Data-driven Analysis of Dynamical Systems Using Recurrent Sets

Towards a GPU-based Approach to Control

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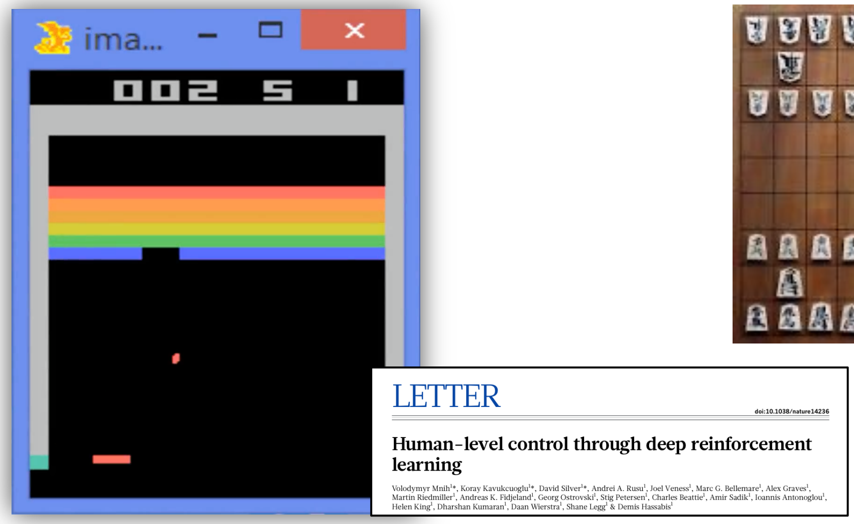
A World of Success Stories

2017 Google DeepMind's DQN

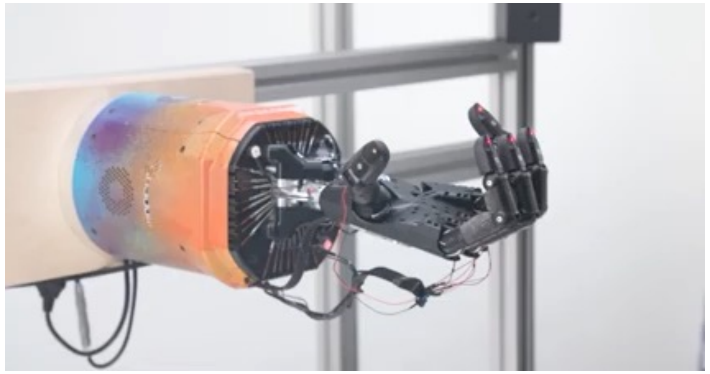


2017 AlphaZero – Chess, Shogi, Go

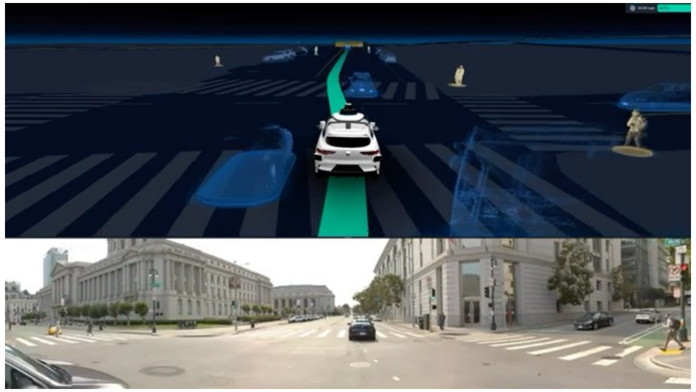
2019 AlphaStar – Starcraft II

Boston Dynamics

OpenAI – Rubik's Cube



Waymo



Reality Kicks In

Angry Residents, Abrupt Stops: Waymo Vehicles Are Still Causing Problems in Arizona

RAY STERN | MARCH 31, 2021 | 8:26AM

GARY MARCUS BUSINESS 08.14.2019 09:00 AM

DeepMind's Losses and the Future of Artificial Intelligence

Alphabet's DeepMind unit, conqueror of Go and other games, is losing lots of money. Continued deficits could imperil investments in AI.

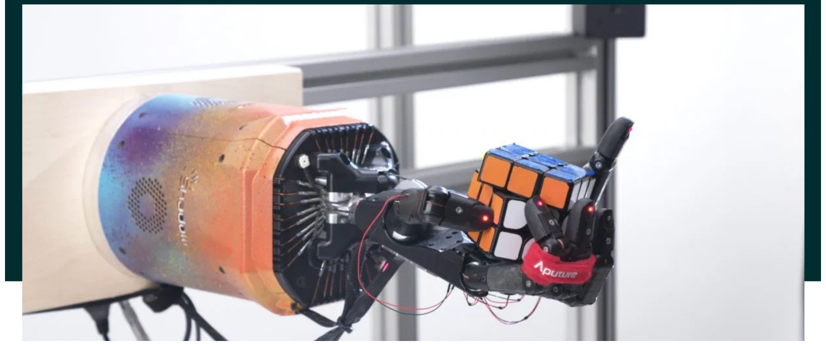
AARIAN MARSHALL BUSINESS 12.07.2020 04:06 PM

Uber Gives Up on the Self-Driving Dream

The ride-hail giant invested more than $1 billion in autonomous vehicles. Now it's selling the unit to Aurora, which makes self-driving tech.

OpenAI disbands its robotics research team

Kyle Wiggers @Kyle_L_Wiggers July 16, 2021 11:24 AM

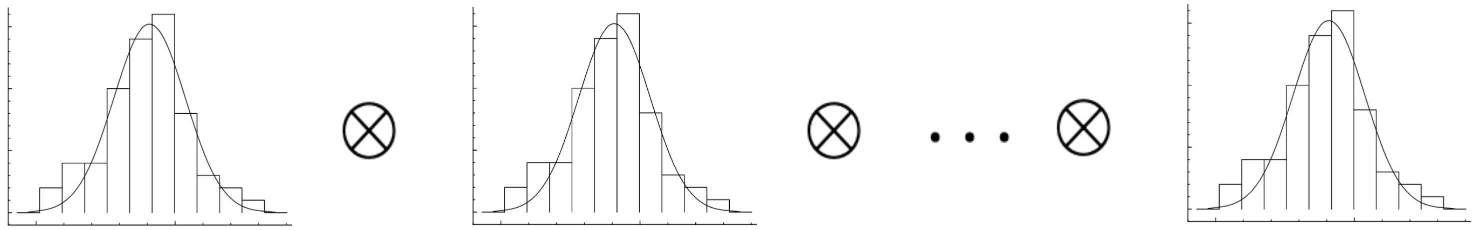


Self-driving Uber car that hit and killed woman did not recognize that pedestrians jaywalk

The automated car lacked "the capability to classify an object as a pedestrian unless that object was near a crosswalk," an NTSB report said.

Core challenge: The curse of dimensionality

- **Statistical: Sampling in d dimension with resolution ϵ**



Sample complexity:

$$O(\varepsilon^{-d})$$

For $\epsilon = 0.1$ and $d = 100$, we would need 10^{100} points.

Atoms in the universe: 10^{78}

- **Computational: Verifying non-negativity of polynomials**

Copositive matrices:

$$[x_1^2 \dots x_d^2] A [x_1^2 \dots x_d^2]^{\mathrm{T}} \geq 0$$

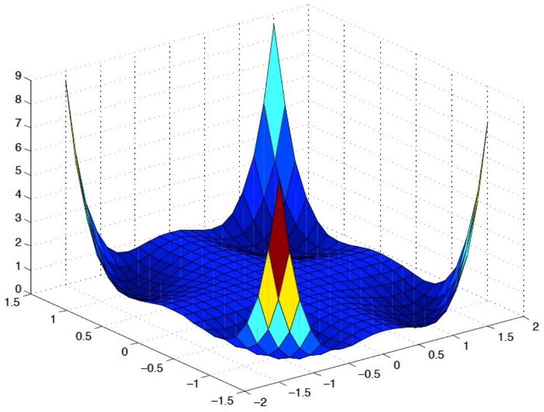
Murty&Kadabi [1987]: Testing co-positivity is NP-Hard

Sum of Squares (SoS):

$$z(x)^T Q z(x) \geq 0, \quad z_i(x) \in \mathbb{R}[x], \; x \in \mathbb{R}^d, Q \succcurlyeq 0$$

Artin [1927] (Hilbert's 17th problem):

Non-negative polynomials are sum of square of *rational* functions



Motzkin [1967]:

$p = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$

is nonnegative,

not a sum of squares,

but $(x^2 + y^2)^2 p$ is SoS

Question: Are we asking too much?

- Analysis tools build on a strict and exhaustive notion of ***invariance***

  Q: Can we substitute invariance with less restrictive notions?

  [arXiv ‘22] Shen, Bichuch, M - [CDC ’23] Siegelmann, Shen, Paganini, M

- Certificates impose conditions on the entire duration of the trajectory

  Q: Can we provide guarantees based on only localized trajectory information?

  [arXiv ‘22] Shen, Bichuch, M - [CDC ’23] Siegelmann, Shen, Paganini, M

- Control synthesis usually aims for the ***best*** (optimal) controller

  Q: Is there any gain in focusing on weaker requirements from the get-go?

  [HSCC 24] Sibai, M - - [CDC ’23] Siegelmann, Shen, Paganini, M

[arXiv 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* **CDC 2022,** journal preprint arXiv:2204.10372.
[CDC 23] Siegelmann, Shen, Paganini, M, *A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions*, **CDC 2023**
[HSCC 24] Sibai, M, *Recurrence of nonlinear control systems: Entropy and bit rates,* **HSCC, 2024**

Outline

- **Invariance: Merits and trade-offs**
- Letting things go, and come back: *Recurrent sets*
- Approximating regions of attractions via recurrent sets
- Stability analysis via non-monotonic Lyapunov functions

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Problem setup

Continuous time dynamical system: $\dot{x}(t) = f(x(t))$

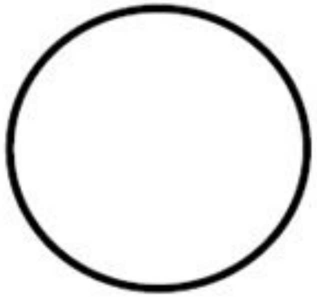
- Initial condition $x_0 = x(0)$, solution at time t: $\phi(t, x_0)$.

Ω-Limit Set $\Omega(f)$:

$$x \in \Omega(f) \iff \exists\, x_0, \{t_n\}_{n\geq 0}, \text{ s.t. } \lim_{n\to\infty} t_n = \infty \text{ and } \lim_{n\to\infty} \phi(t_n, x_0) = x$$

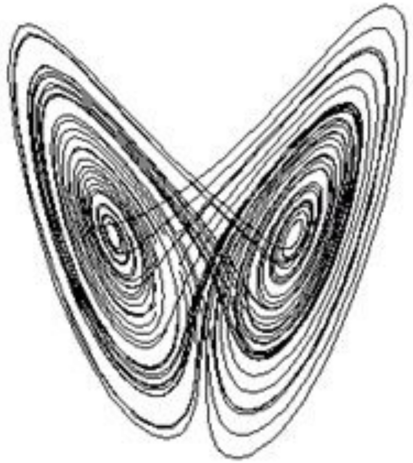
Types of Ω-limit set

equilibrium



limit cycle

limit torus



chaotic attractor

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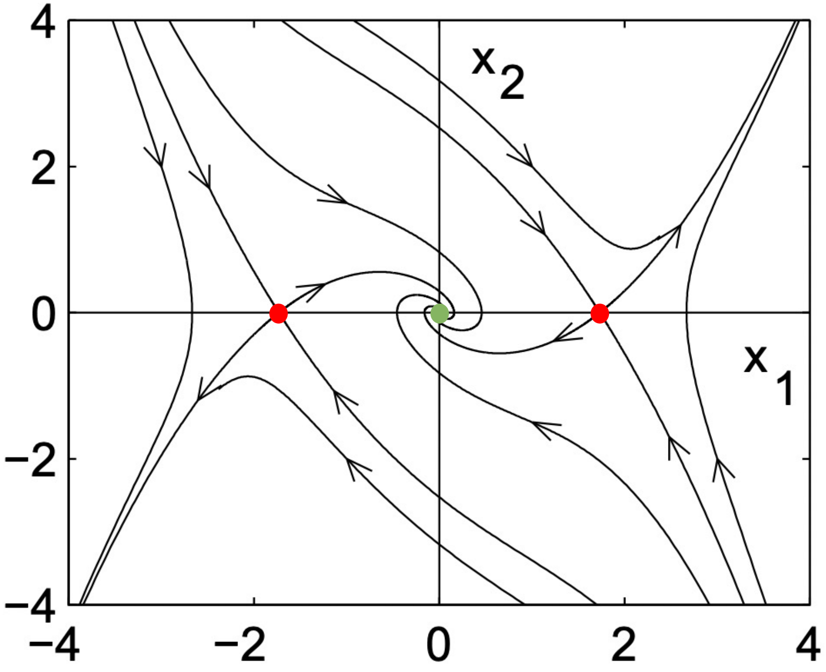
Illustrative Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 + \frac{1}{3}x_1^3 - x_2 \end{bmatrix}$$

$$\Omega(f) = \{(0,0), (-\sqrt{3}, 0), (\sqrt{3}, 0)\}$$ (equilibria)



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Region of attraction (ROA) of a set $S \subseteq \Omega(f)$:

$$\mathcal{A}(S) := \left\{ x \in \mathbb{R}^d \middle| \liminf_{t \to \infty} d(\phi(t, x), S) = 0 \right\}$$

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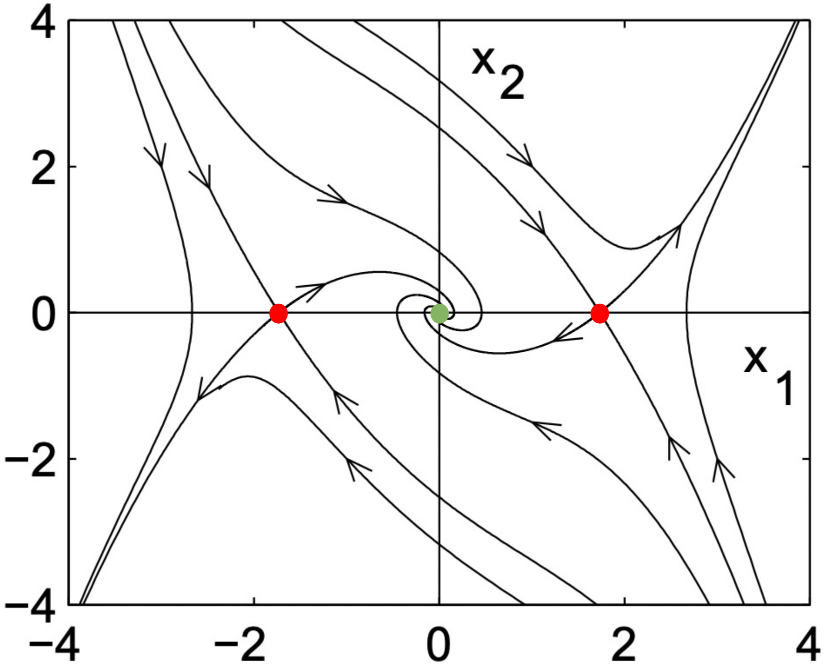

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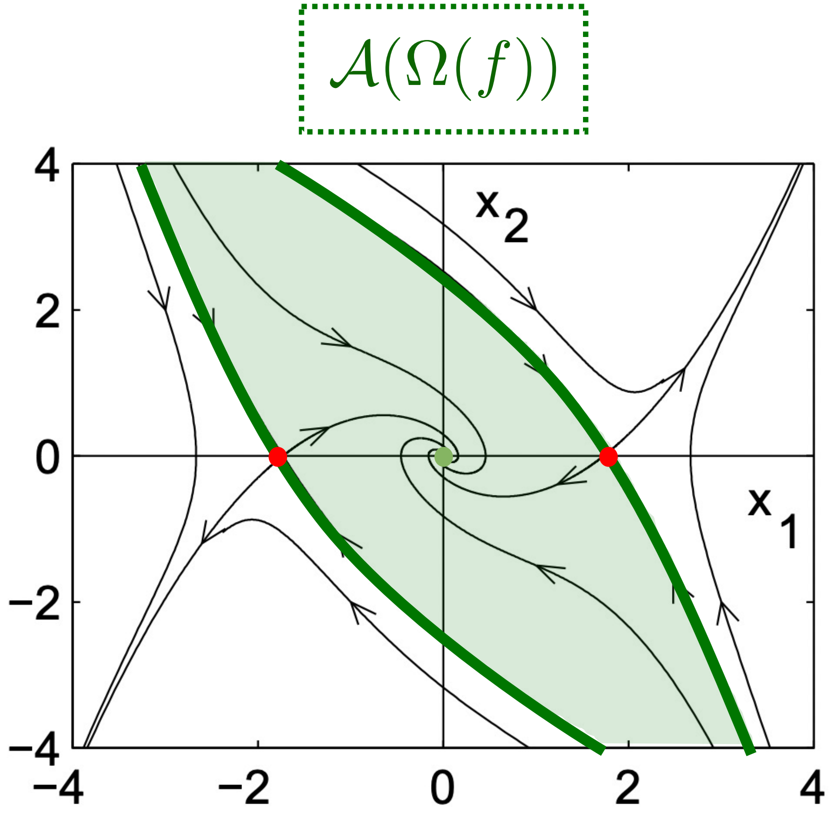

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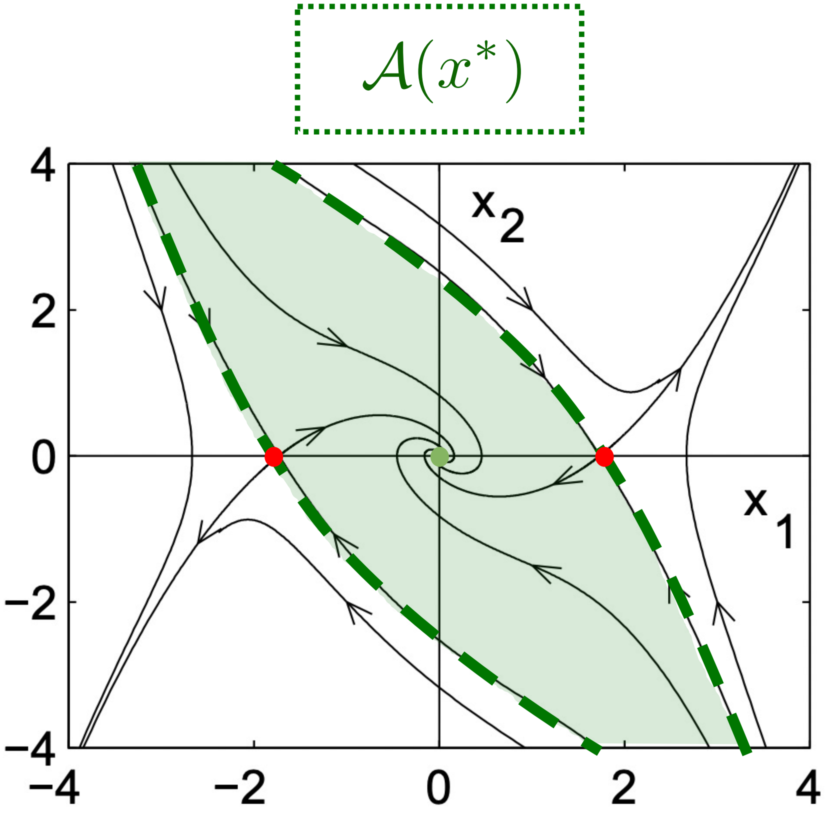
Asymptotically stable equilibrium at $x^* = (0,0)$



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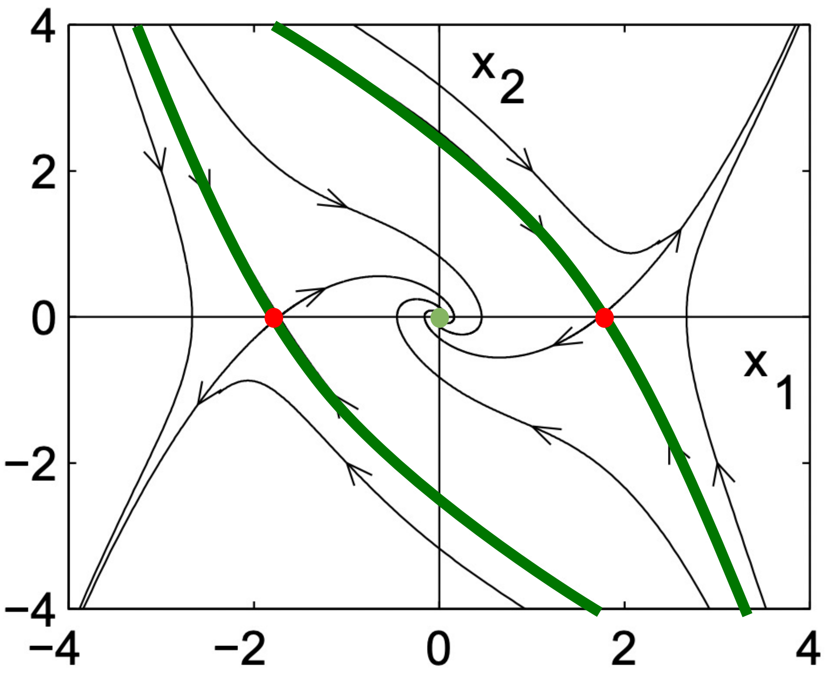
Unstable equilibria $\{(\sqrt{3}, 0), (-\sqrt{3}, 0)\}$

