Model-Free Analysis of Dynamical Systems Using Recurrent Sets

Towards a GPU-based Approach to Control

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JOHNS HOPKINS UNIVERSITY

2024 INFORMS International Conference

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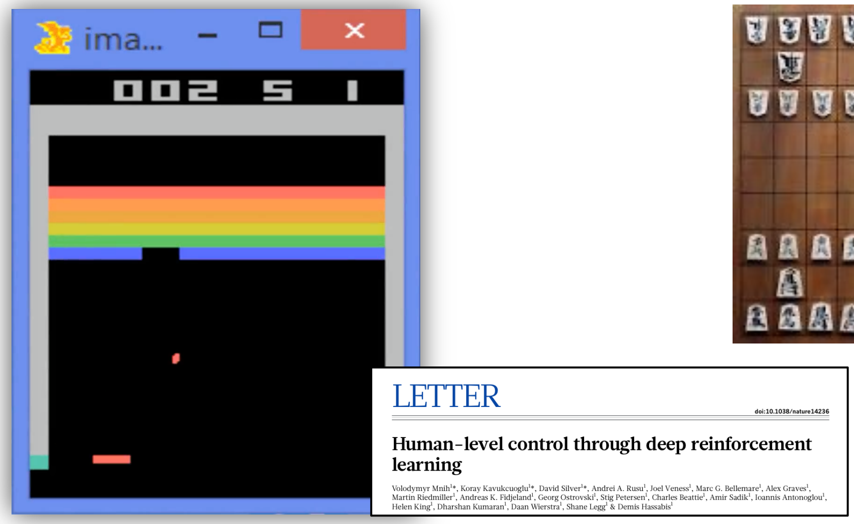
A World of Success Stories

2017 Google DeepMind's DQN

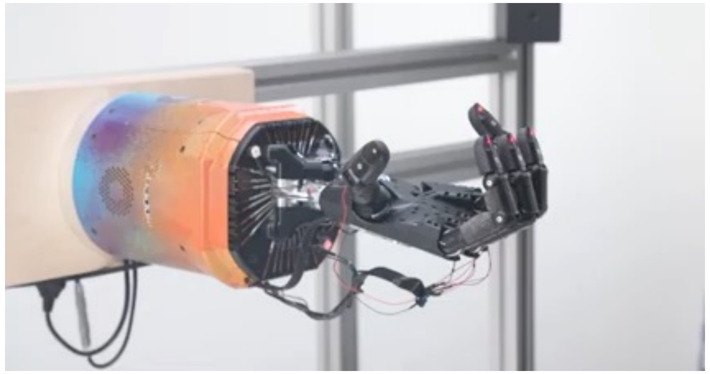


2017 AlphaZero – Chess, Shogi, Go

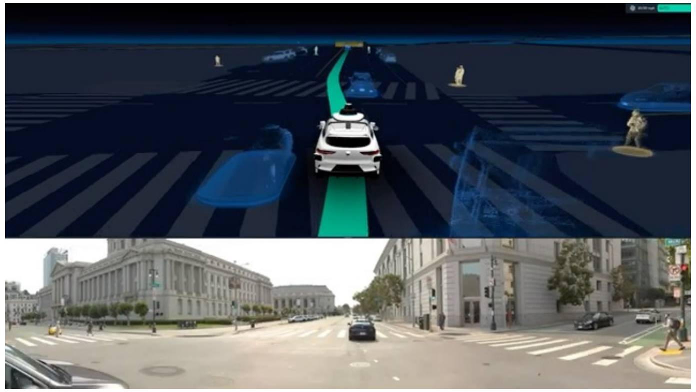
2019 AlphaStar – Starcraft II

Boston Dynamics

OpenAI – Rubik's Cube



Waymo



Reality Kicks In

Angry Residents, Abrupt Stops: Waymo Vehicles Are Still Causing Problems in Arizona

RAY STERN | MARCH 31, 2021 | 8:26AM

GARY MARCUS BUSINESS 08.14.2019 09:00 AM

DeepMind's Losses and the Future of Artificial Intelligence

Alphabet's DeepMind unit, conqueror of Go and other games, is losing lots of money. Continued deficits could imperil investments in AI.

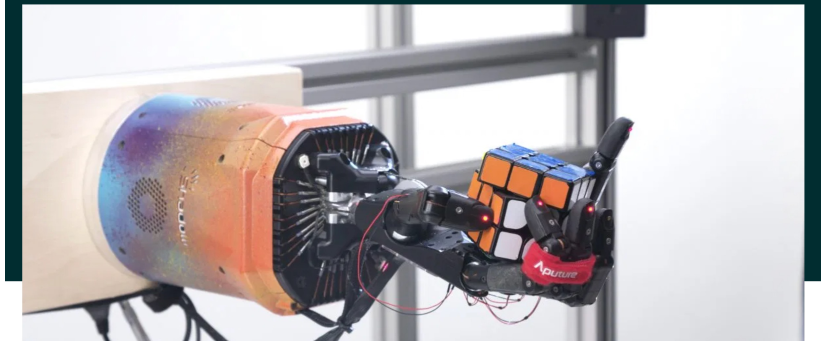
AARIAN MARSHALL BUSINESS 12.07.2020 04:06 PM

Uber Gives Up on the Self-Driving Dream

The ride-hail giant invested more than $1 billion in autonomous vehicles. Now it's selling the unit to Aurora, which makes self-driving tech.

