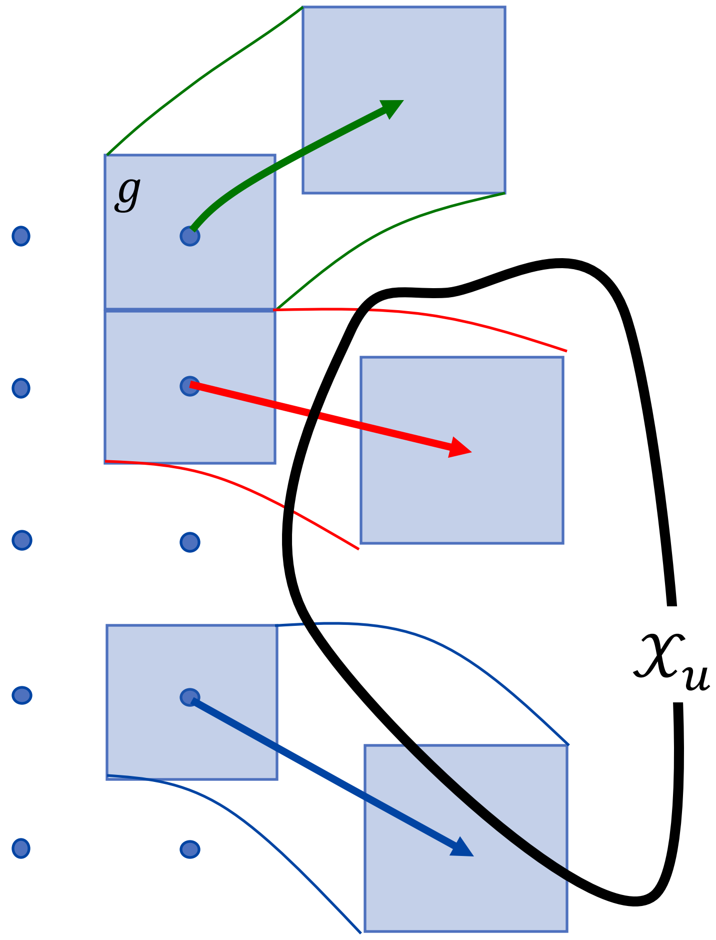
- $\hat{h}(x) \coloneqq -\,\mathrm{sd}\,(\,x\,, S\,)$ is a Recurrent Exponential Barrier Function



X_u :

- Cover the region with a grid G
  - For each point $g \in G$, S_g represents its cell
- We build $S = \cup_{g \in G^s} S_g$, with G^s representing safe points
  - Initialize $G^s \leftarrow G$
- Check both conditions using only one trajectory for each cell!

A GPU based algorithm



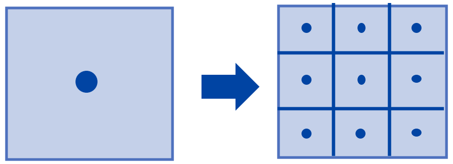
Checking Reachability Condition

Starting from centers of grid cell g, simulate the trajectories for τ-seconds in parallel using a GPU

If $\phi(t,x) \notin \mathcal{X}_u, \forall x \in S_g, t \in [0,\tau]$
"$\mathcal{R}_{[0,\tau]}(S_g) \cap X_u = \emptyset$" → Keep g in G^s

If $\exists t \in [0,\tau]$ s.t. $\phi(t,x) \in \mathcal{X}_u, \forall x \in S_g$
"$\mathcal{R}_t(S_g) \subseteq X_u$ for some $t \in [0,\tau]$ " → Remove g from G^s, Add g to G^u