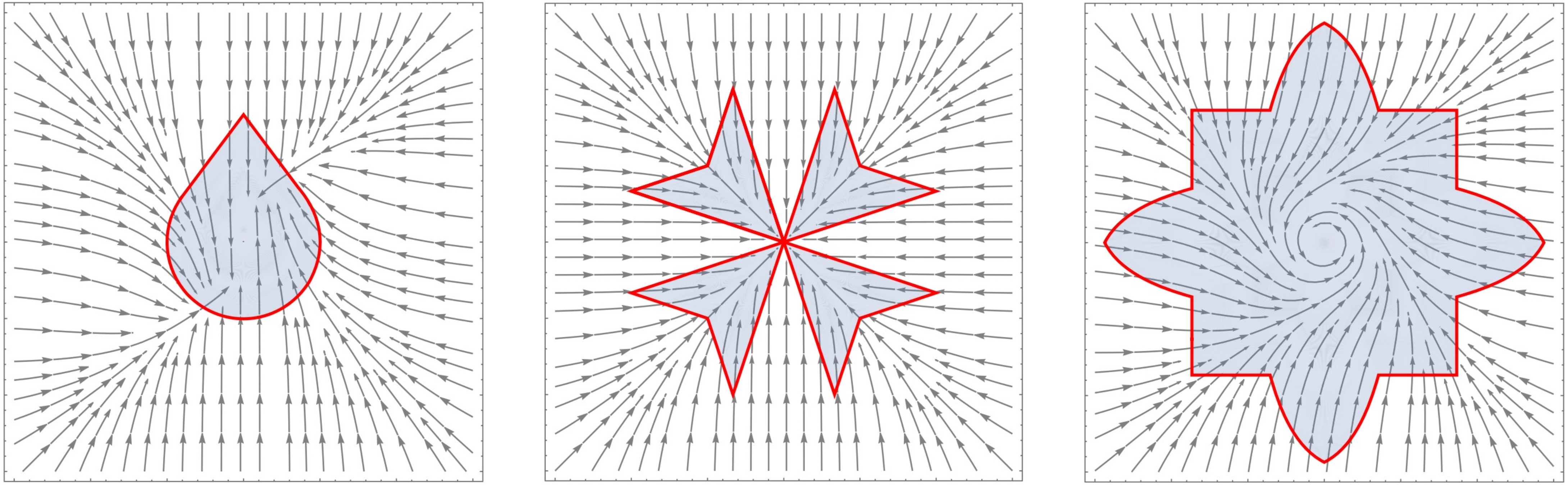
$\mathcal{A}(\{(\sqrt{3}, 0), (-\sqrt{3}, 0)\})$



Invariant sets

A set $\mathcal{S} \subseteq \mathbb{R}^d$ is **positively invariant** if and only if: $x_0 \in \mathcal{S} \rightarrow \phi(t, x_0) \in \mathcal{S},\ \forall t \geq 0$

Any trajectory starting in the set remains in inside it for all times



Source: K. Ghorbal, K. and A. Sogokon, Characterizing positively invariant sets: Inductive and topological methods. Journal of Symbolic Computation, 2022

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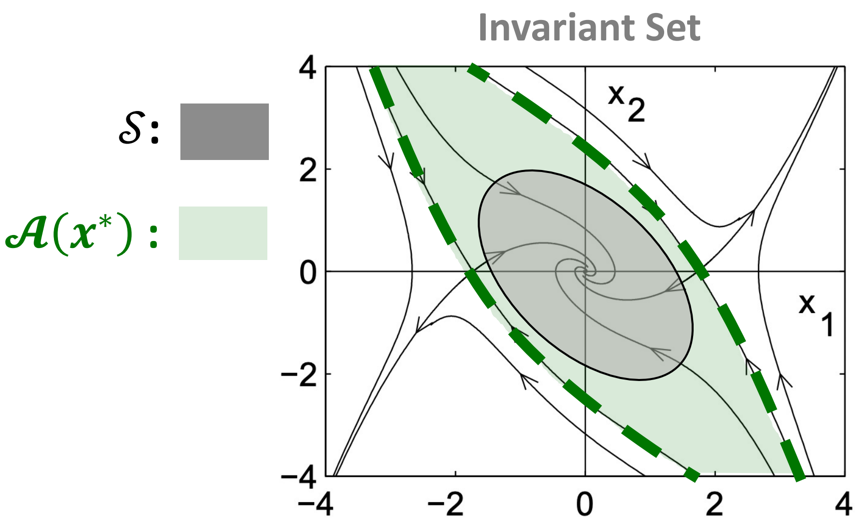
- **Invariant sets approximate regions of attraction**
  Compact invariant set $\mathcal{S}$, containing **only** $\{x^*\} = \Omega(f) \cap \mathcal{S}$ must be in the region of attraction $\mathcal{A}(x^*)$ ($\mathcal{S} \subset \mathcal{A}(x^*)$)



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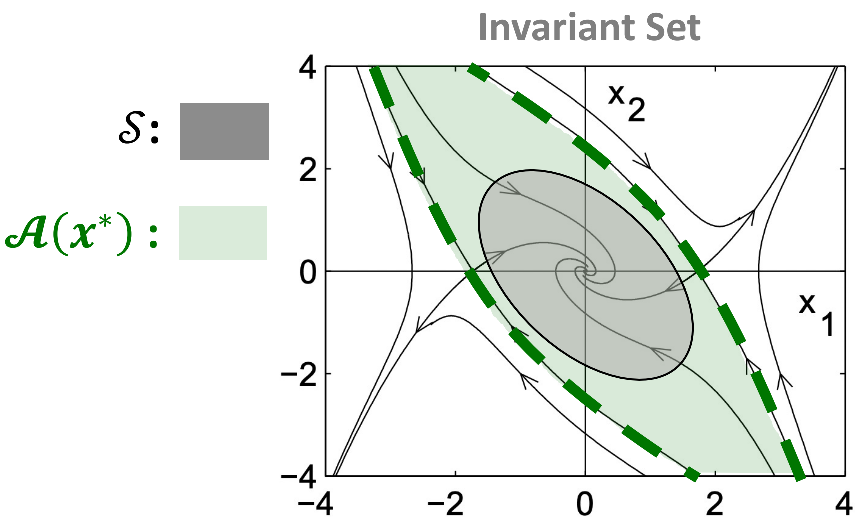
  Proof sketch:
  - Take $x_0 \in \mathcal{S}$, and assume to $x_0 \notin \mathcal{A}(x^*)$
  - Then, **no bounded seq** satisfies $\phi(t_n, x_0) \to x^*$
  - Since $\mathcal{S}$ is invariant and compact: Bolzano-Weierstrass implies there is bounded sub-seq

  $$\phi\left(t_{n_i}, x_0\right) \to \bar{x} \neq x^*$$

  - Contradiction! x^* is the only limiting point in $\mathcal{S}$



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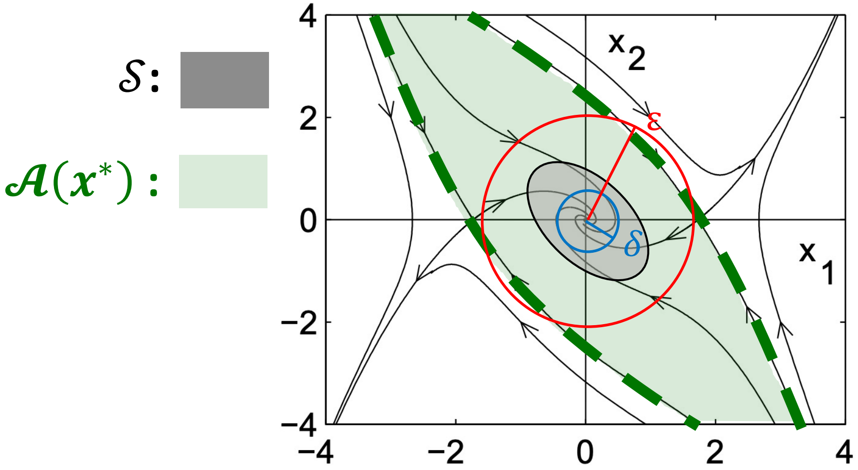
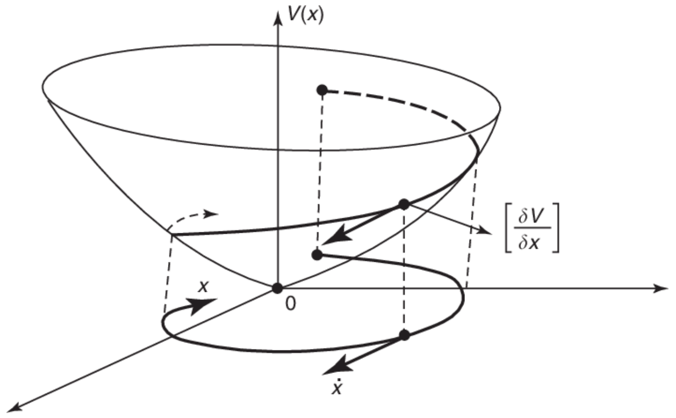
- **Invariant sets guarantee stability**
  Lyapunov stability: solutions starting "close enough" to the equilibrium (within a distance δ) remain "close enough" forever (within a distance ε)

- **Invariant sets further certify asymptotic stability via Lyapunov's direct method**
  Asymptotic stability: solutions that start close enough, remain close enough, and eventually converge to equilibrium.

Invariant Set



Lyapunov Functions



Invariant sets: Challenges

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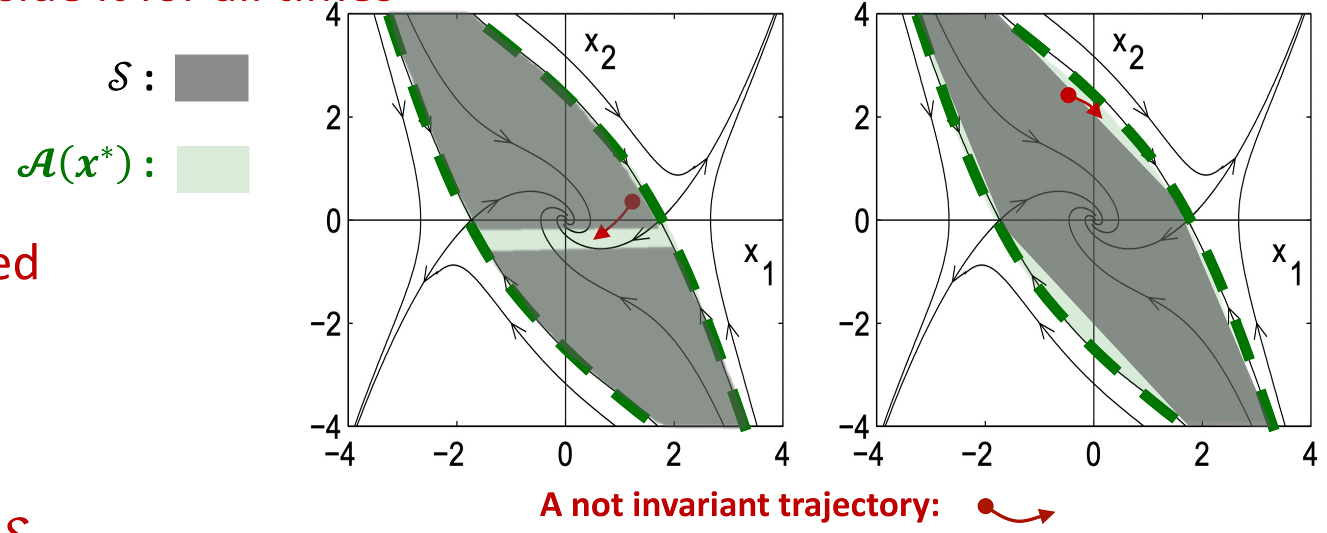
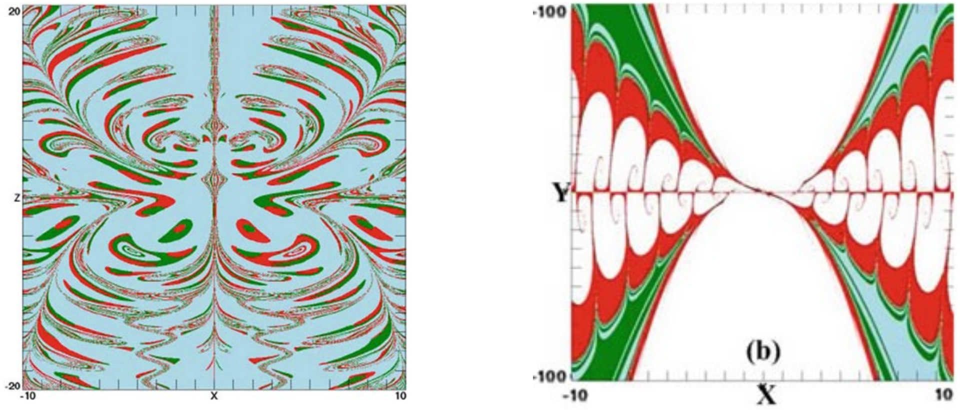


- **$\mathcal{S}$ is topologically constrained**
  - If $\mathcal{S} \cap \Omega(f) = \{x^*\}$, then $\mathcal{S}$ is connected
- **$\mathcal{S}$ is geometrically constrained**
  - f should not point outwards for $x \in \partial\mathcal{S}$
- **$\mathcal{S}$ geometry can be wild**
  - $\mathcal{A}(\Omega(f))$ is not necessarily analytic!

Basin of $\Omega(f)$



Outline

- Invariance: Merits and trade-offs
- **Letting things go, and come back: Recurrent sets**
- Analysis using recurrent sets
  - Approximating regions of attractions
  - Stability analysis via non-monotonic Lyapunov functions
- Recurrence in nonlinear control systems
  - Entropy and bit rates of control recurrent sets

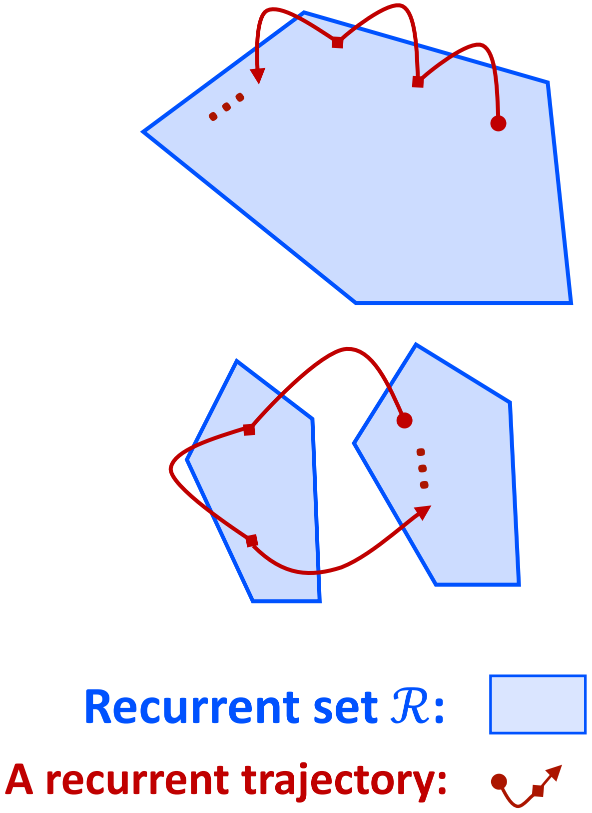
Recurrent sets: Letting things go, and come back

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \geq t$ s.t. $\phi(t', x_0) \in \mathcal{R}$.

Property of Recurrent Sets

- $\mathcal{R}$ need **not** be **connected**
- $\mathcal{R}$ does **not** require f to **point inwards** on all $\partial\mathcal{R}$

> Recurrent sets, while not invariant, guarantee that solutions that start in this set, will come back **infinitely often, forever!**



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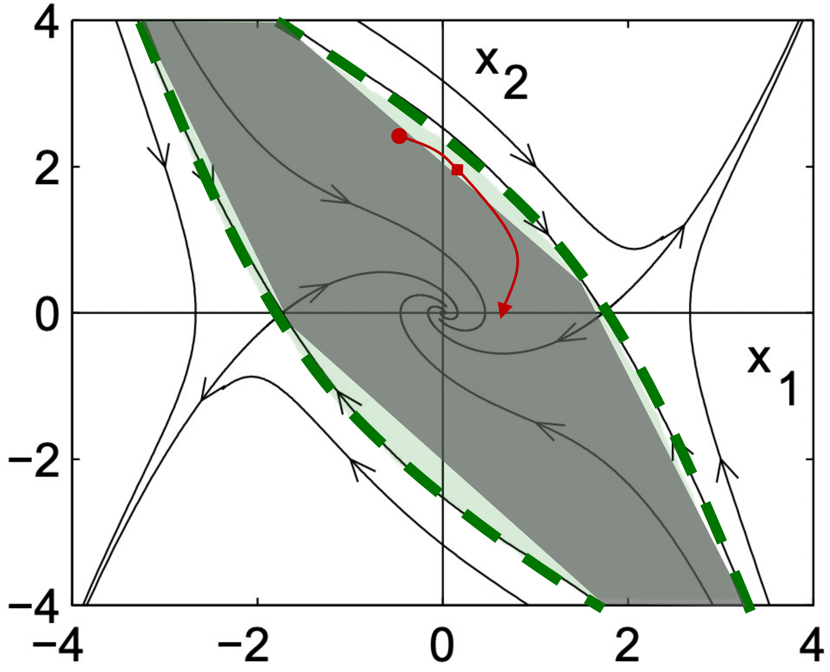
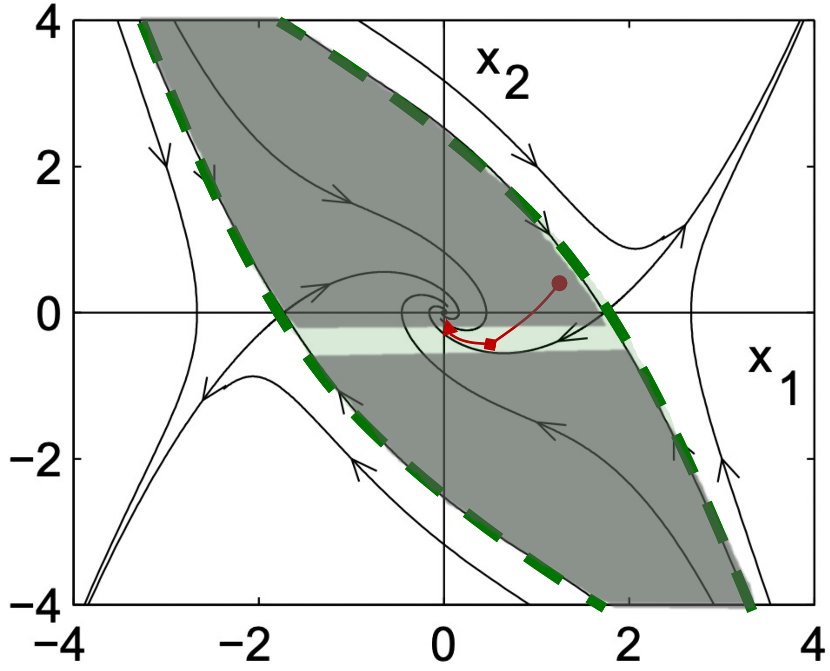
Previous two good inner approximations of $\mathcal{A}(x^*)$ are recurrent sets





[arXiv 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* **CDC 2022,** journal preprint arXiv:2204.10372.

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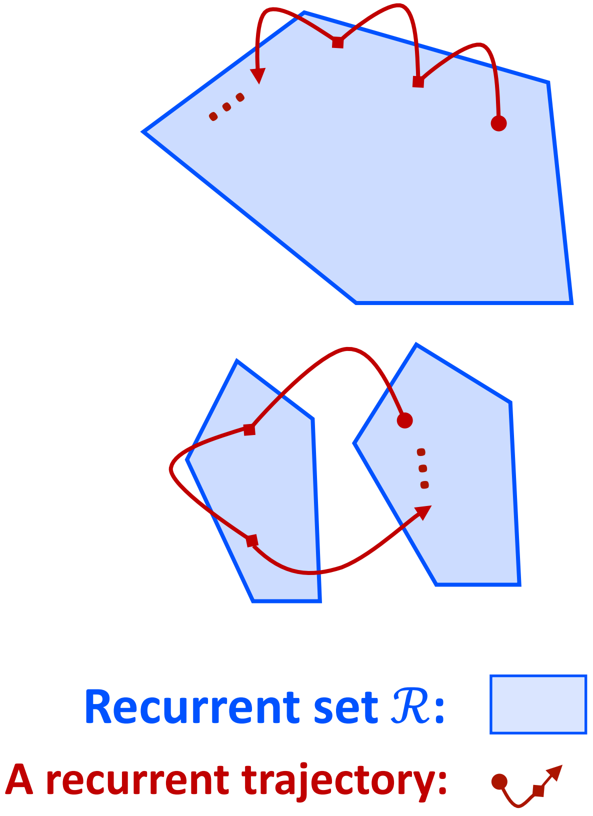

Question: Can we use recurrent sets as functional substitutes of invariant sets?