OpenAI disbands its robotics research team

Kyle Wiggers @Kyle_L_Wiggers July 16, 2021 11:24 AM

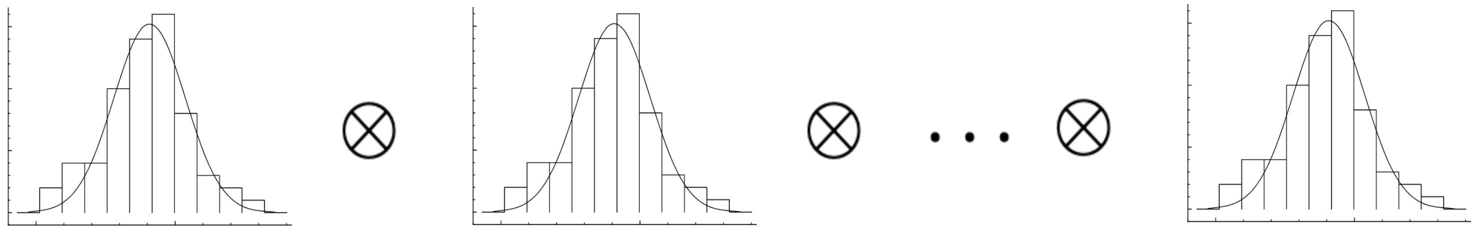


Self-driving Uber car that hit and killed woman did not recognize that pedestrians jaywalk

The automated car lacked "the capability to classify an object as a pedestrian unless that object was near a crosswalk," an NTSB report said.

Core challenge: The curse of dimensionality

- **Statistical: Sampling in d dimension with ϵ accuracy**



Sample complexity:

$$O(\varepsilon^{-d})$$

For $\epsilon = 0.1$ and $d = 100$, we would need 10^{100} points.

Atoms in the universe: 10^{78}

- **Computational: Verifying non-negativity of polynomials**

Copositive matrices:

$$[x_1^2 \dots x_d^2] A [x_1^2 \dots x_d^2]^{\mathrm{T}} \geq 0$$

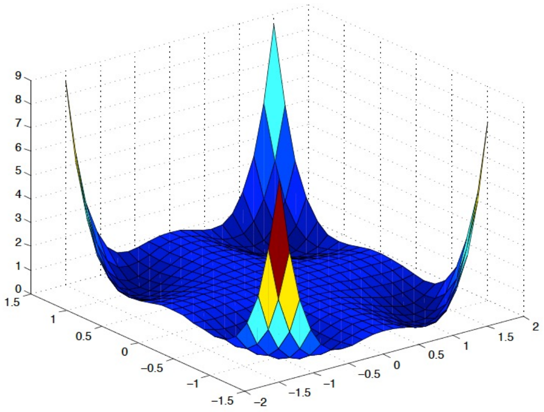
Murty&Kadabi [1987]: Testing co-positivity is NP-Hard

Sum of Squares (SoS):

$$z(x)^T Q z(x) \geq 0, \quad z_i(x) \in \mathbb{R}[x], \; x \in \mathbb{R}^d, Q \succcurlyeq 0$$

Artin [1927] (Hilbert's 17th problem):

Non-negative polynomials are sum of square of *rational* functions



Motzkin [1967]:
$p = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$
is nonnegative,
not a sum of squares,
but $(x^2 + y^2)^2 p$ is SoS

Question: Are we asking too much?

- Analysis tools build on a strict and exhaustive notion of ***invariance***

  Q: Can we substitute invariance with less restrictive notions?

  [arXiv ‘22] Shen, Bichuch, M - [CDC ’23] Siegelmann, Shen, Paganini, M

- Certificates impose conditions on the entire duration of the trajectory

  Q: Can we provide guarantees based on only localized trajectory information?

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- Control synthesis usually aims for the ***best*** (optimal) controller

  Q: Is there any gain in focusing on weaker requirements from the get-go?

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Outline

- Invariance: Merits and trade-offs
- Letting things go, and come back: Recurrent sets
- Analysis using recurrent sets
  - Approximating regions of attractions
  - Stability analysis via non-monotonic Lyapunov functions

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Problem setup

Continuous time dynamical system: $\dot{x}(t) = f(x(t))$

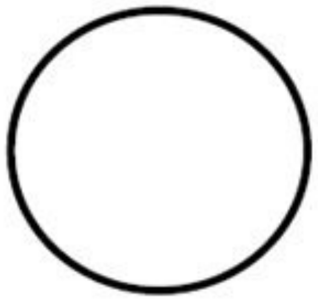
- Initial condition $x_0 = x(0)$, solution at time t: $\phi(t, x_0)$.