Else
Undetermined → Split g



*Stop splitting g and mark it as unsafe whenever g is too small

A GPU based algorithm

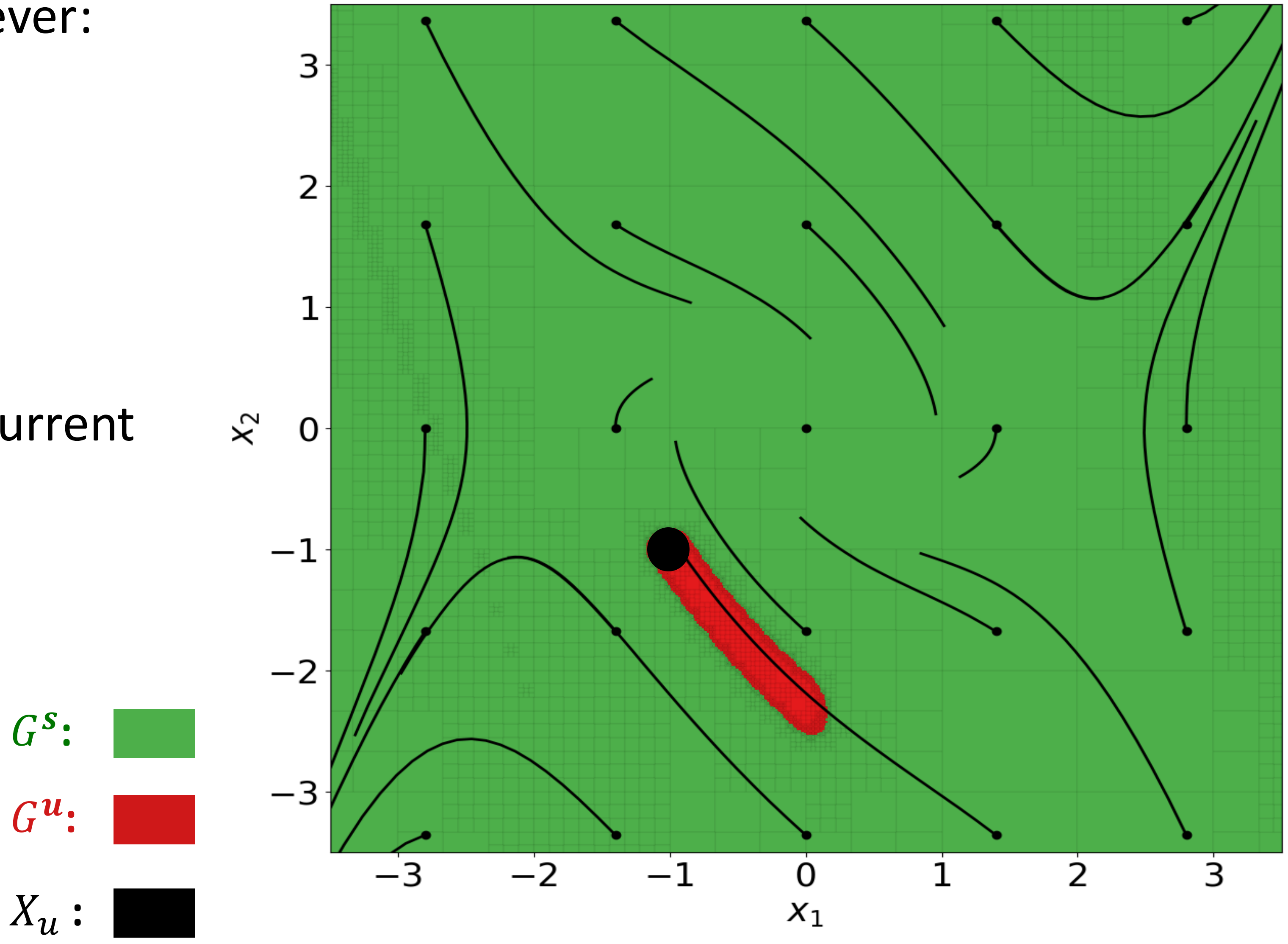
A set $S = \cup_{s \in G^s} S_g$ is safe whenever:

Reachability Condition

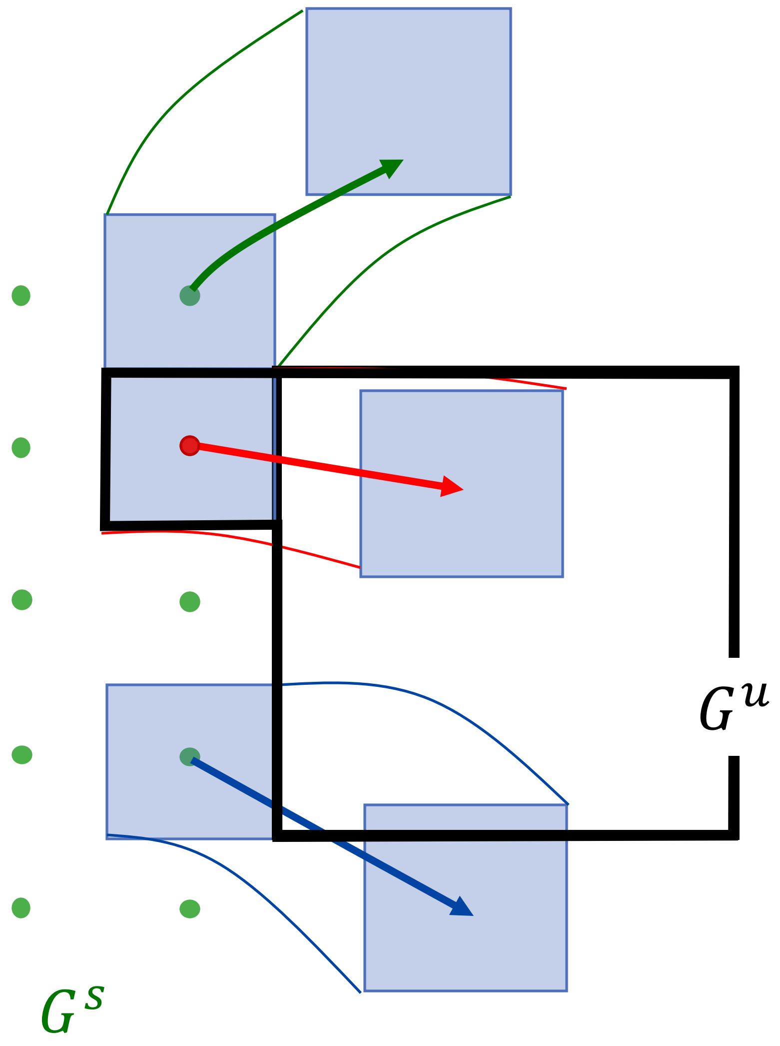
- $\mathcal{R}_{[0,\tau]}(S) \cap X_u = \emptyset$

Recurrent Condition

- $\hat{h}(x) := -\,\mathrm{sd}\,(\,x\,, S\,)$ is a Recurrent exponential barrier function



A GPU based algorithm



Check the Recurrent condition

- Let $S = \cup_{g \in G^s} S_g$, [redacted]
- Starting from centers of grid cell $g \in G^s$, simulate the trajectories for τ-seconds in parallel using a GPU

If $\max_{t \in (0,\tau]} e^{\hat{\alpha} t}\, \hat{h}(\phi(t,x)) \geq \hat{h}(x), \forall x \in S_g$

"REBF condition is satisfied within g"

→ Keep g in G^s

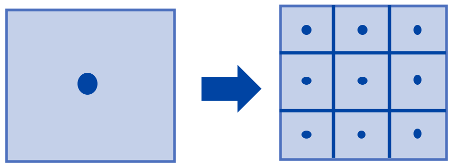
If $\max_{t \in (0,\tau]} e^{\hat{\alpha} t}\, \hat{h}(\phi(t,x)) < \hat{h}(x), \forall x \in S_g$

"REBF condition is NOT satisfied within S_g"

→ Remove g from G^s
Add g to G^u

Else
Undetermined

→ Split g



*Stop splitting g and mark it as unsafe whenever g is too small

A GPU based algorithm

A set $S = \cup_{s \in G^s} S_g$ is safe whenever:

Reachability Condition

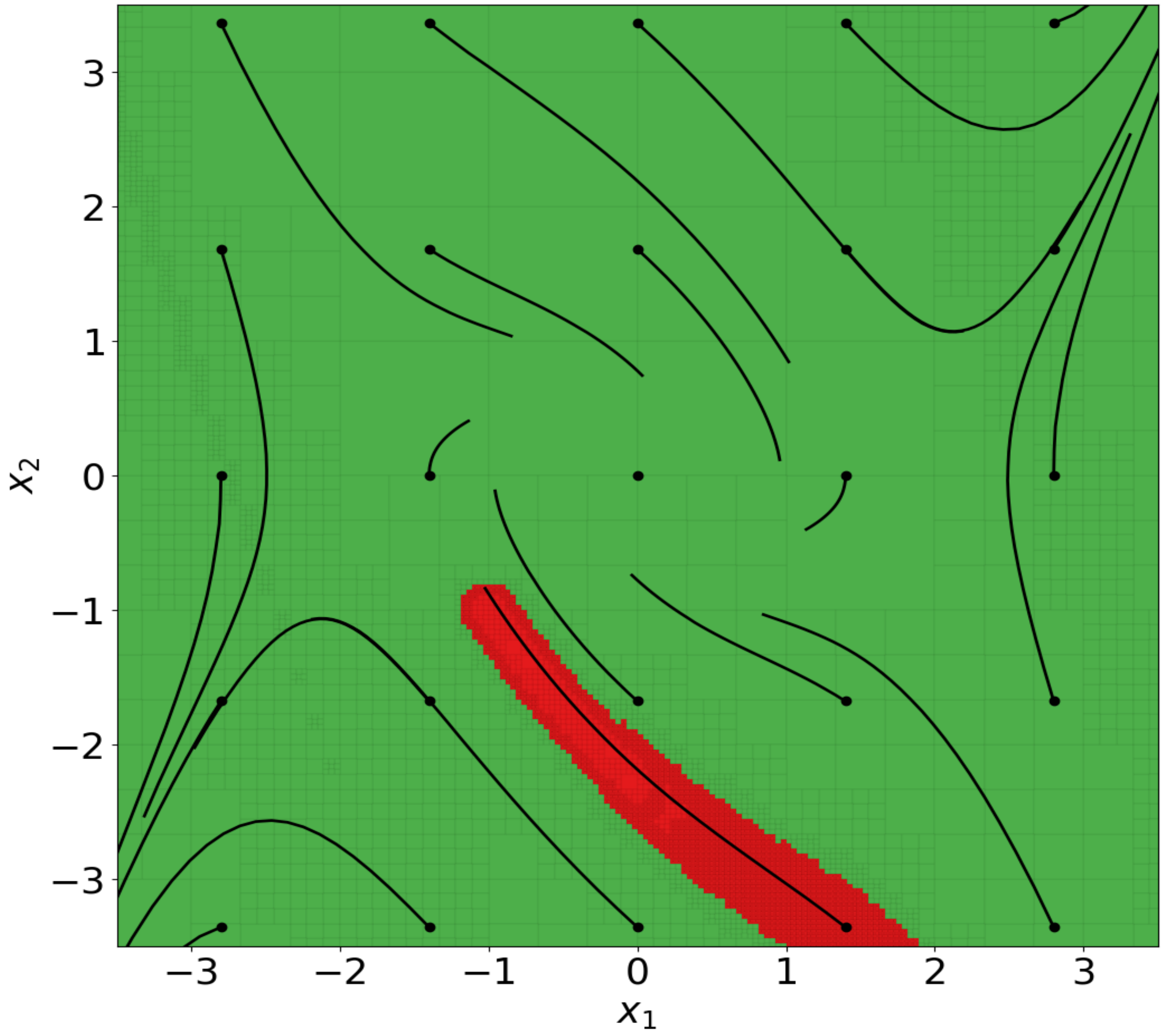
- $\mathcal{R}_{[0,\tau]}(S) \cap X_u = \emptyset$