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Yue Shen
Johns Hopkins University

Maxim Bichuch
University at Buffalo

Model-free Learning of Regions of Attractions via Recurrent Sets

Y Shen, M. Bichuch, and E Mallada, "Model-free Learning of regions of attraction via recurrent sets." CDC 2022.

Recurrent sets are subsets of the region of attraction

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \geq t$ s.t. $\phi(t', x_0) \in \mathcal{R}$.

Theorem. *Let $\mathcal{R} \subset \mathbb{R}^d$ be a <u>compact</u> set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$.*

Then:

$\mathcal{R}$ is *invariant* ➡ $\mathcal{R} \cap \Omega(f) \neq \emptyset$, $\mathcal{R} \subset \mathcal{A}(\mathcal{R} \cap \Omega(f))$

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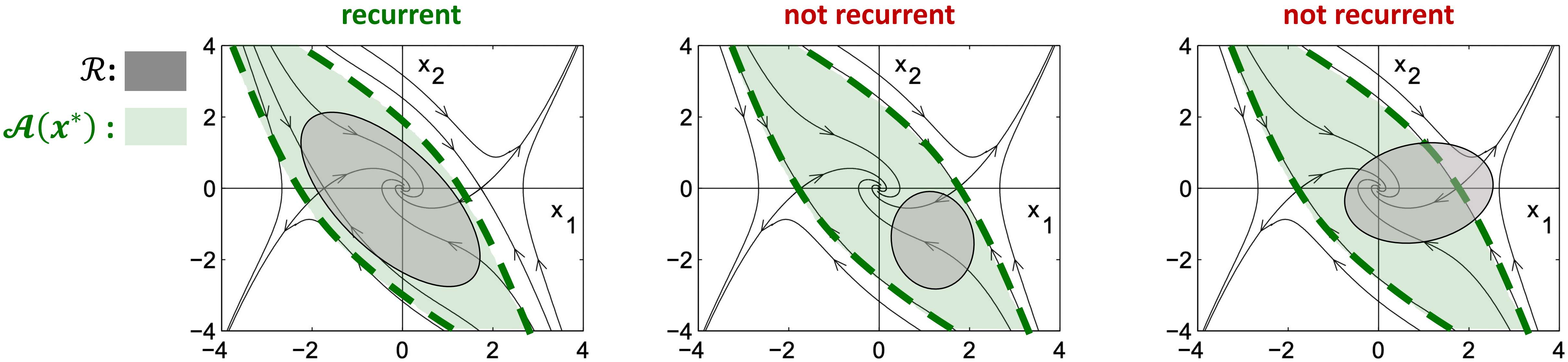

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Proof: *[Sketch]*

(⟹)
- *If $x_0 \in \mathcal{R}$, the solution $\phi(t, x_0)$ visits $\mathcal{R}$ infinitely often, forever.*
- *We can build a sequence $\{x(t_n)\}_{n=0}^{\infty} \in \mathcal{R}$ with $\lim_{n\to+\infty} t_n = +\infty$*
- *Bolzano-Weierstrass $\Longrightarrow$ convergent subsequence $x(t_{n_i}) \to \bar{\boldsymbol{x}} \in \Omega(f) \cap \mathcal{R} \neq \emptyset$*
- $\boldsymbol{\partial\mathcal{R}} \cap \boldsymbol{\Omega(f)} = \emptyset$ and $\mathcal{R}$ *recurrent* $\Longrightarrow$ $\phi(t, x_0)$ *leaves $\mathcal{R}$ finitely many times*
- *$\mathcal{R}$ is eventually invariant, and it follows that $x_0 \in \mathcal{A}(\mathcal{R} \cap \Omega(f))$*

(⟸)
- $\boldsymbol{\partial\mathcal{R}} \cap \boldsymbol{\Omega(f)} = \emptyset$ and $\mathcal{R} \subset \mathcal{A}(\mathcal{R} \cap \Omega(f)) \Longrightarrow$ *$\mathcal{R}$ is eventually invariant $\Longrightarrow$ $\mathcal{R}$ recurrent*

Recurrent sets are subsets of the region of attraction

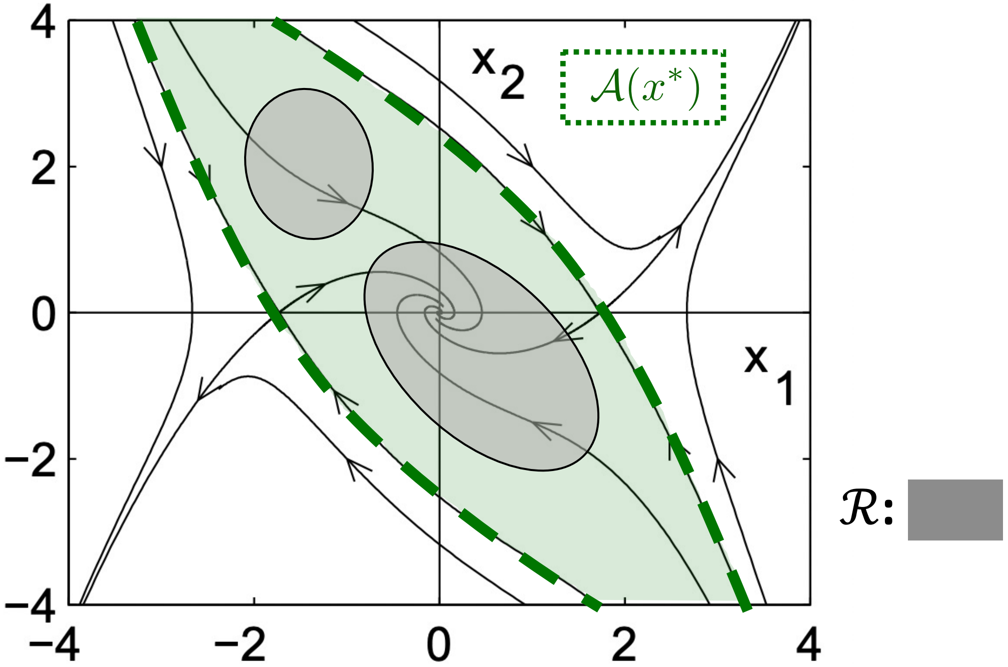
A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \geq t$ s.t. $\phi(t', x_0) \in \mathcal{R}$. **?**

Corollary. *Let Assumption 1 hold, and let $\mathcal{R} \subset \mathbb{R}^d$ be a compact set satisfying:*

$$\partial\mathcal{R} \cap \Omega(f) = \emptyset \text{ and } \mathcal{R} \cap \Omega(f) = \{x^*\}$$

Then:

$\mathcal{R}$ is recurrent ⟷ $\mathcal{R} \subset \mathcal{A}(x^*)$



Idea: Use recurrence as a mechanism for finding inner approximations of $\mathcal{A}(x^*)$

Potential Issues:

- We do not know how long it takes to come back!
- We need to adapt results to trajectory samples

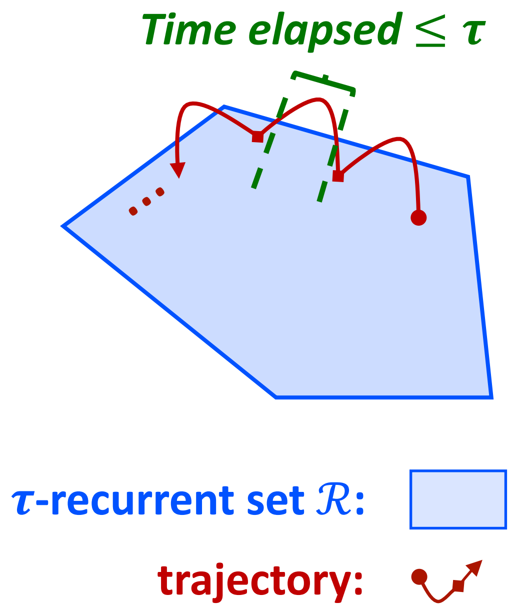
τ-recurrent sets

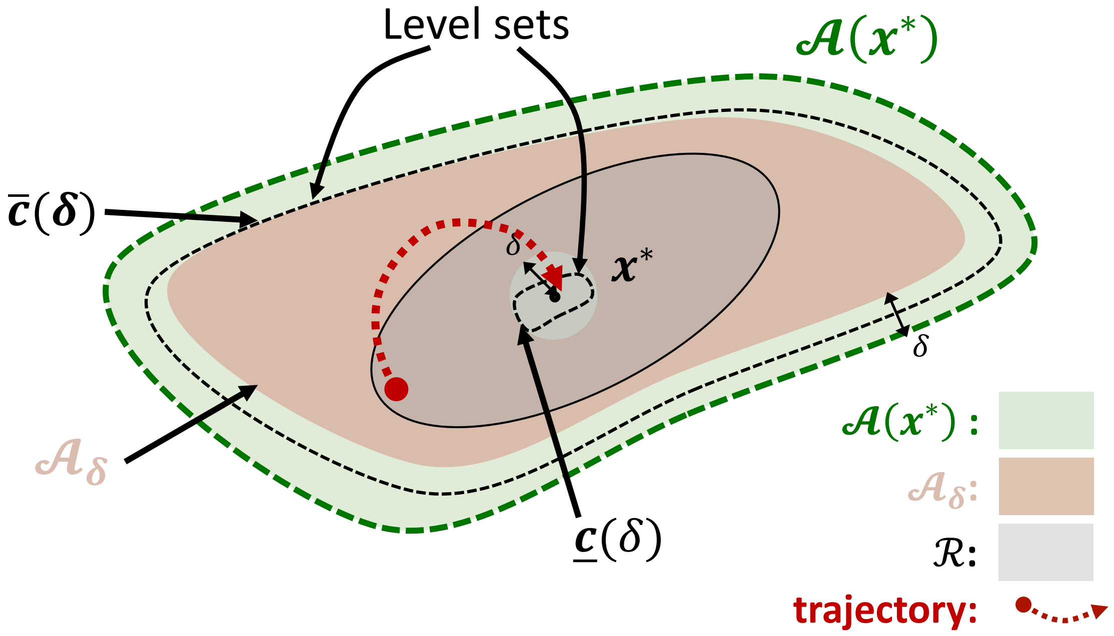
A set $\mathcal{R}$ is **τ-recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists\, t' \in [t, t+\tau]$ such that $\phi(t', x_0) \in \mathcal{R}$

Theorem. *Under mild assumptions, any compact set* $\mathcal{R}$ satisfying:

$$x^* + \mathcal{B}_\delta \subseteq \mathcal{R} \subseteq \mathcal{A}_\delta$$

is τ-recurrent for $\tau \geq \bar{\tau}(\delta) := \frac{\underline{c}(\delta) - \bar{c}(\delta)}{a(\delta)}$.





Recurrent sets are subsets of the region of attraction

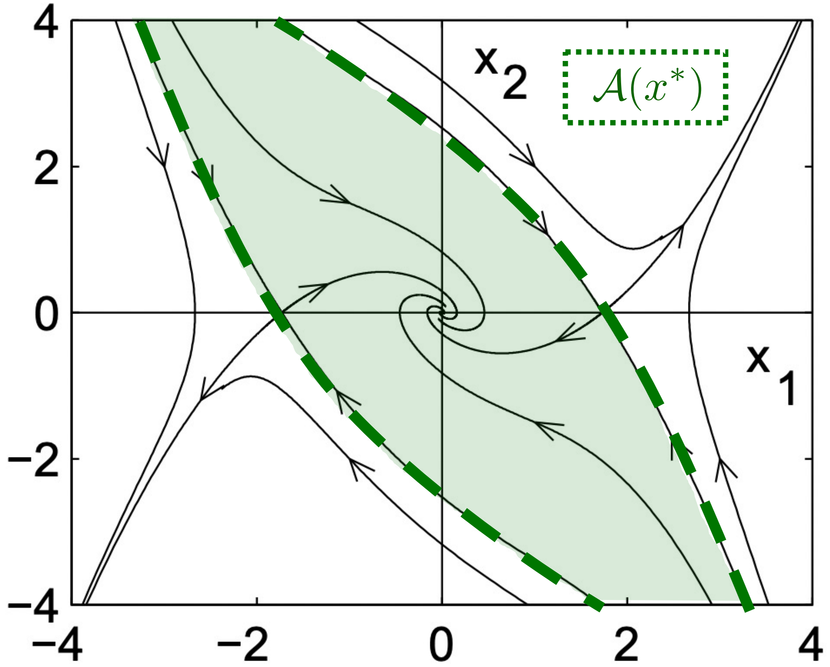
A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for $x_0 \in \mathcal{R}$, for any t $\geq 0 \Rightarrow \exists t' > t,$ s.t. $\phi(t', x_0) \in \mathcal{R}$?

***Corollary**. Let Assumption 1 hold, and let $\mathcal{R} \subset \mathbb{R}^d$ be a compact set satisfying:*

$$\partial\mathcal{R} \cap \Omega(f) = \emptyset \text{ and } \mathcal{R} \cap \Omega(f) = \{x^*\}$$

Then:

$\mathcal{R}$ is recurrent ⬌ $\mathcal{R} \subset \mathcal{A}(x^*)$



Idea: Use recurrence as a mechanism for finding inner approximations of $\mathcal{A}(x^*)$

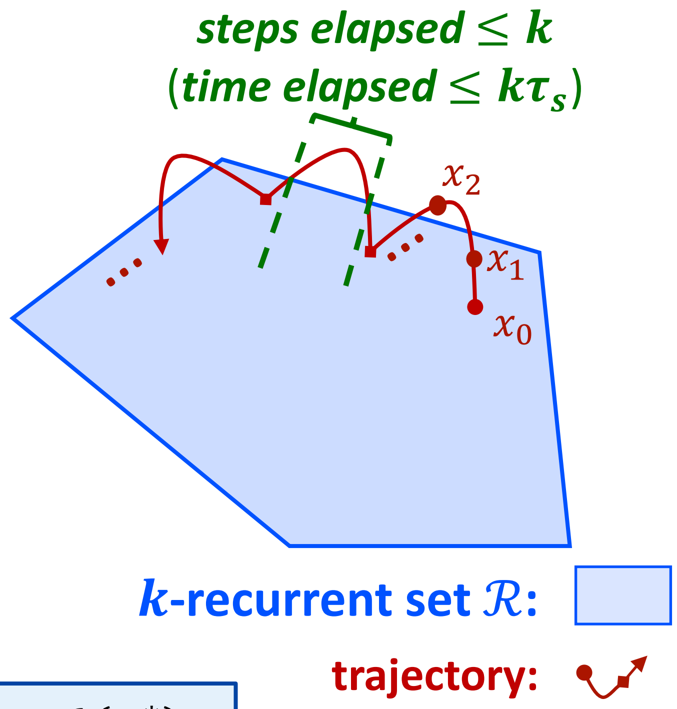
Potential Issues:

- We do not know how long it takes to come back! ✔
- We need to adapt results to trajectory samples

Learning recurrent sets from k-length trajectory samples

- Consider **finite length** trajectories:
  $$x_n = \phi(n\tau_s, x_0), \qquad x_0 \in \mathbb{R}^d, n \in \mathbb{N},$$
  where $\tau_s > 0$ is the sampling period.

- A set $\mathcal{R} \subseteq \mathbb{R}^d$ is k**-recurrent** if whenever $x_0 \in \mathcal{R}$, then $\exists\, n \in \{1, \dots, k\}$ s.t. $x_n \in \mathcal{R}$



Sufficiency:

$\mathcal{R}$ is k-recurrent ➡ $\mathcal{R}$ is τ-recurrent with $\tau = k\tau_s$ ➕ $\mathcal{R}$ is compact, $\mathcal{R} \cap \Omega(f) = \{x^*\}$, $x^* \in \text{int}\,\mathcal{R}$ ➡ $\mathcal{R} \subset \mathcal{A}(x^*)$

Necessity:

Theorem *3. Under mild assumptions, any compact set* $\mathcal{R}$ satisfying:
$$\mathcal{B}_\delta + x^* \subseteq \mathcal{R} \subseteq \mathcal{A}_\delta$$
is k*-recurrent for any* $k > \bar{k} := \bar{\tau}(\delta)/\tau_s$.

Recurrent sets are subsets of the region of attraction

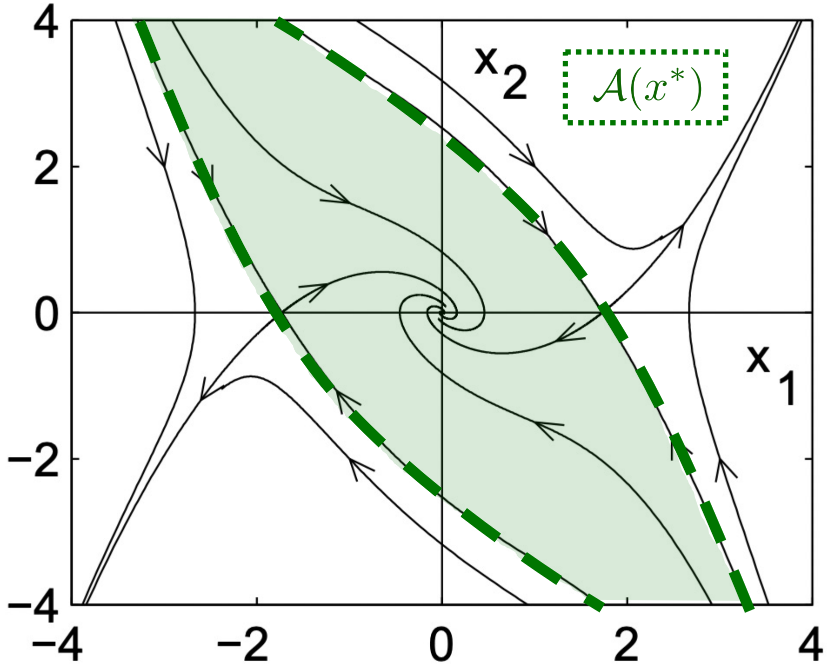
A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for $x_0 \in \mathcal{R}$, $\phi(t, x_0) \notin \mathcal{R} \Rightarrow \exists t' > t,$ s.t. $\phi(t', x_0) \in \mathcal{R}$?

Corollary. *Let Assumption 1 hold, and let $\mathcal{R} \subset \mathbb{R}^d$ be a* <u>*compact*</u> *set satisfying:*

$$\partial\mathcal{R} \cap \Omega(f) = \emptyset \text{ and } \mathcal{R} \cap \Omega(f) = \{x^*\}$$

Then:

$\mathcal{R}$ is recurrent $\Leftrightarrow$ $\mathcal{R} \subset \mathcal{A}(x^*)$



Idea: Use recurrence as a mechanism for finding inner approximations of $\mathcal{A}(x^*)$

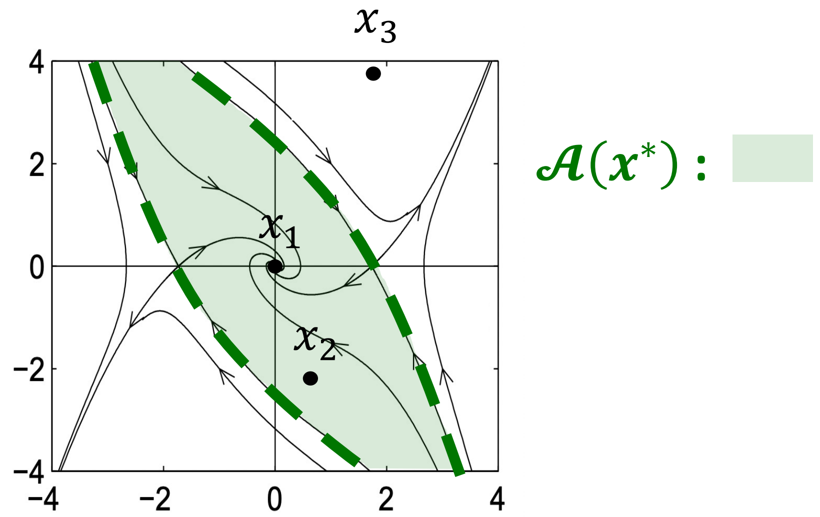
Potential Issues:

- We do not know how long it takes to come back! ✔
- We need to adapt results to trajectory samples ✔

Learning Regions of Attractions via Recurrent Sets

Algorithm: Given h, k, and $\varepsilon > 0$:

- *Build approximation using unions of balls centered at* $x_1, \dots, x_h$, *with* $x_1 = x^*$



[arXiv 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* **CDC 2022,** journal preprint arXiv:2204.10372.