Recurrent Condition

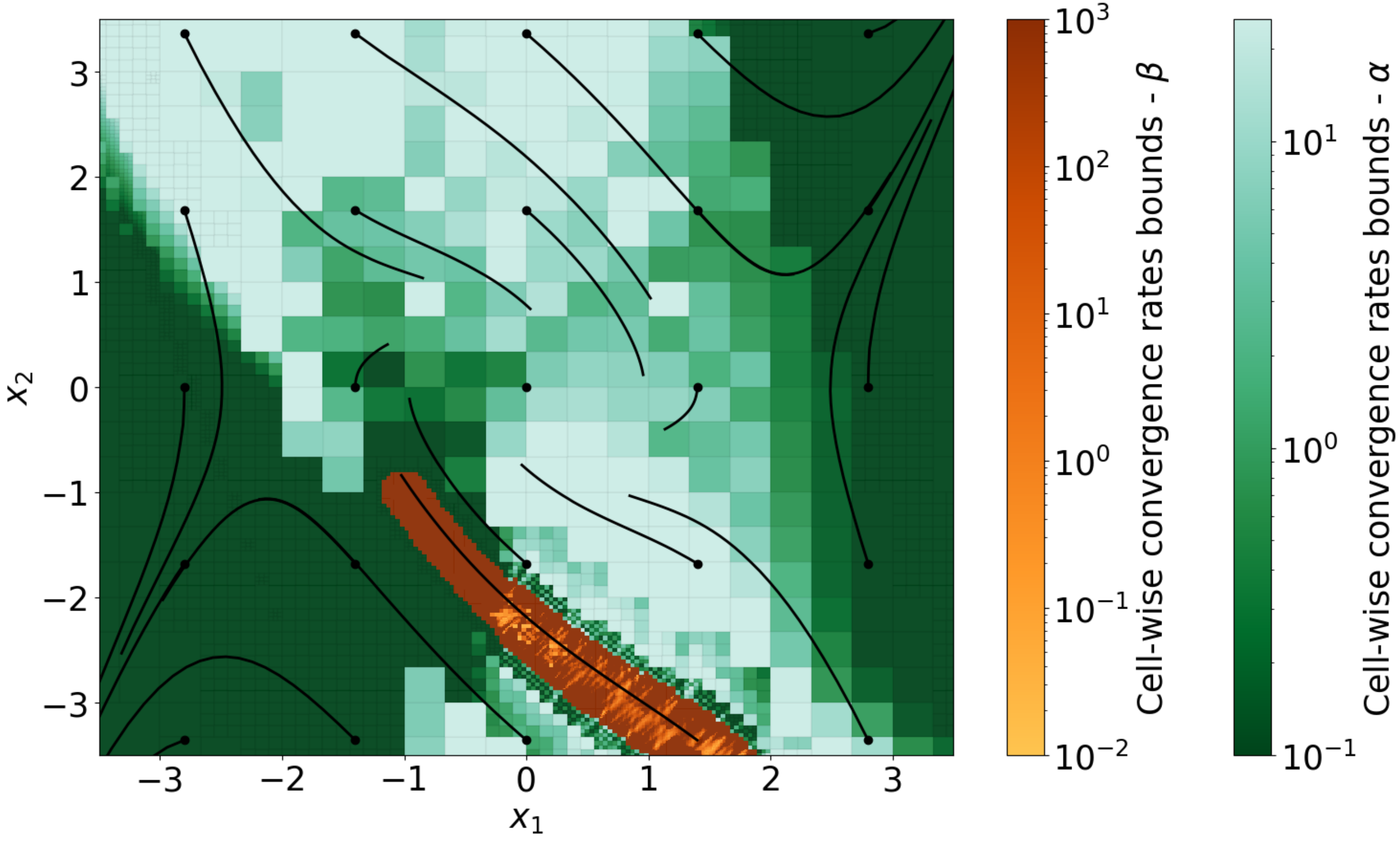
- $\hat{h}(x) := -\,\mathrm{sd}\,(\,x\,, S)$ is a Recurrent exponential barrier function

G^s: (green)

G^u: (red)



A GPU based algorithm



Conclusions and Future work

- **Takeaways**
  - Proposed a **relaxed notion of invariance** known as **recurrence**
  - Introduced **Recurrent Barrier Functions** using recurrence ideas
  - **Signed norms** on many sets are RBFs!
  - Develop **parallelizable algorithms using GPUs**
- **Ongoing work**
  - **Recurrent Sets:** Smart choice of multi-points, control recurrent sets, GPU implementation
  - **Function Certificates:** Generalize other Lyapunov notions, Control Lyapunov Functions, Control Barrier Functions, Contraction, etc.
  - **Recurrence Entropy:** Understanding the complexity of making a set recurrent when compared with invariance

Thanks!

Related Publications:

[arXiv 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* **CDC 2022, journal preprint arXiv:2204.10372**.
[CDC 23] Siegelmann, Shen, Paganini, M, *A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions*, **CDC 2023**
[HSCC 24] Sibai, M, *Recurrence of nonlinear control systems: Entropy and bit rates,* **HSCC, 2024**
[Allerton 24] Shen, Sibai, M, *Generalized Barrier Functions: Integral conditions and recurrent relaxations,* **Allerton 2024**

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