Learning Regions of Attractions via Recurrent Sets

Algorithm: Given h, k, and $\varepsilon > 0$:

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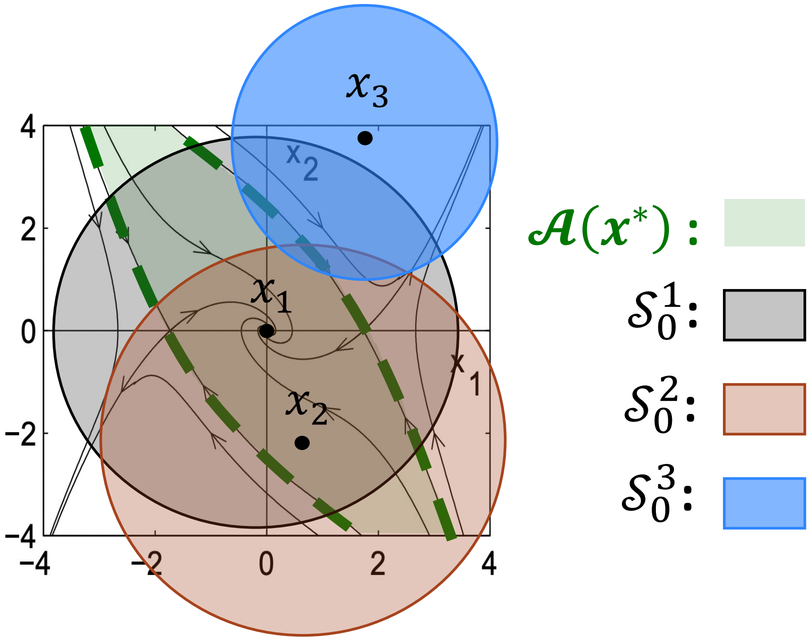
- *Initial approximation:* $\mathcal{S}_0 = \cup_{q=1}^{h} \mathcal{S}_0^q$, *where* $\mathcal{S}_0^q = \left\{x \colon \left\|x - x_q\right\| \le b_0^q\right\}$



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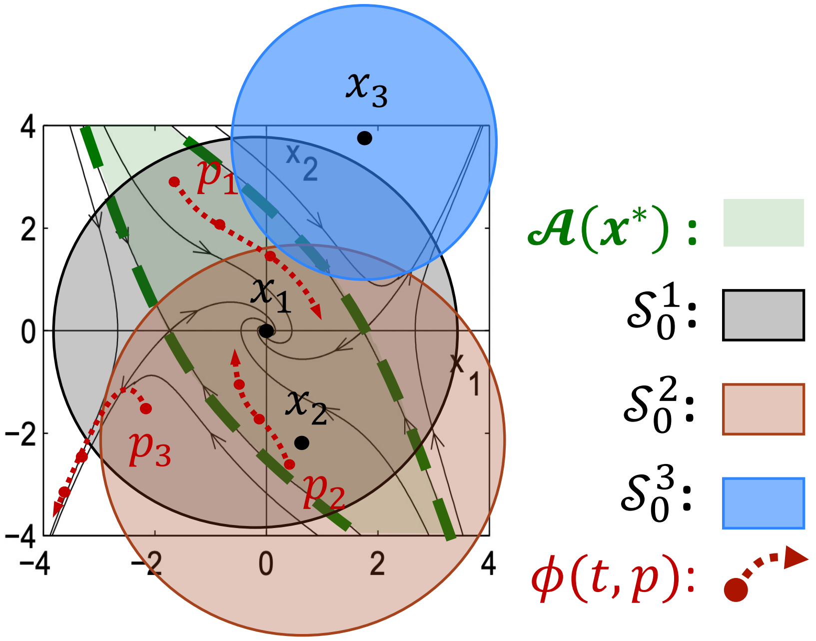
At each iteration l

- *Sample trajectories of duration τ from $\mathcal{S}_l$ until recurrence is violated (counter-example)*



p_3: **counter example**

[arXiv 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* **CDC 2022,** journal preprint arXiv:2204.10372.

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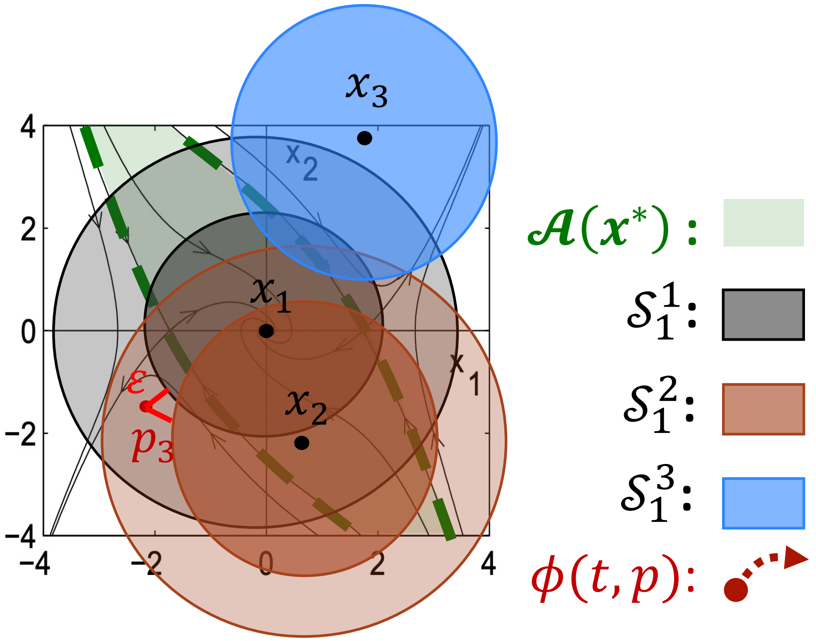
At each iteration l

- *Sample trajectories of duration* τ *from* $\mathcal{S}_l$ *until recurrence is violated (counter-example)*
- *Update approximation* $\mathcal{S}_{l+1}$ *to exclude counter-example neighborhood:* $p_j + \mathrm{B}_\varepsilon$

Sample complexity: $m \ge \frac{\mathrm{V}(\mathcal{S}_l + \mathrm{B}_\varepsilon)}{\mathrm{V}(\mathrm{B}_\varepsilon)} \log\left(\frac{1}{\delta}\right)$



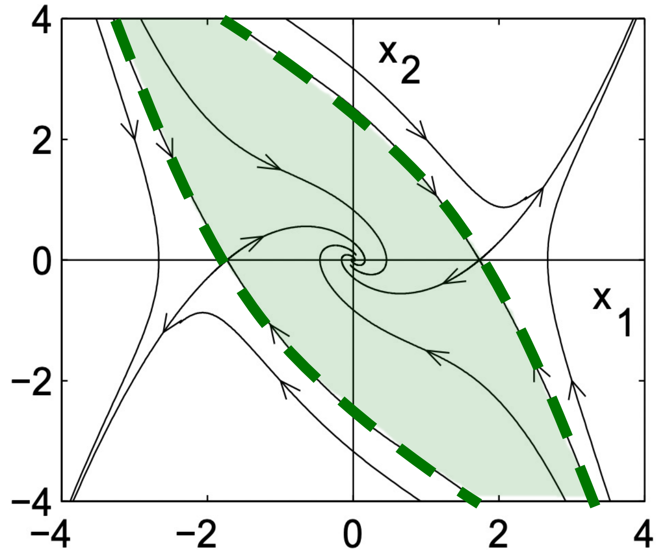
p_3: counter example

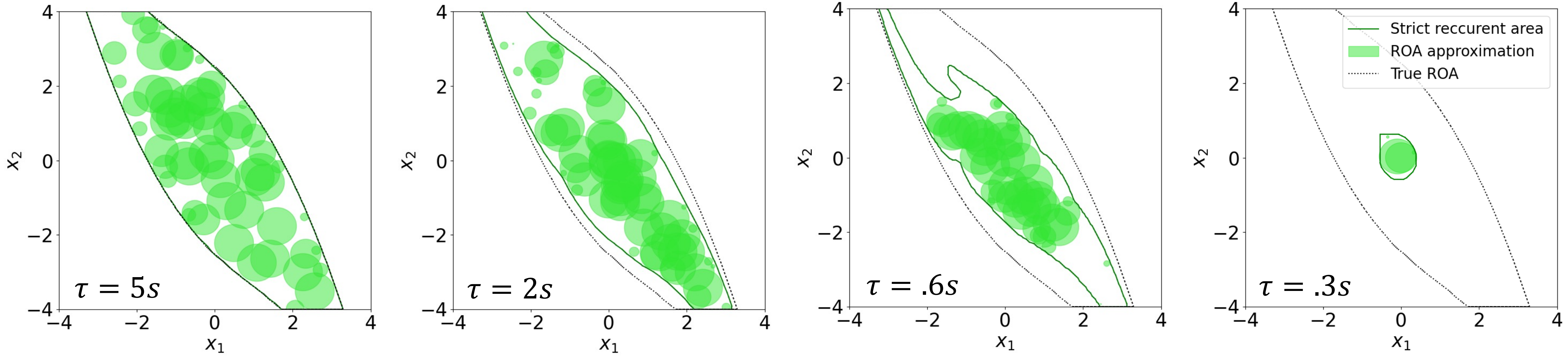
[arXiv 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* **CDC 2022,** journal preprint arXiv:2204.10372.

Numerical illustrations

- **Run:** 200 center points sampled (uniformly)
- **Stopping criteria:** $\rho = 10^{-5}$

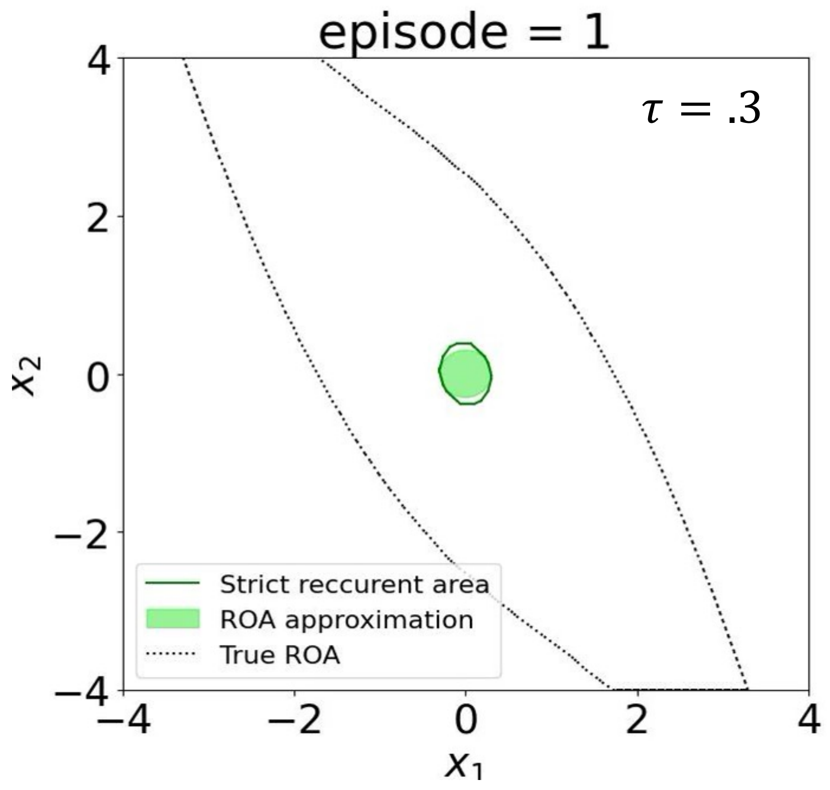
τ (s)	Running time	Volume %
5	57.7	72.0%
2	55.8	51.2%
.6	47.1	31.2%
.3	28.7	3.24%

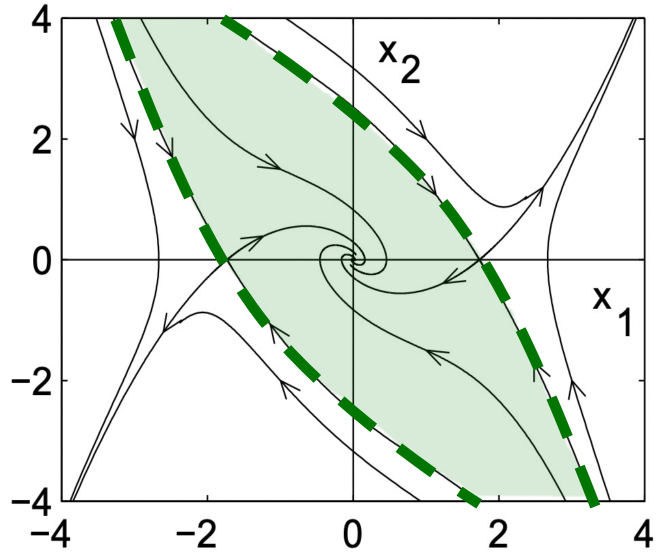


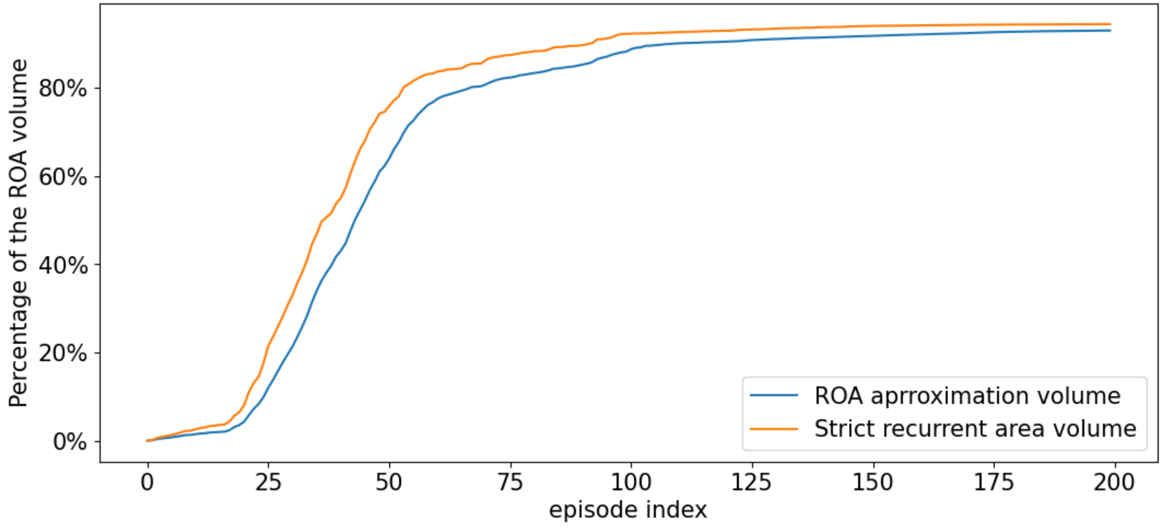


Using multiple runs to increase volume

- At Each Episode:
  - **Sample 50** center points (uniformly)
  - **Stopping criteria:** $\rho = 10^{-5}$

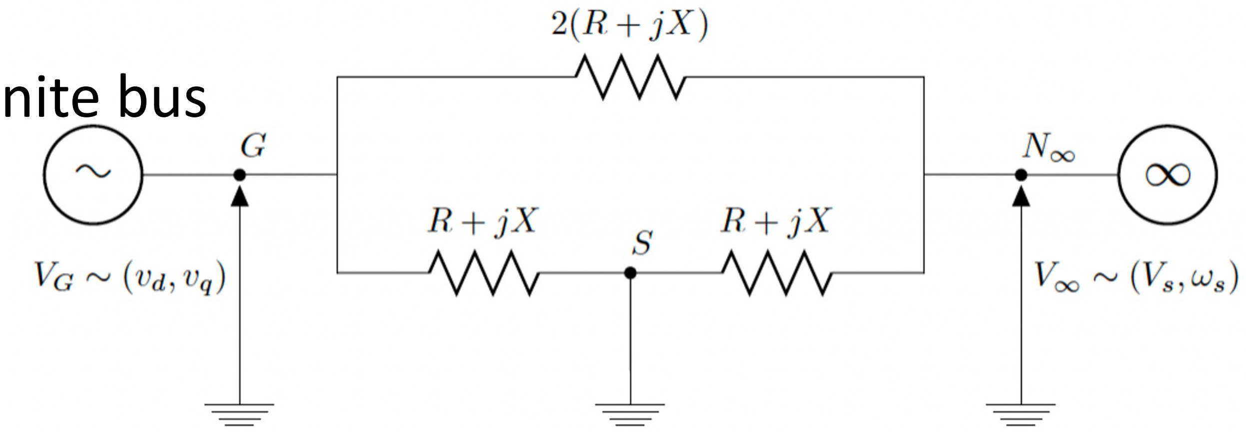






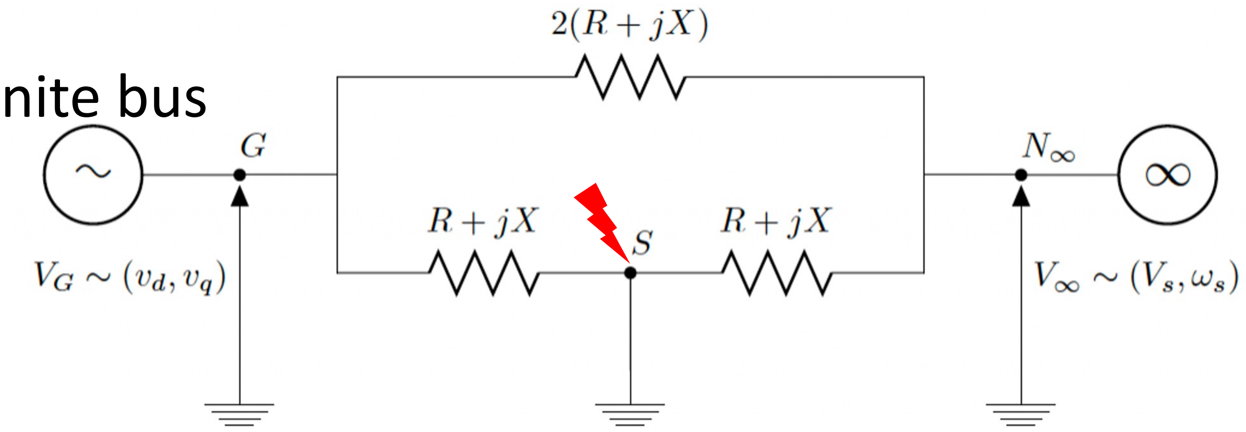
Transient Stability Analysis

- Synchronous machine connected to infinite bus



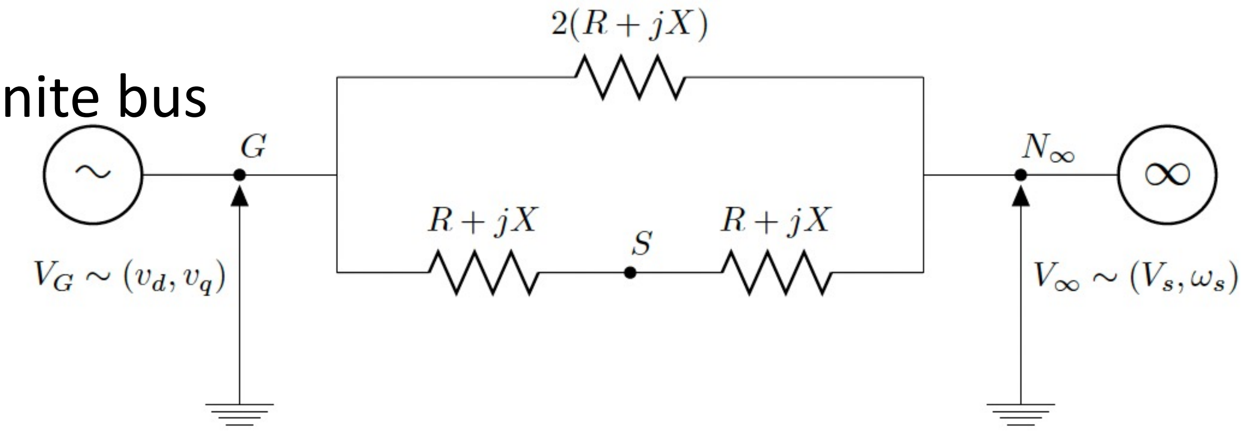
Transient Stability Analysis

- Synchronous machine connected to infinite bus
- t_1 lower line is short-circuited



Transient Stability Analysis

- Synchronous machine connected to infinite bus
- t_1 lower line is short-circuited
- t_2 fault is cleared



Transient Stability Analysis

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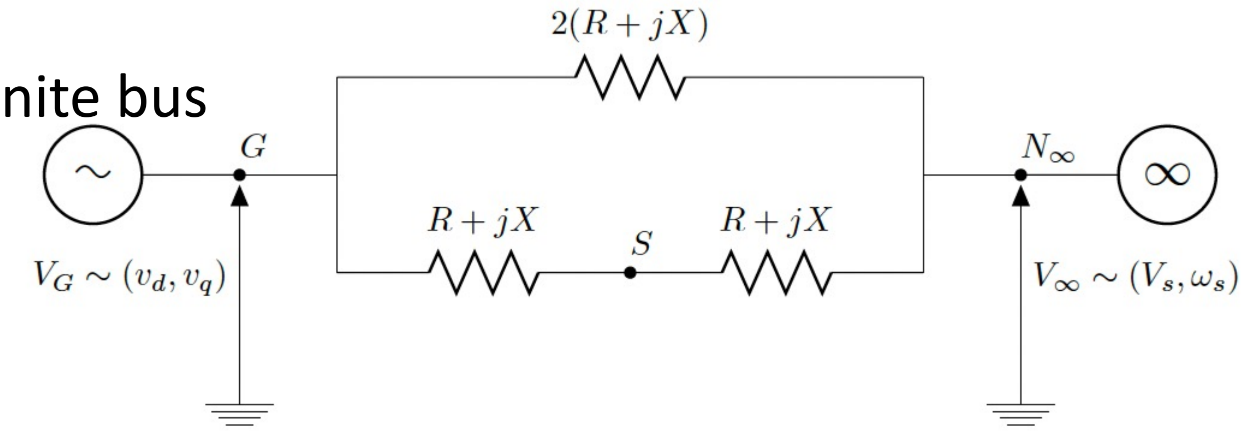


$$\frac{d\delta}{dt} = \omega - \omega_s$$

$$2H\frac{d\omega}{dt} = P_m - (v_d i_d + v_q i_q + e i_d^2 + r i_q^2)$$

$$T'_{d0}\frac{de'_q}{dt} = -e'_q - (x_d - x'_d)i_d + E_{fd}$$

$$T_a\frac{dE_{fd}}{dt} = -E_{fd} + K_a(V_{ref} - V_t)$$

$$T_g\frac{dP_m}{dt} = -P_m + P_{ref} + K_g(\omega_{ref} - \omega)$$

$$i_q = \frac{(X - x'_d)V_s\sin(\delta) - (R+r)(V_s\cos(\delta) - e'_q)}{(R+r)^2 + (X + x'_d)(X + x_q)}$$

$$i_d = \frac{X - x_q}{R+r}i_q - \frac{1}{R+r}V_s\sin(\delta)$$

$$v_d = x_q i_q - r - i_d$$

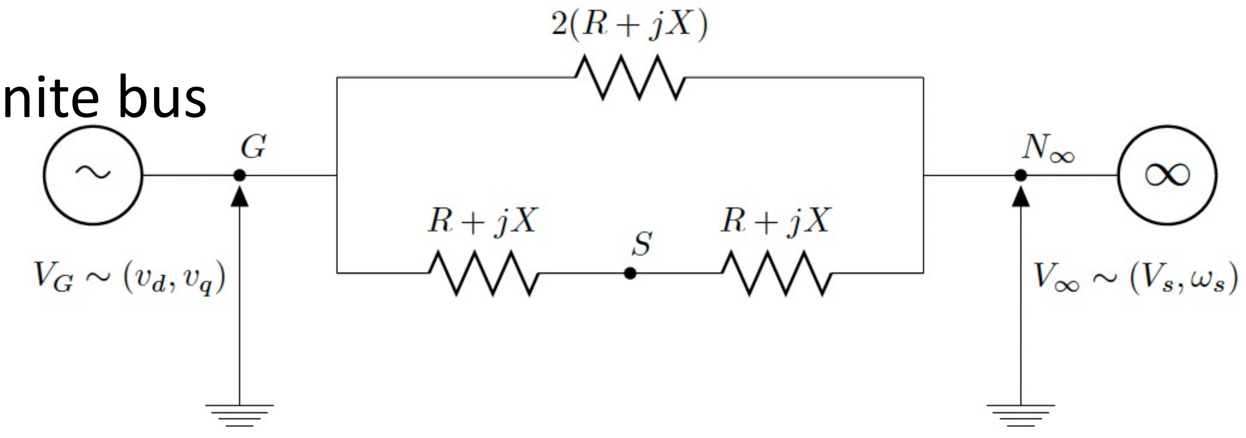
$$v_q = R i_q + X i_d + V_s\cos(\delta)$$

$$V_t = \sqrt{v_d^2 + v_q^2}$$

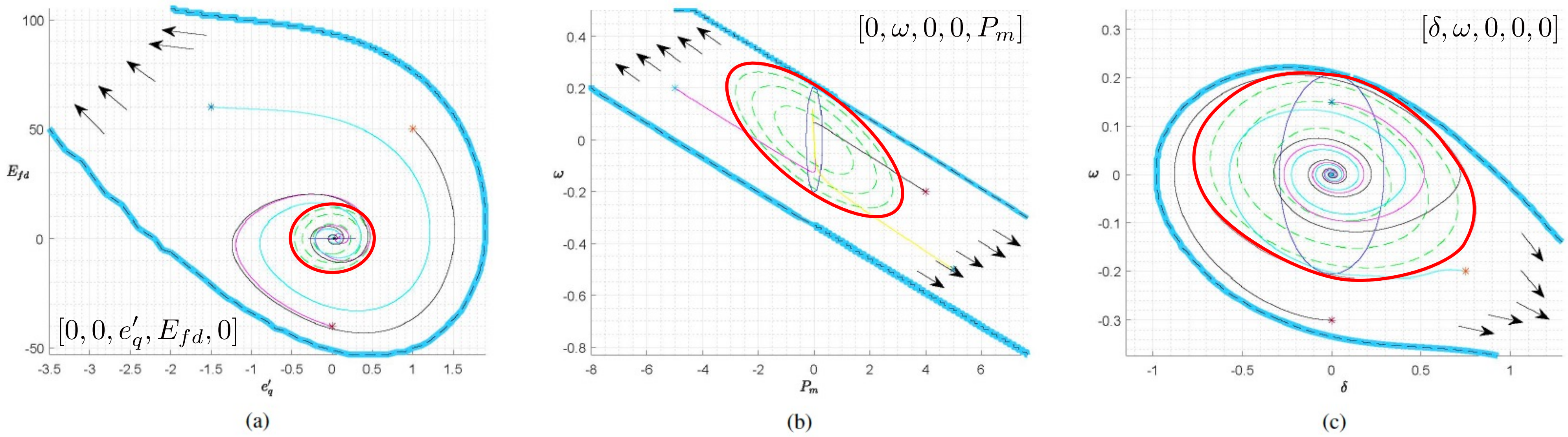
$T'_{d_0} = 9.67$	$x_d = 2.38$	$x'_d = 0.336$	$x_q = 1.21$
$H = 3$	$r = 0.002$	$\omega_s = \omega_{ref} = 1$	$R = 0.01$
$X = 1.185$	$V_s = 1$	$T_a = 1$	$K_a = 70$
$V_{ref} = 1$	$T_g = 0.4$	$K_g = 0.5$	$P_{ref} = 0.7$

Transient Stability Analysis

- Synchronous machine connected to infinite bus
- t_1 lower line is short-circuited
- t_2 fault is cleared



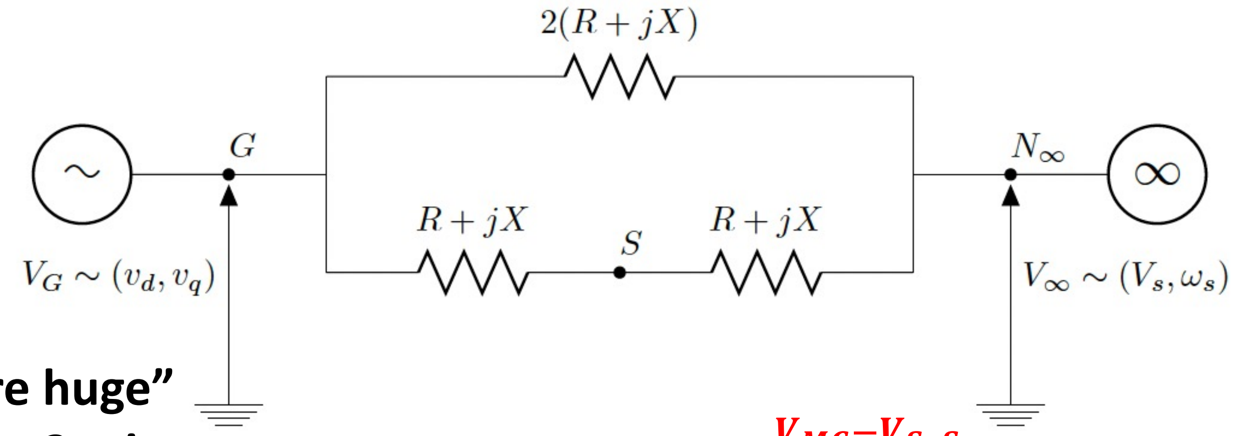
SoS approx. in red (2d-sections)



M. Tacchi et al "Power system transient stability analysis using SoS programming" Power System Computation Conference (PSCC) 2018

Transient Stability Analysis

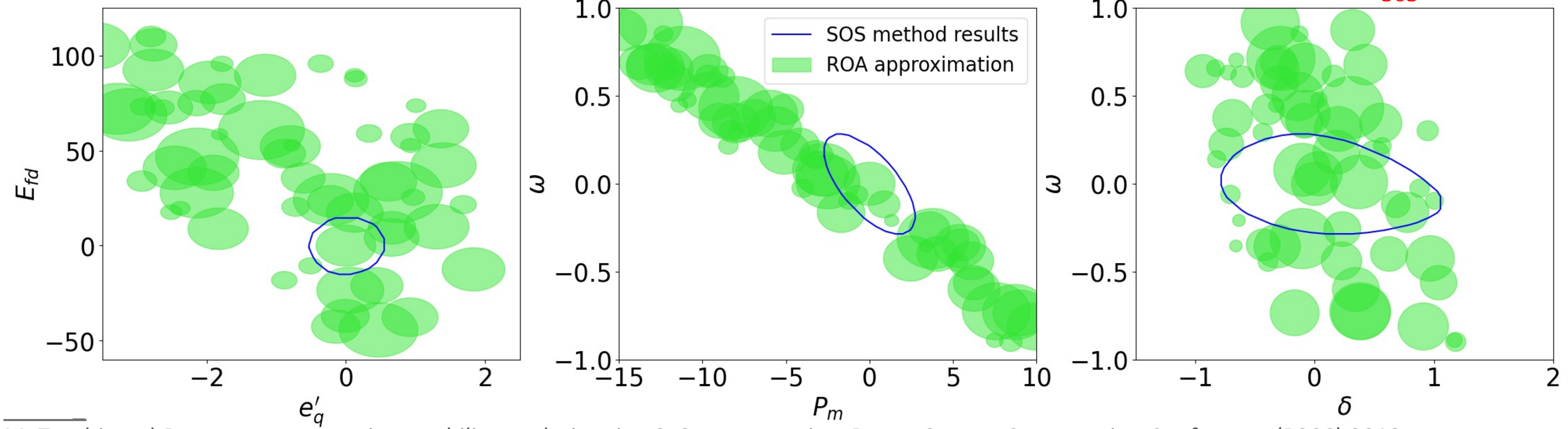
- **Algorithm parameters:**
  - **Centers:** 1000 per episode
  - **Failure prob.:** $\rho = 10^{-5}$
  - **Time constant:** $\tau = 100$ s



SoS in **blue**: [Tacchi 18] **vol = 0.05%,** run time **"they are huge"**

Multi-center in **green**: **vol = 0.23%, 1** episode, run time **3 min**

Percent vol. gain: $\frac{V_{MC} - V_{SoS}}{V_{SoS}} = 360\%$



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Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* Control and Decision Conference (**CDC) 2022**

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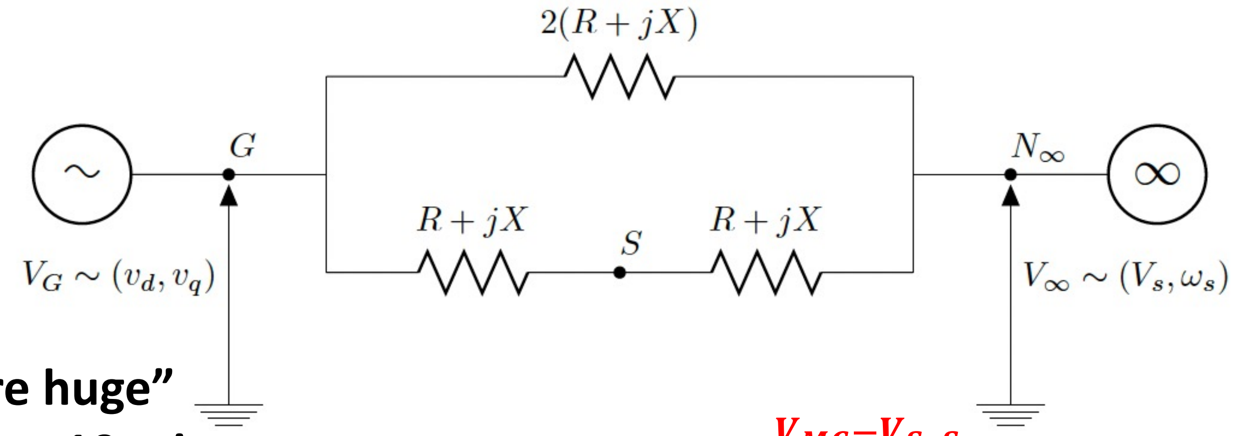
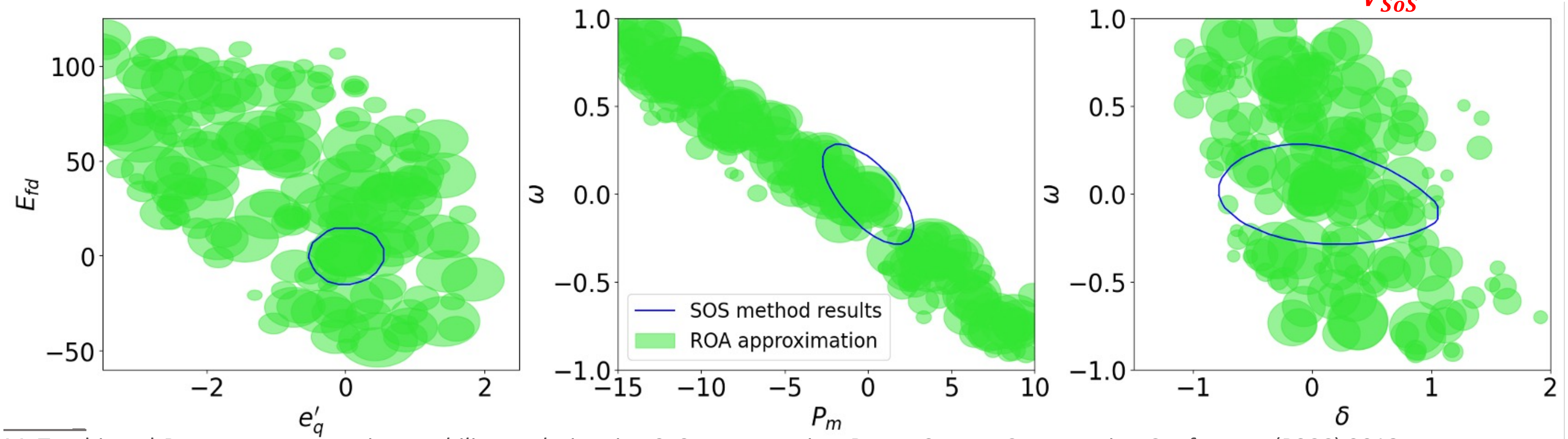


SoS in **blue**: [Tacchi 18] **vol = 0.05%,** run time **"they are huge"**
Multi-center in **green**: **vol = 0.45%, 3** episodes, run time **10 min**

Percent vol. gain: $\frac{V_{MC} - V_{SoS}}{V_{SoS}} = 800\%$



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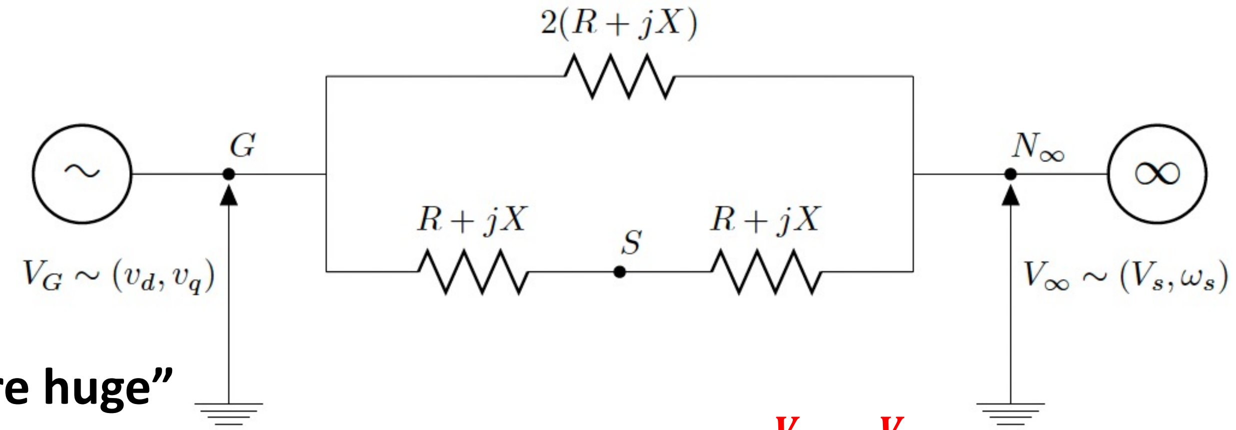
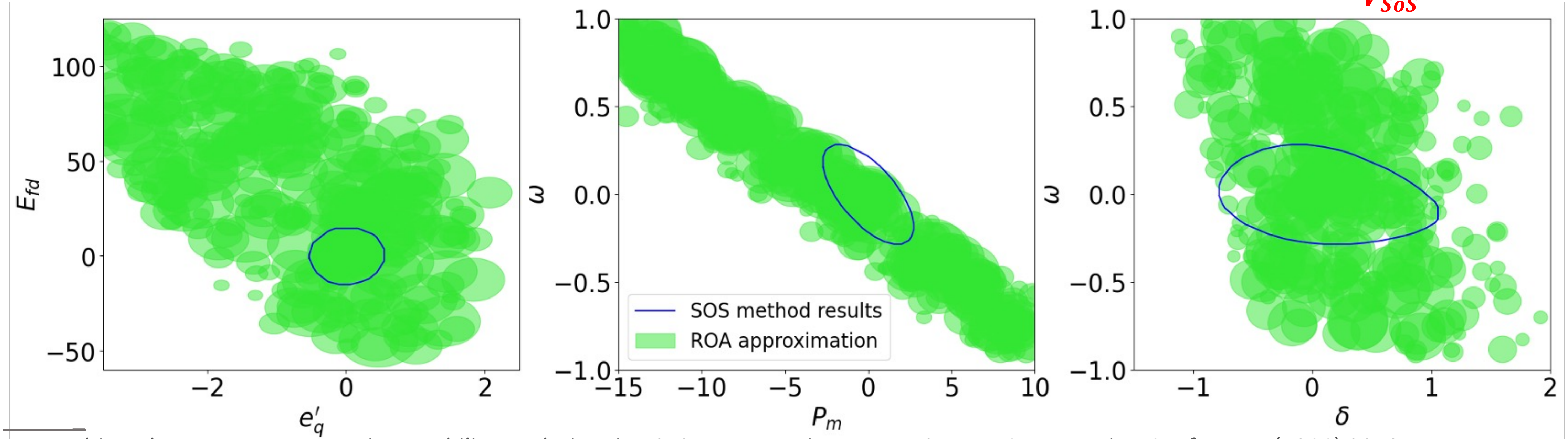


SoS in **blue**: [Tacchi 18] **vol = 0.05%,** run time **"they are huge"**
Multi-center in **green**: **vol = 0.74%, 5** episode, run time **17.5 min**

Percent vol. gain: $\frac{V_{MC} - V_{SoS}}{V_{SoS}} = $ **1380%**



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Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* Control and Decision Conference (**CDC) 2022**

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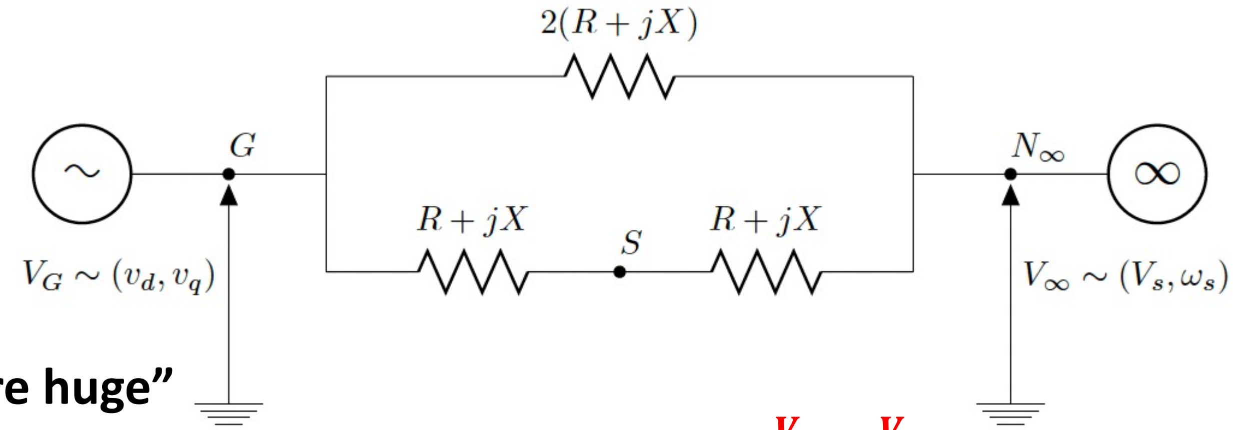
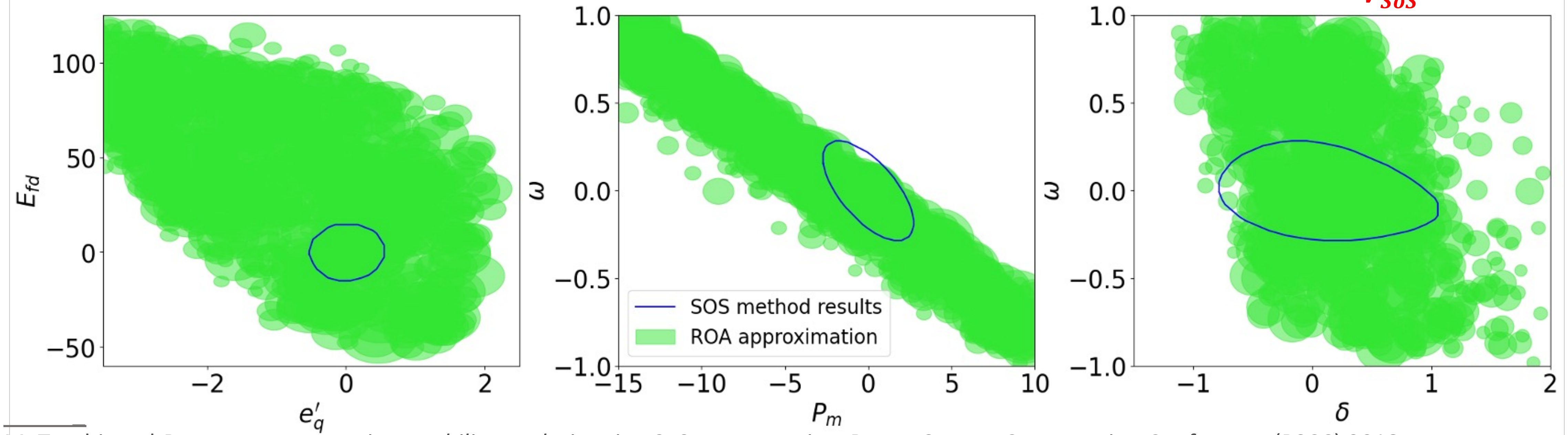


SoS in **blue**: [Tacchi 18] **vol = 0.05%,** run time **"they are huge"**

Multi-center in **green**: **vol = 1.56%, 10** episodes, run time **39.5 min**

Percent vol. gain: $\frac{V_{MC} - V_{SoS}}{V_{SoS}} = \mathbf{3020\%}$



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Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* Control and Decision Conference (**CDC) 2022**

Outline

- Invariance: Merits and trade-offs
- Letting things go, and come back: *Recurrent sets*
- Approximating regions of attractions via recurrent sets
- **Stability analysis via non-monotonic Lyapunov functions**

Roy Siegelmann

Johns Hopkins University

Yue Shen

Johns Hopkins University

Fernando Paganini

Universidad ORT Uruguay

Recurrently Non-Increasing Lyapunov Functions

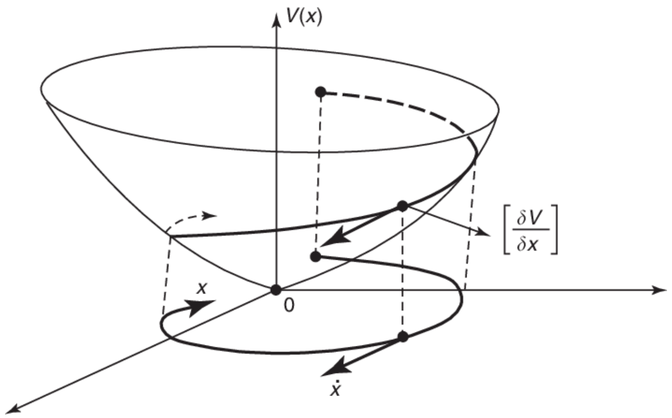
R. Siegelmann, Y. Shen, F. Paganini, and E. Mallada, “A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions”, submitted CDC 2023

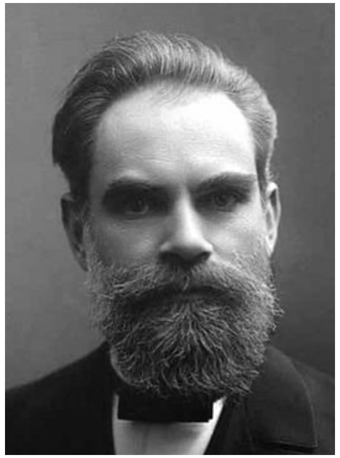
Lyapunov's Direct Method

Key idea: Make sub-level sets invariant to trap trajectories

Theorem [Lyapunov '1892]*. Given $V: \mathbb{R}^d \to \mathbb{R}_{\geq 0}$, with $V(x) > 0, \forall x \in \mathbb{R}^d \setminus \{x^*\}$, then:*

- $\dot{V} \leq 0 \to x^*$ *stable*
- $\dot{V} < 0 \to x^*$ *as. stable*





Challenge: Couples shape of V and vector field f

- Towards decoupling the $V - f$ geometry
  - Controlling regions where $\dot{V} \geq 0$ **[Karafyllis '09, Liu et al '20]**
  - Higher order conditions: $g(V^{(q)}, \ldots, \dot{V}, V) \leq 0$ **[Butz '69, Gunderson '71, Ahmadi '06, Meigoli '12]**
  - Discretization approach: $V\big(x(T)\big) \leq V(x(0))$ **[Coron et al '94, Aeyels et. al '98, Karafyllis '12]**
  - Multiple Lyapunov Functions: $\{V_j : j \in [k]\}$ **[Ahmadi et al '14]**

A Butz. Higher order derivatives of Lyapunov functions. IEEE Transactions on automatic control, 1969
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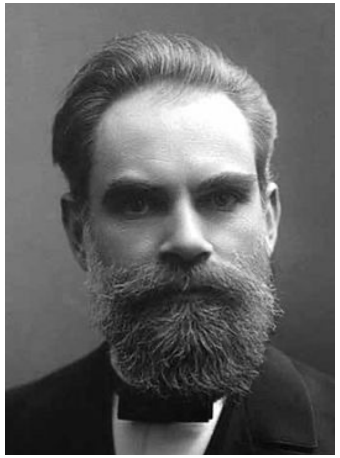
Lyapunov's Direct Method



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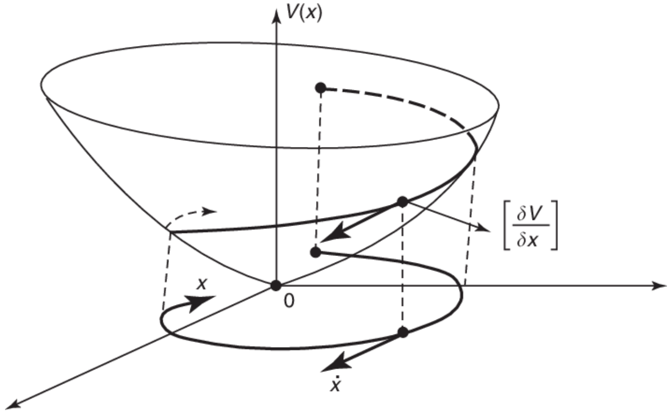

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- Towards decoupling the $V - f$ geometry
  - Controlling regions where $\dot{V} \geq 0$ **[Karafyllis '09, Liu et al '20]**
  - Higher order conditions: $g(V^{(q)}, \dots, \dot{V}, V) \leq 0$ **[Butz '69, Gunderson '71, Ahmadi '06, Meigoli '12]**
  - Discretization approach: $V\big(x(T)\big) \leq V(x(0))$ **[Coron et al '94, Aeyels et. al '98, Karafyllis '12]**
  - Multiple Lyapunov Functions: $\{V_j : j \in [k]\}$ **[Ahmadi et al '14]**

Question: Can we provide stability conditions based on recurrence?

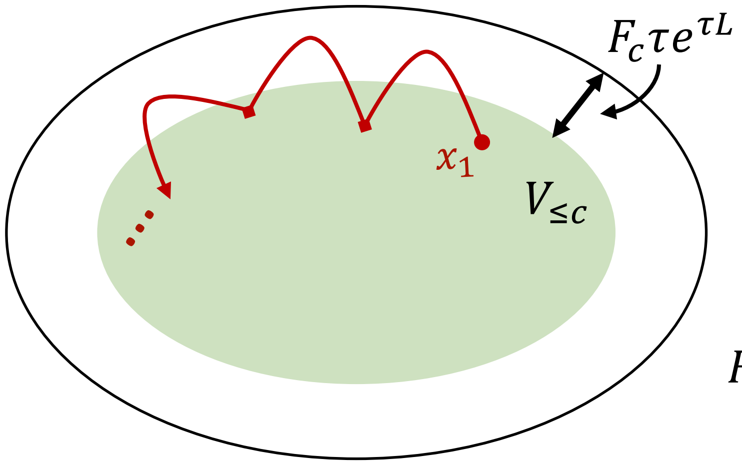
Recurrently Decreasing Lyapunov Functions

A continuous function $V: \mathbb{R}^d \to \mathbb{R}_+$ is a **recurrently non-increasing Lyapunov function** over intervals of length $\boldsymbol{\tau}$ if

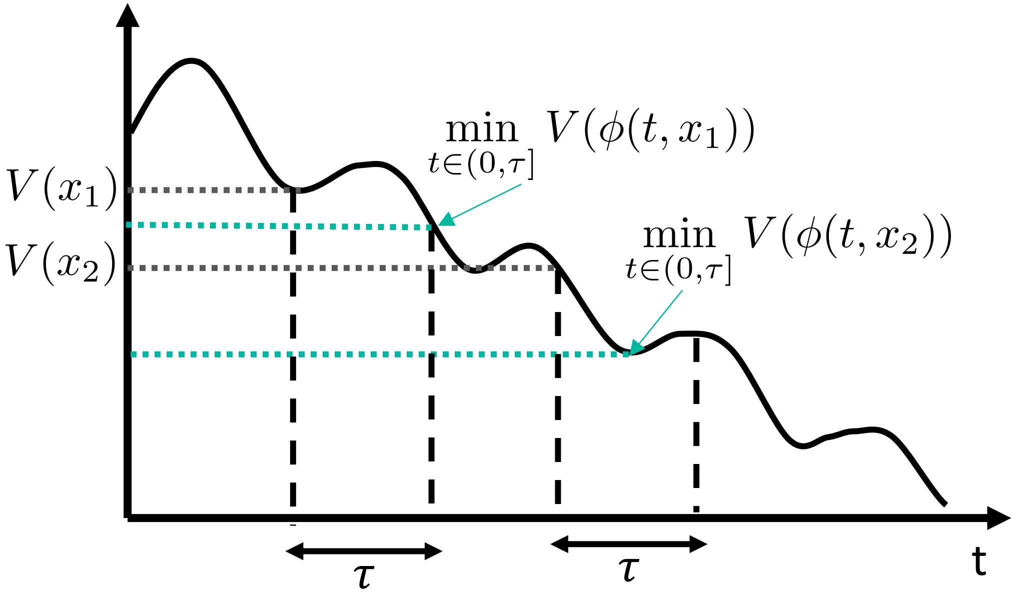
$$\mathcal{L}_f^{(0,\tau]} V(x) := \min_{t \in (0,\tau]} V(\phi(t,x)) - V(x) \leq 0 \quad \forall x \in \mathbb{R}^d$$

Preliminaries:

- Sub-level sets $\{V(x) \leq c\}$ are $\boldsymbol{\tau}$-recurrent sets.
- When f is L-Lipschitz, one can trap trajectories.



$$F_c = \max_{x \in V_{\leq c}} ||f(x)||$$



Recurrently Non-Increasing Lyapunov Functions

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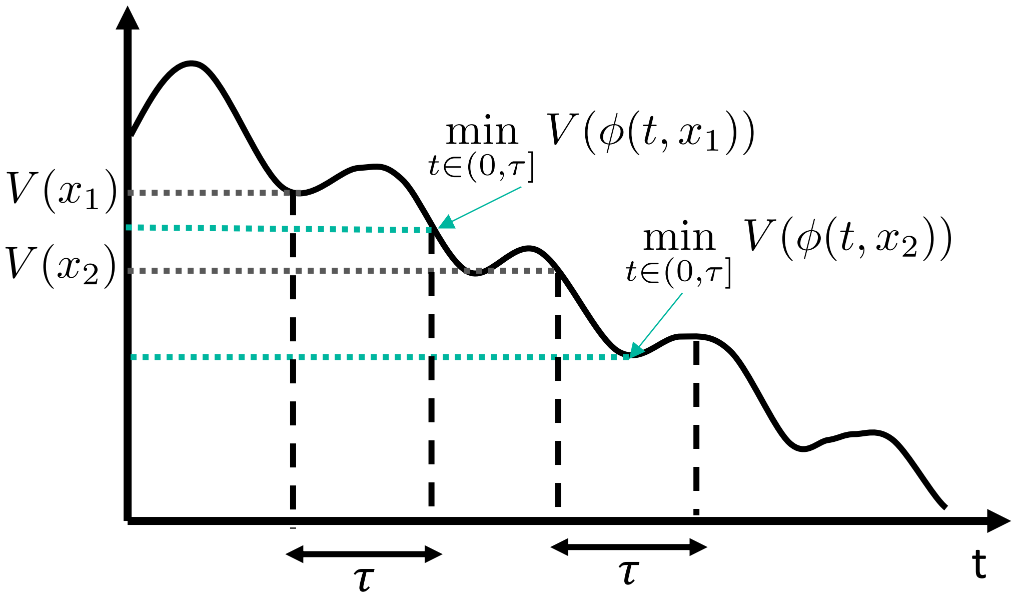
Theorem [CDC 23]: *Let $V\colon \mathbb{R}^d \to \mathbb{R}_{\ge 0}$ be a recurrently non-increasing Lyapunov function over intervals of length τ. Let f be L-Lipschitz*

- *Then the equilibrium x^* is stable.*
- *Further, if the* ***inequality is strict****, then x^* is asymptotically stable!*



Siegelmann, Shen, Paganini, M, *A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions*, **CDC 2023**

Exponential Stability Analysis

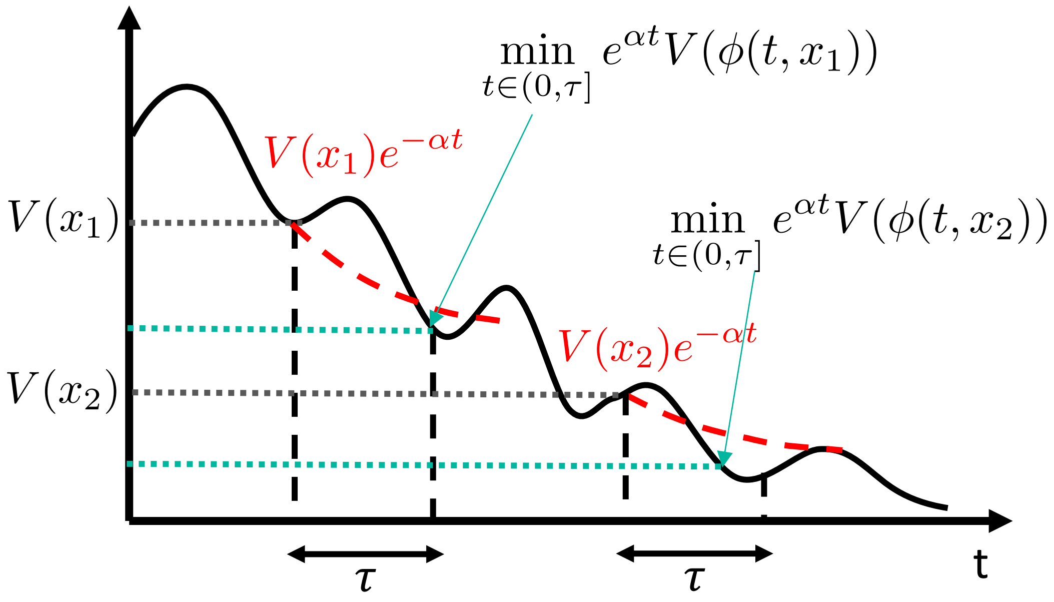
The function $V: \mathbb{R}^d \to \mathbb{R}_+$ is ***α*-exponentially recurrently non-increasing Lyapunov function** over intervals of length $\boldsymbol{\tau}$ if

$$\mathcal{L}_{f,\alpha}^{(0,\tau]} V(x) := \min_{t\in(0,\tau]} e^{\alpha t} V(\phi(t,x)) - V(x) \leq 0 \quad \forall x \in \mathbb{R}^d$$

Theorem [CDC 23]: *Let* $V: \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ *satisfy*

$$\alpha_1 ||x - x^*|| \leq V(x) \leq \alpha_2 ||x - x^*||.$$

Then, if V is α-exponentially recurrently τ-decreasing Lyapunov function, then x^* is exponentially stable with rate α.



Siegelmann, Shen, Paganini, M, *A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions*, **CDC 2023**

A Converse Theorem

Theorem: Assume x^* is λ-exponentially stable: $\exists\ K, \lambda > 0$ such that:

$$\left|\left|\phi(t,x) - x^*\right|\right| \leq Ke^{-\lambda t}\left|\left|x_0 - x^*\right|\right|, \qquad \forall x \in \mathbb{R}^d.$$

Then, $V(x) = \left|\left|x - x^*\right|\right|$ *is* α*-exponentially recurrently* τ*-decreasing , i.e.,*

$$\min_{t \in (0,\tau]} e^{\alpha t}\left|\left|\phi(t,x) - x^*\right|\right| - \left|\left|x - x^*\right|\right| \leq 0, \qquad \forall x \in \mathbb{R}^d,$$

whenever $\alpha < \lambda$ *and* $\tau \geq \frac{1}{\lambda - \alpha} \ln K.$

Remarks:

- The rate α must be strictly smaller than the rate of convergence c (giving up optimality).
- *Any norm is a Lyapunov function!*

Question: Is the struggle for its search over?

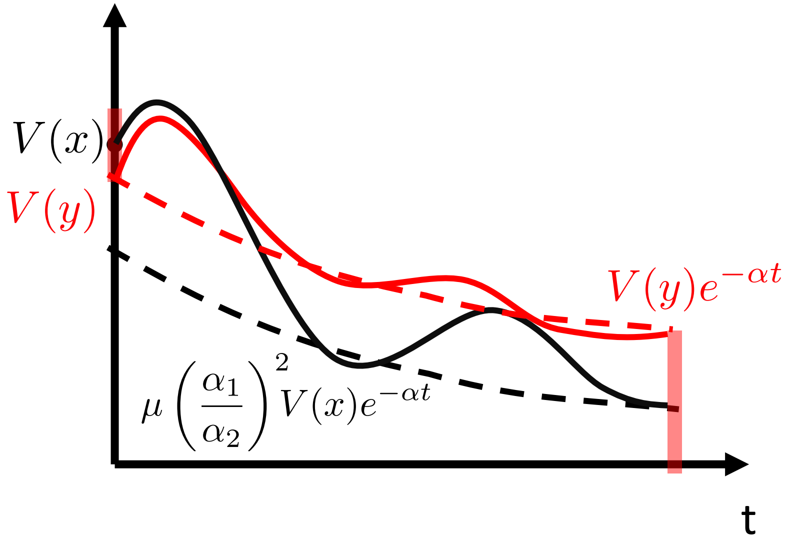
Verification of Exponential Stability

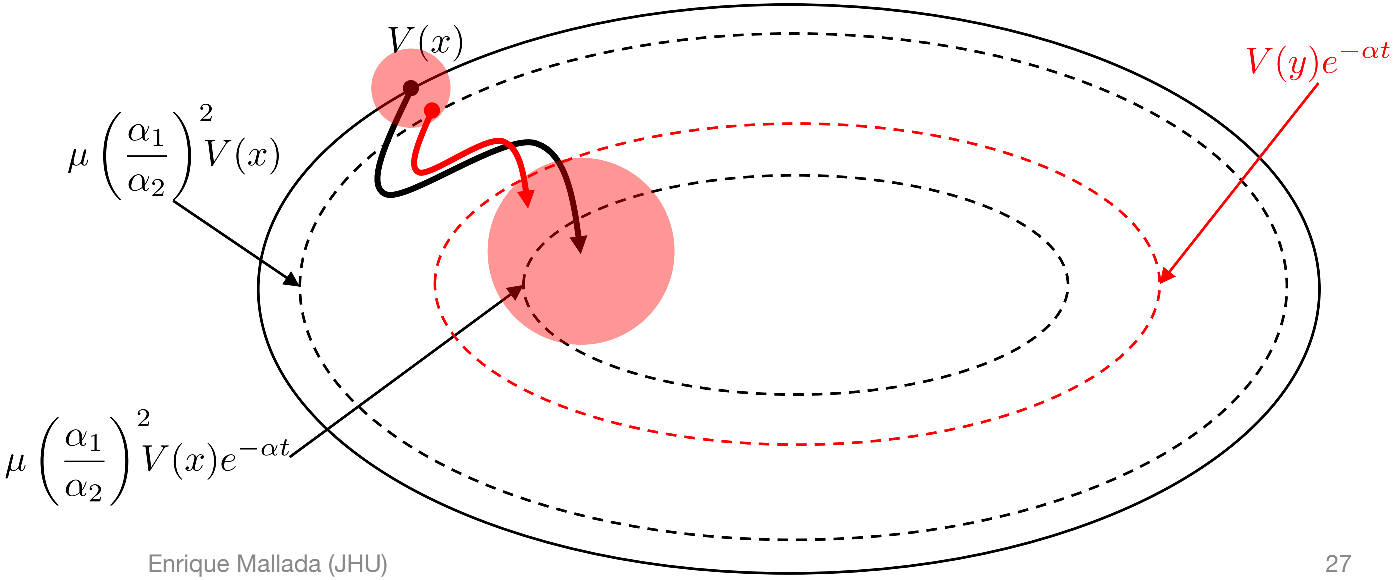
Proposition [CDC 23*]: *Let* $V: \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ *satisfy* $\alpha_1||x - x^*|| \leq V(x) \leq \alpha_2||x - x^*||$, *and* $0 < \mu < 1$. *Then, whenever*

$$\min_{t\in(0,\tau]} e^{\alpha t} V(\phi(x,t)) \leq \mu \left(\frac{\alpha_1}{\alpha_2}\right)^2 V(x)$$

for all y *with* $||y - x|| \leq r := \frac{V(x)}{\alpha_2} g(\mu)$

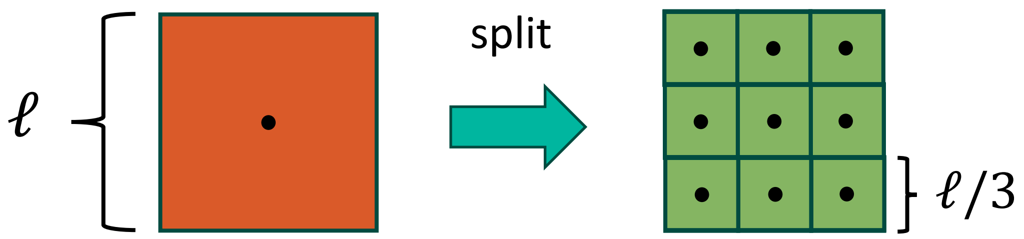
$$\min_{t\in(0,\tau]} e^{\alpha t} V(\phi(y,t)) \leq V(y)$$

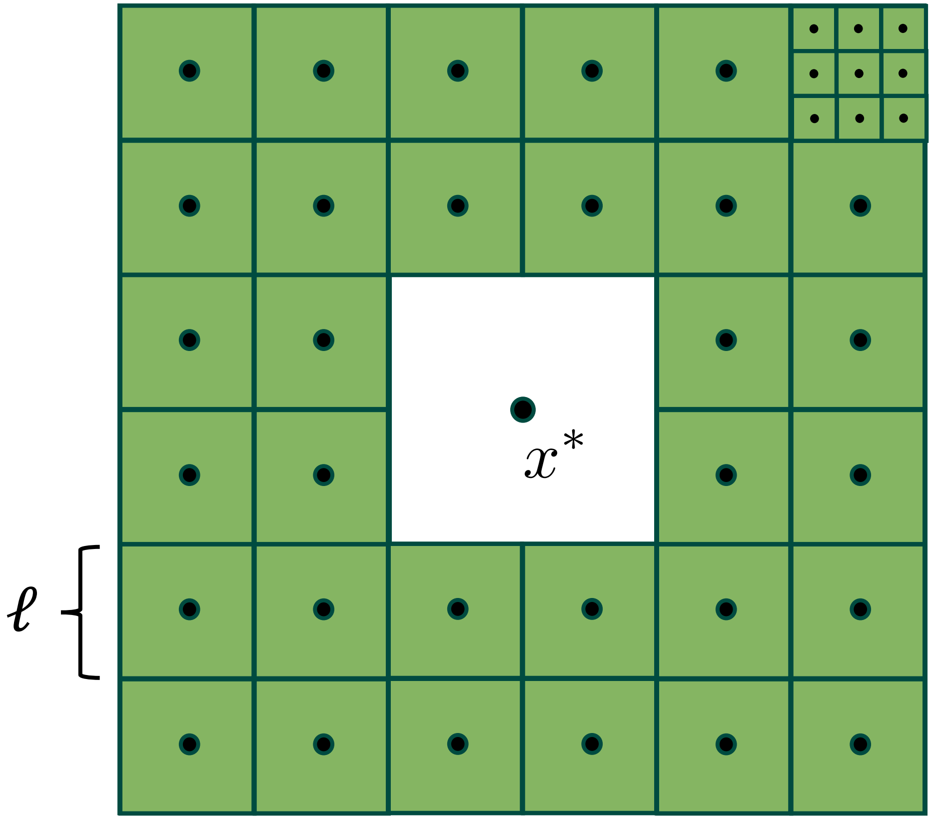




GPU-based Algorithm

- **Basic Algorithm:**
  - Consider $V(x) = ||x - x^*||_\infty$
  - Build a grid of hypercubes surrounding x^*
  - Test the center point and find α s.t. the verified radius is $r \geq \ell/2$
  - Hypercube **not verified**, **split in $\mathbf{3^d}$** parts
  - Repeat testing of new points





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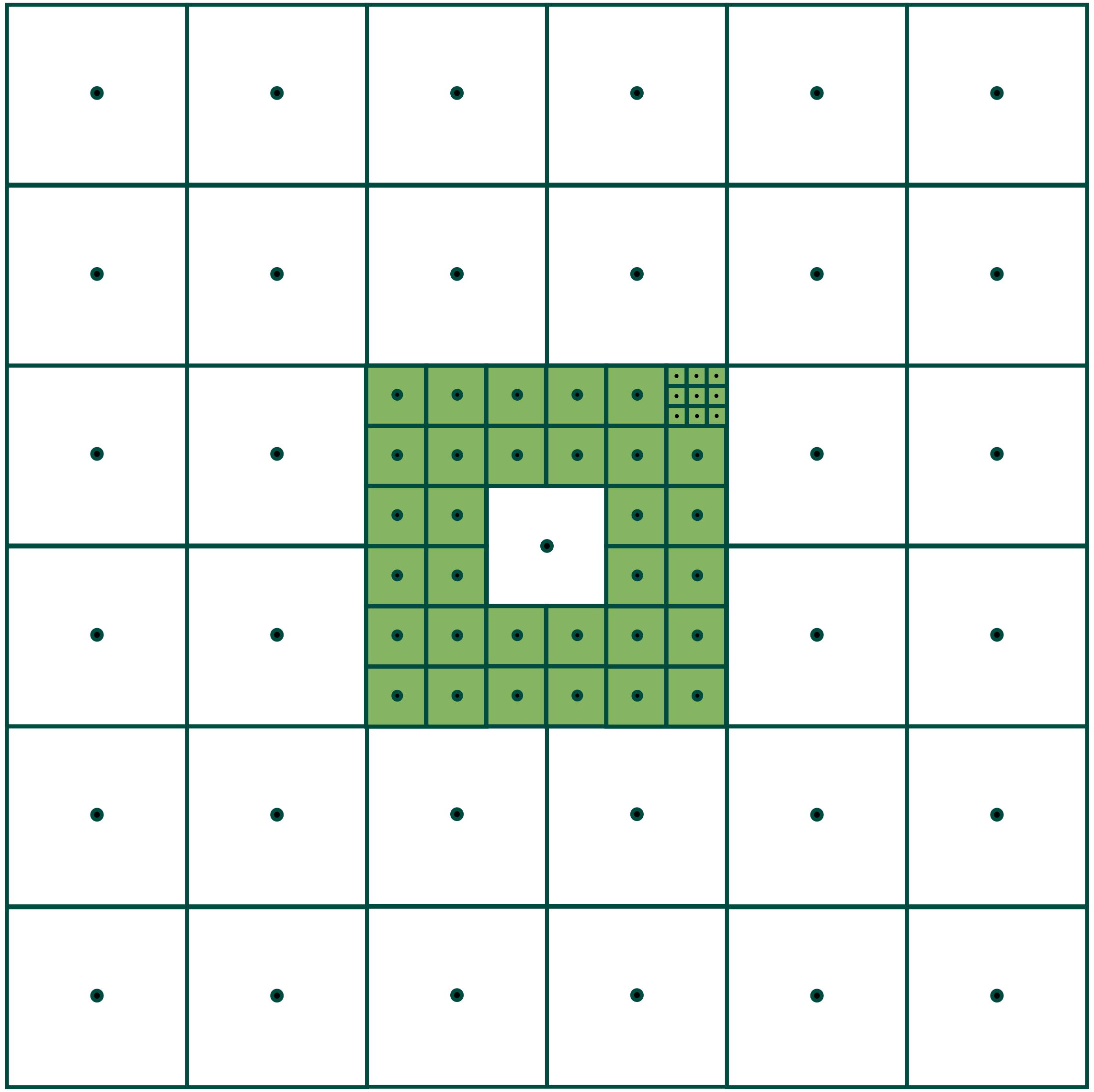
  - Repeat testing of new points
  - Exponentially expand to outer layer
  - Repeat testing in new layer



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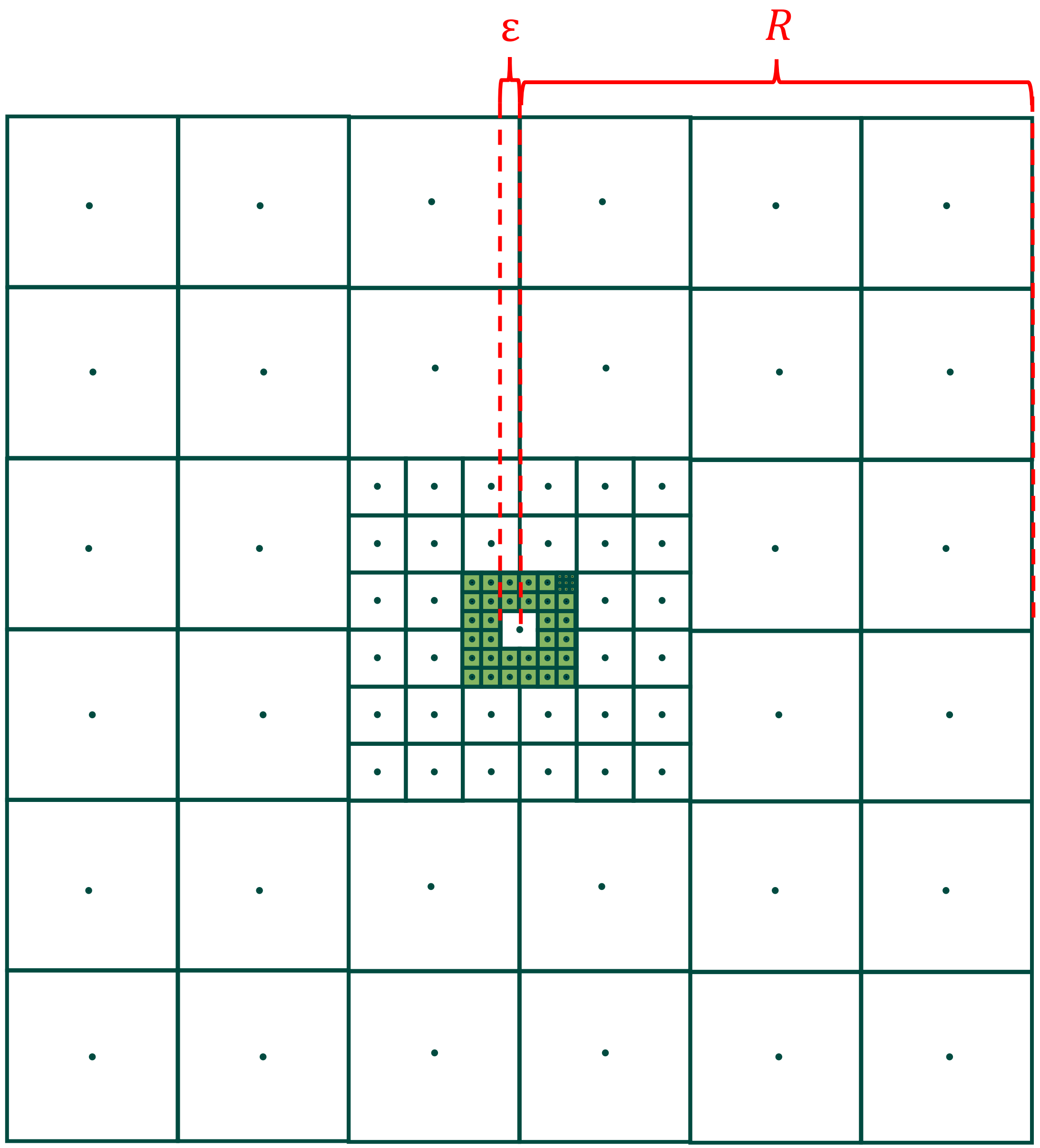


Q: How many samples are needed?

If x^* is λ-exp. stable

$$\mathcal{O}\left(q^{-d}\log\left(\frac{R}{\varepsilon}\right)\right)$$

with $q = \frac{1-Ke^{(\alpha-\lambda)\tau}}{1+e^{(L+\alpha)\tau}}$.

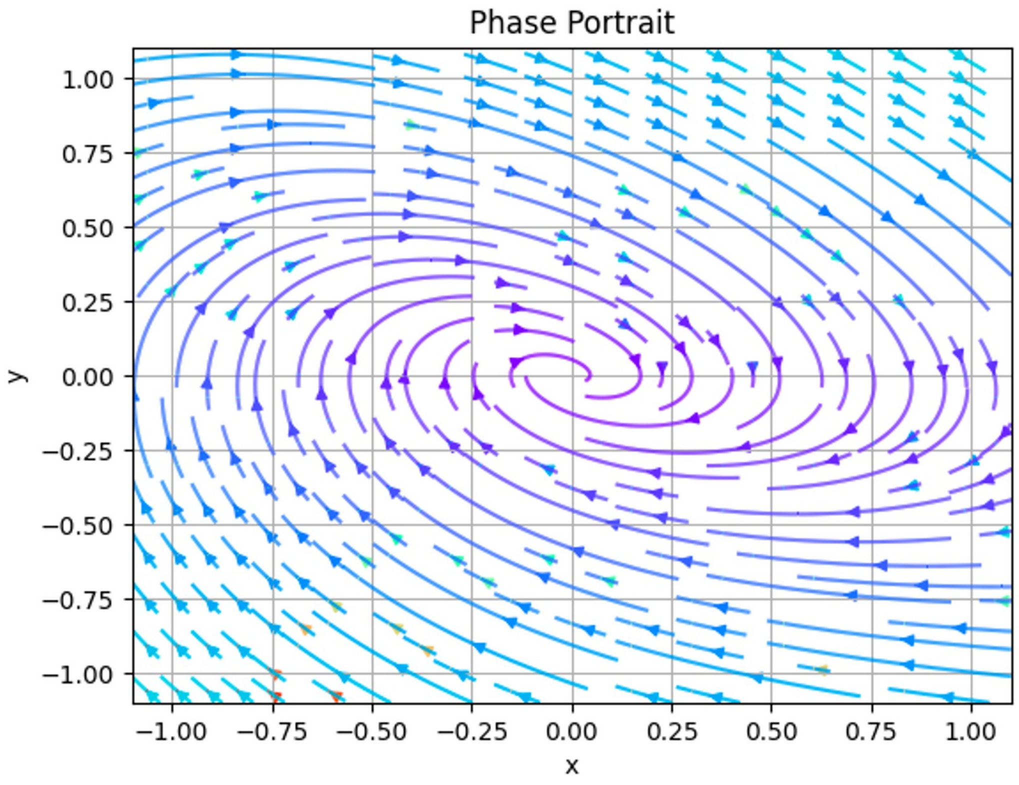
Numerical Illustration

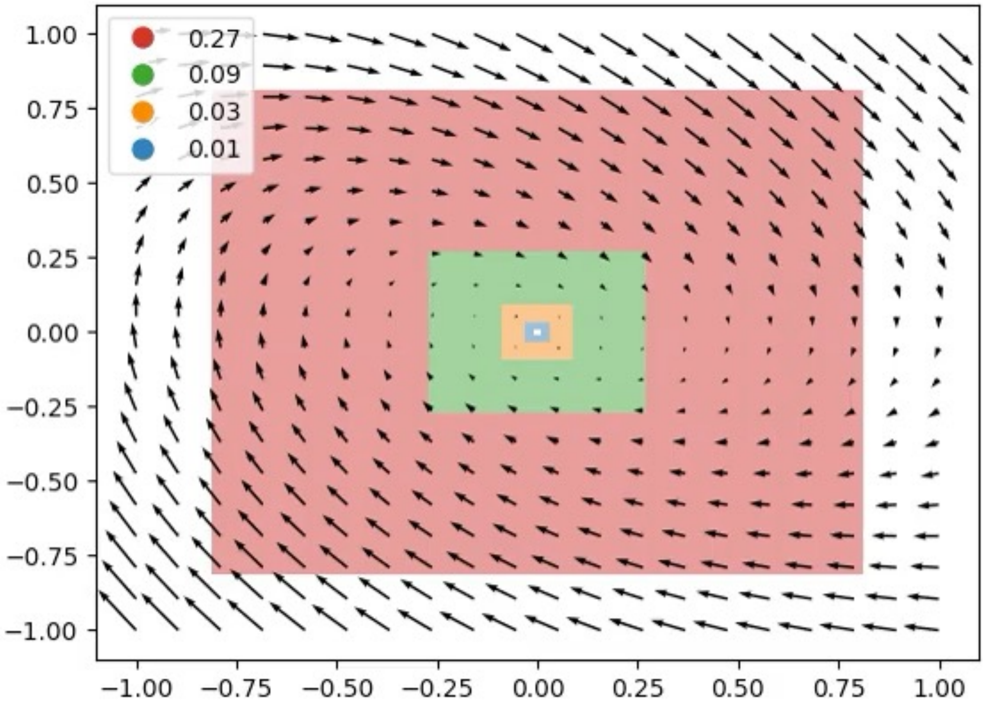
Consider the 2-d non-linear system: $\dot{x} = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} x + B \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$

with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

Parameter	Value
L	1.8
τ	1.5
ℓ	0.01

$\sigma = 0.2$





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L	1.8
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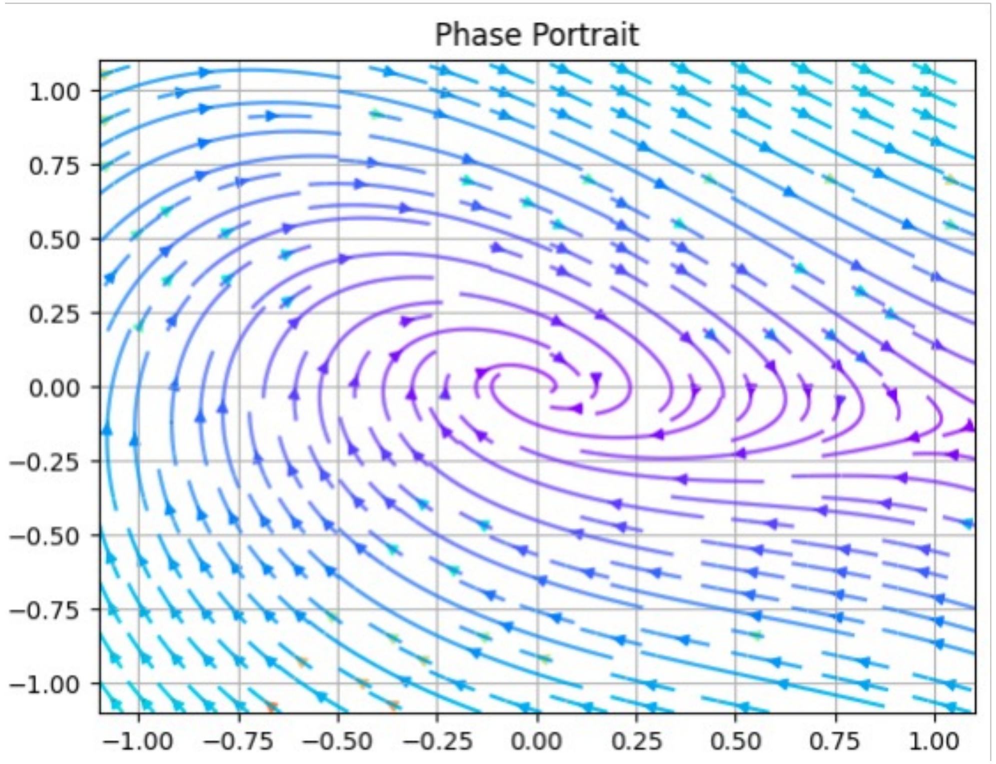

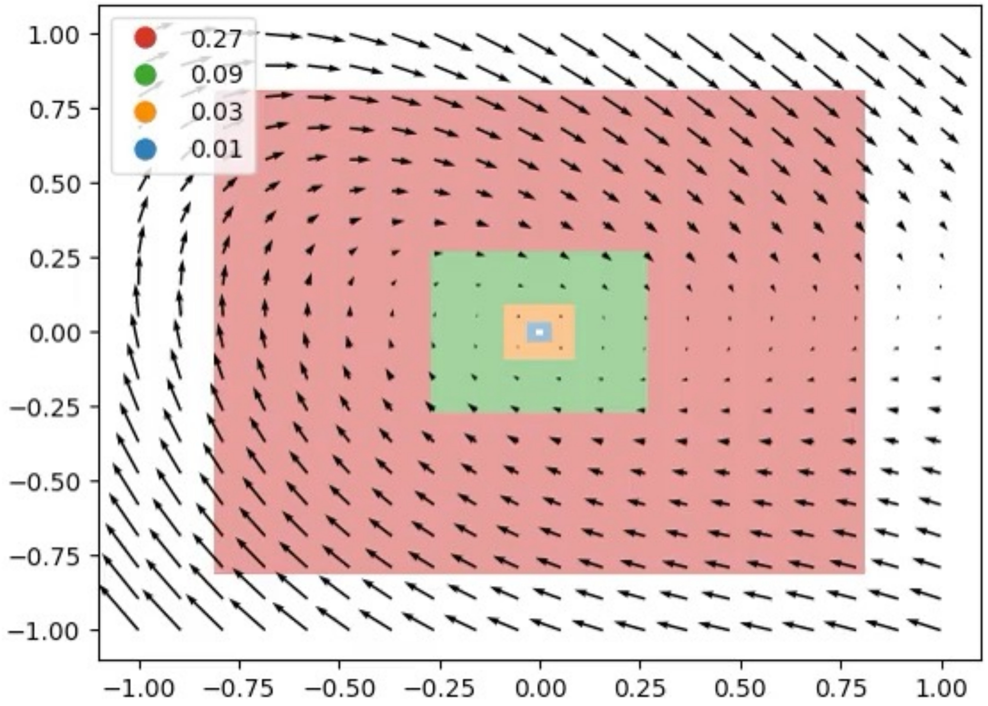

Comparison with SoS

Consider the 2-d non-linear system: $\dot{x} = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} x + B \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$

with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

Parameter	Value
L	1.8
τ	1.5
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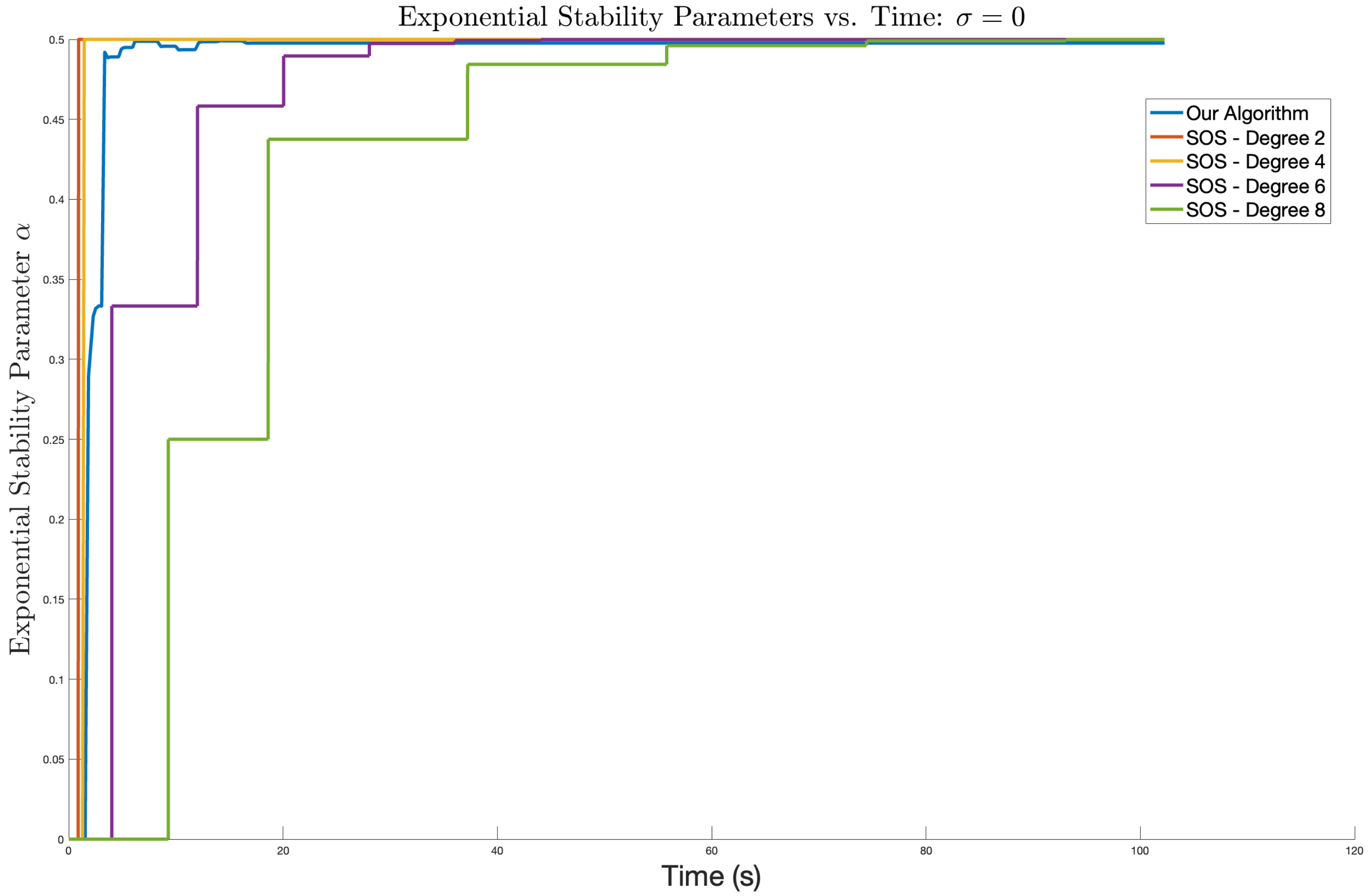

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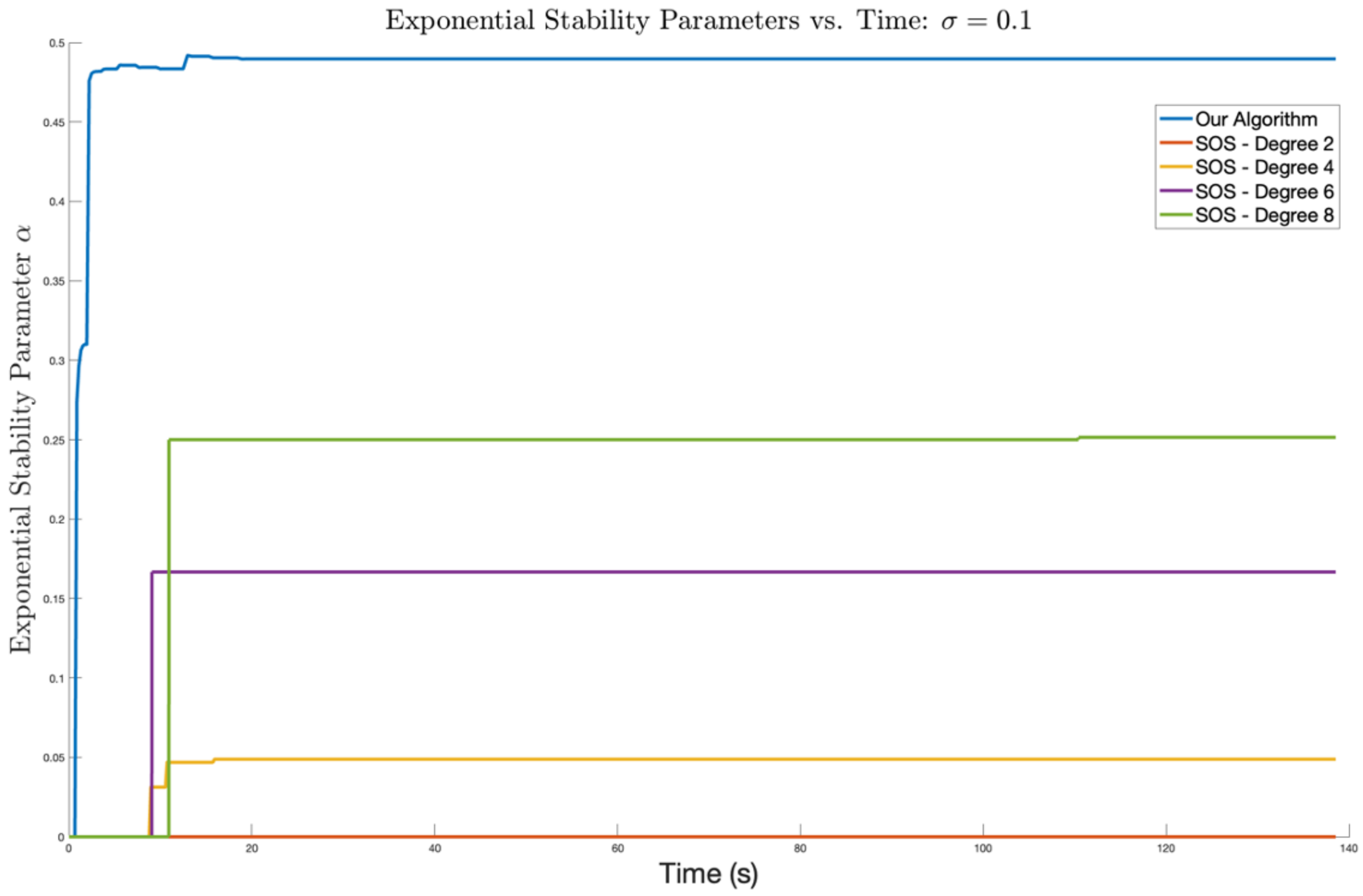
with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

Parameter	Value
L	1.8
τ	1.5
ℓ	0.01

$\sigma = 0.1$



Comparison with SoS

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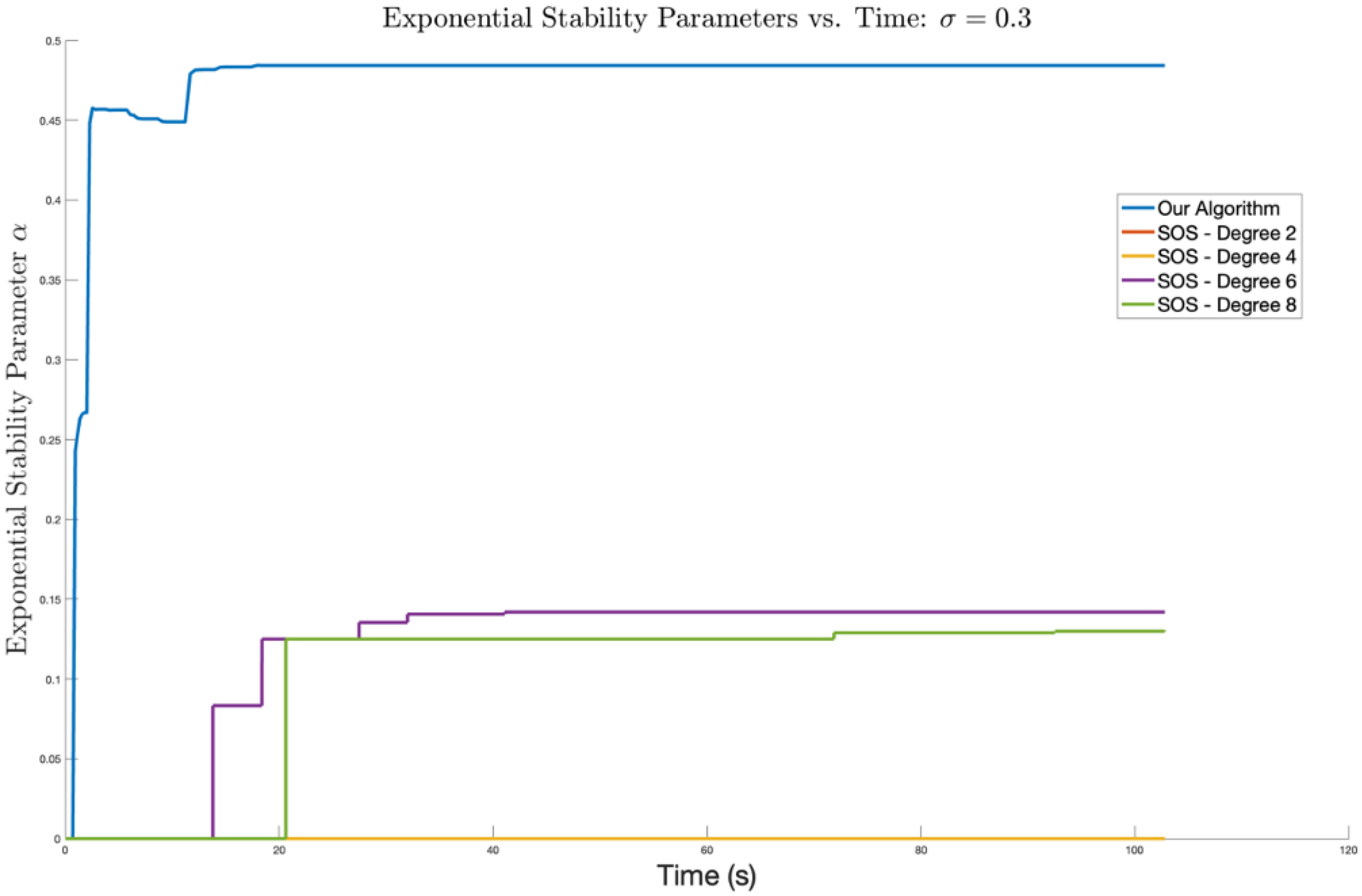
with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

Parameter	Value
L	1.8
τ	1.5
ℓ	0.01

$\sigma = 0.3$



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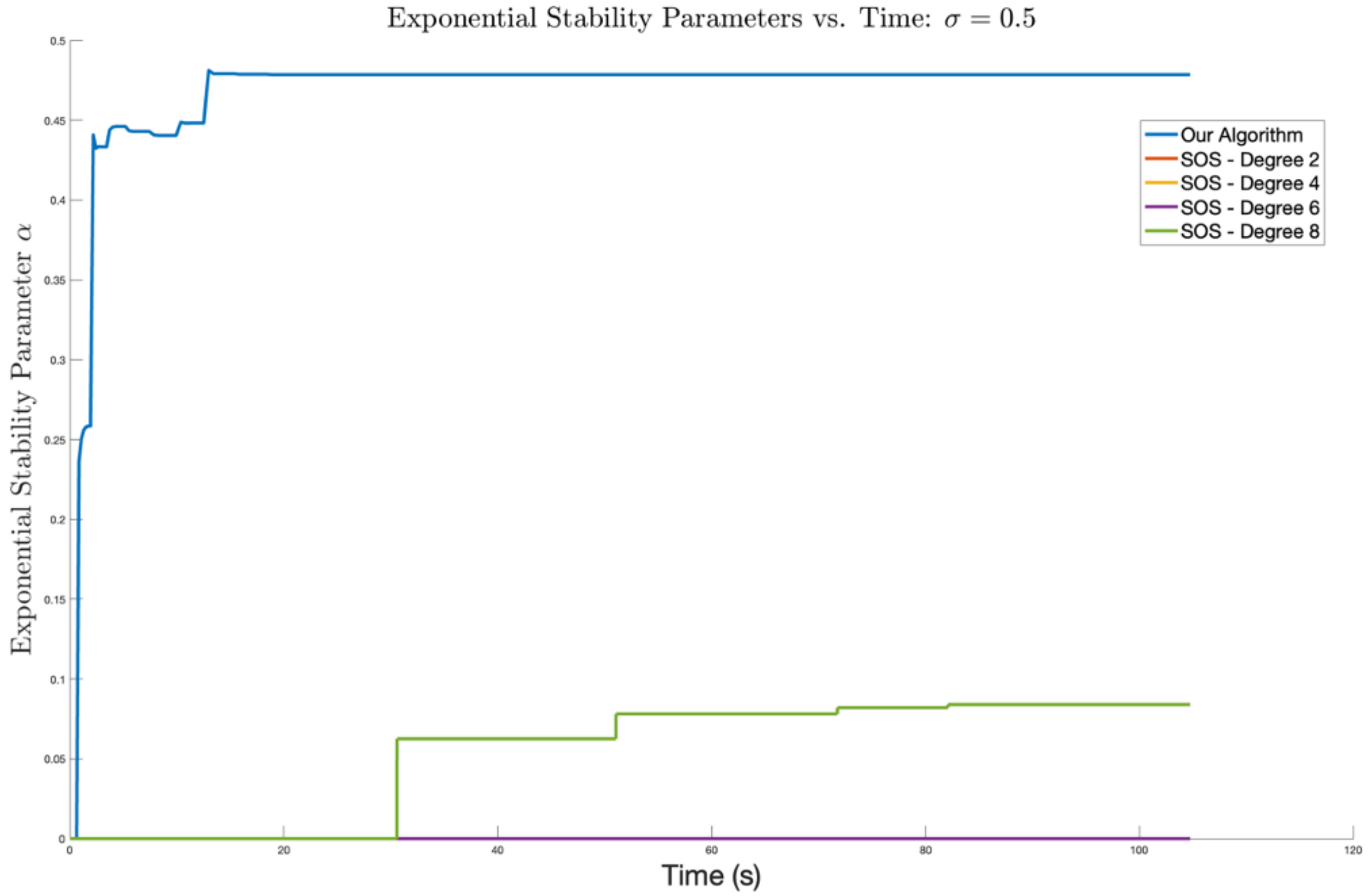
with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

Parameter	Value
L	1.8
τ	1.5
ℓ	0.01

$\sigma = 0.5$



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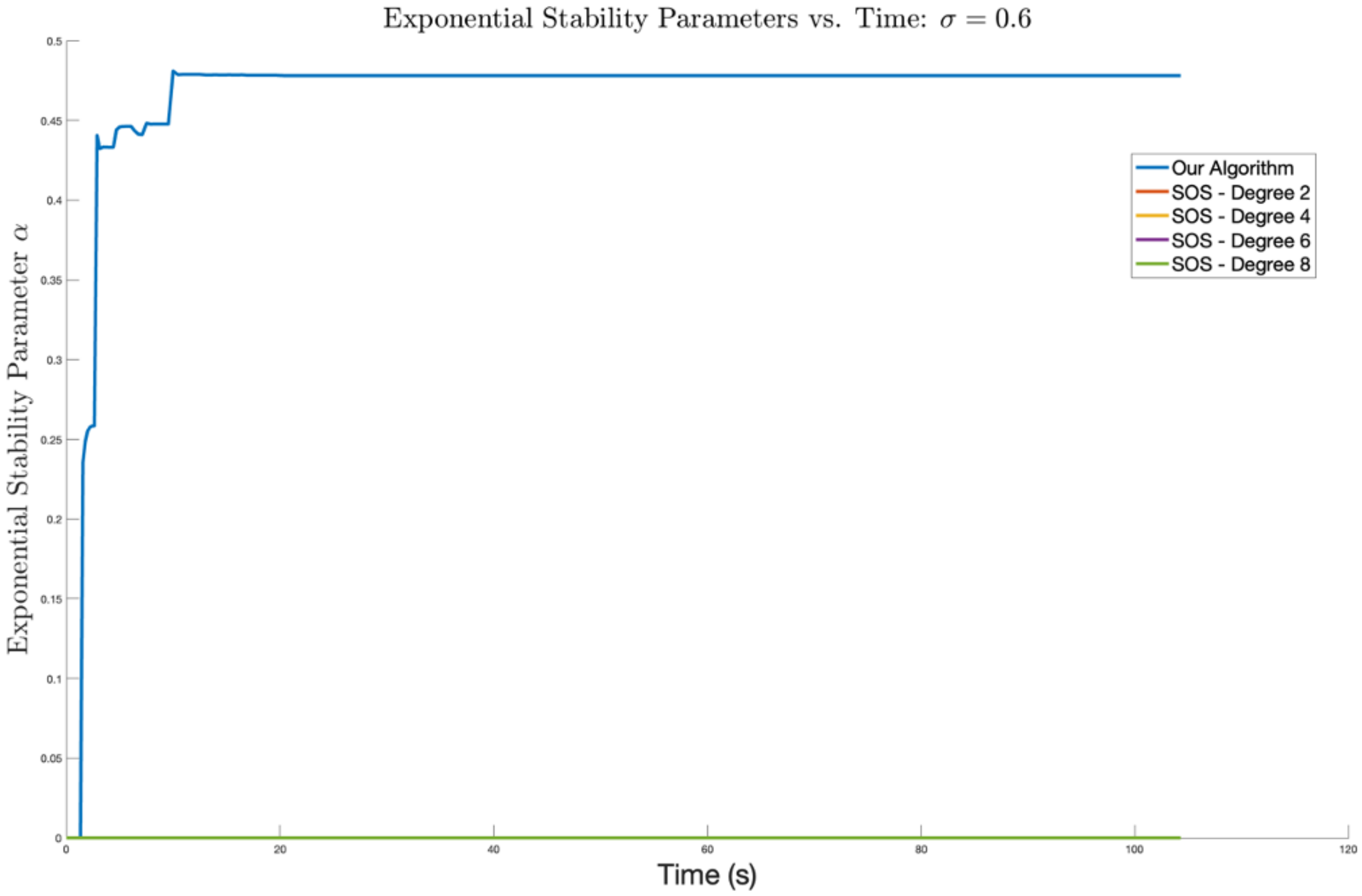
with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

Parameter	Value
L	1.8
τ	1.5
ℓ	0.01

$\sigma = 0.6$



Conclusions and Future work

- **Takeaways**
  - Proposed a **relaxed notion of invariance** known as **recurrence.**
  - Provide **necessary and sufficient conditions** for a recurrent set to be an **inner approximation** of the ROA.
  - Generalized Lyapunov Theory **for recurrently decreasing functions** using recurrent sets
  - Our algorithms are **parallelizable via GPUs and progressive/sequential.**
- **Ongoing work**
  - **Recurrent Sets:** Smart choice of multi-points, control recurrent sets, GPU implementation
  - **Lyapunov Functions:** Generalize other Lyapunov notions, Control Lyapunov Functions, Barrier Functions, Control Barrier Functions, Contraction, etc.
  - **Recurrence Entropy:** Understanding the complexity of making a set recurrent when compared with invariance.

Thanks!

Related Publications:

[arXiv 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* **CDC 2022, journal preprint arXiv:2204.10372**.

[CDC 23] Siegelmann, Shen, Paganini, M, *A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions*, **CDC 2023**

[HSCC 24] Sibai, M, *Recurrence of nonlinear control systems: Entropy and bit rates,* **HSCC, 2024**

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