Ω-Limit Set $\Omega(f)$:

$$x \in \Omega(f) \iff \exists\, x_0, \{t_n\}_{n\geq 0}, \text{ s.t. } \lim_{n\to\infty} t_n = \infty \text{ and } \lim_{n\to\infty} \phi(t_n, x_0) = x$$

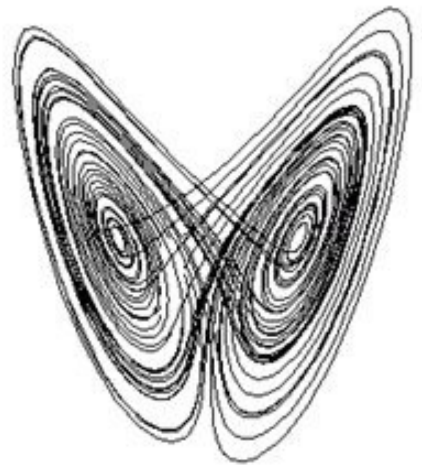
Types of Ω-limit set

equilibrium



limit cycle

limit torus



chaotic attractor

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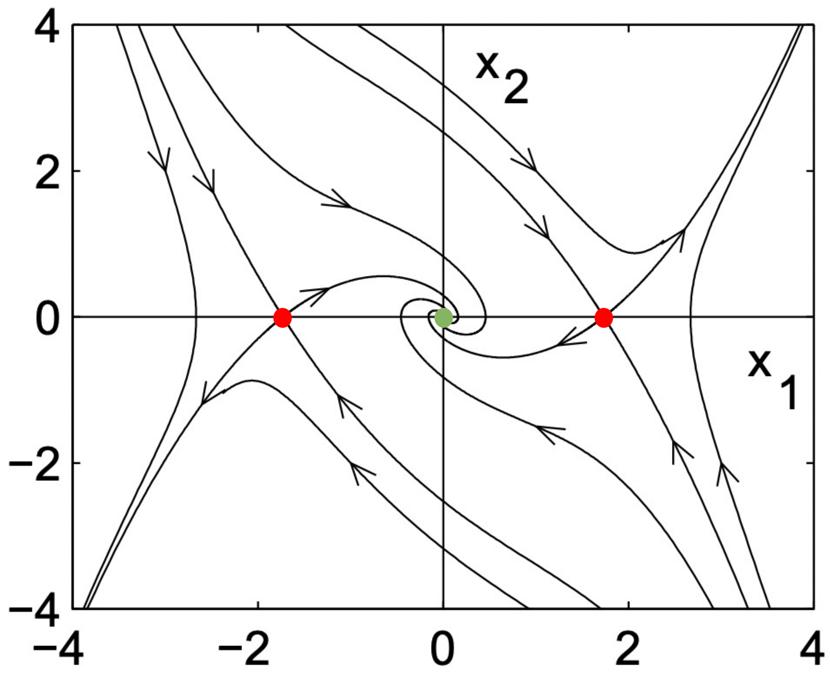
Illustrative Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 + \frac{1}{3}x_1^3 - x_2 \end{bmatrix}$$

$$\Omega(f) = \{(0,0), (-\sqrt{3}, 0), (\sqrt{3}, 0)\}$$ (equilibria)



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Region of attraction (ROA) of a set $S \subseteq \Omega(f)$:

$$\mathcal{A}(S) := \left\{ x \in \mathbb{R}^d \middle| \liminf_{t \to \infty} d(\phi(t,x), S) = 0 \right\}$$

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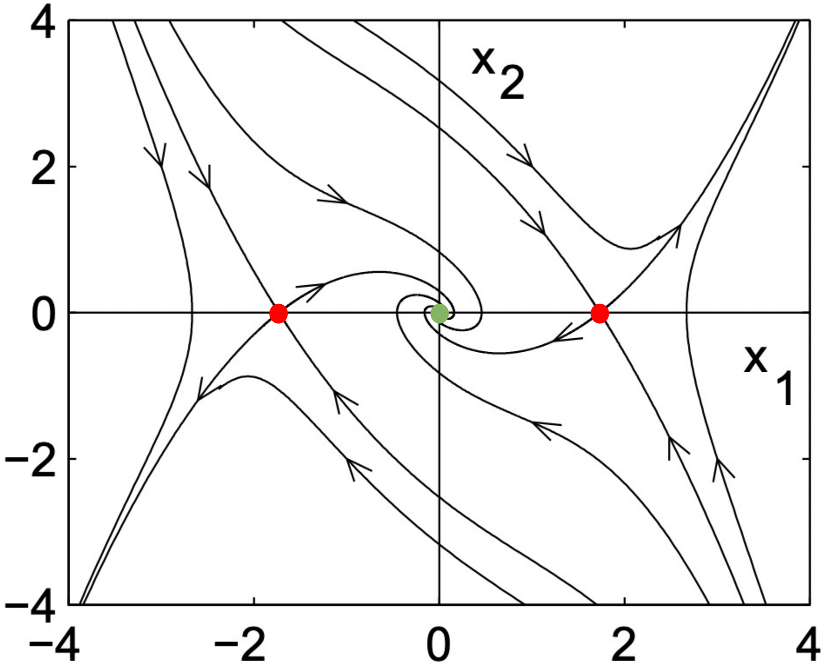

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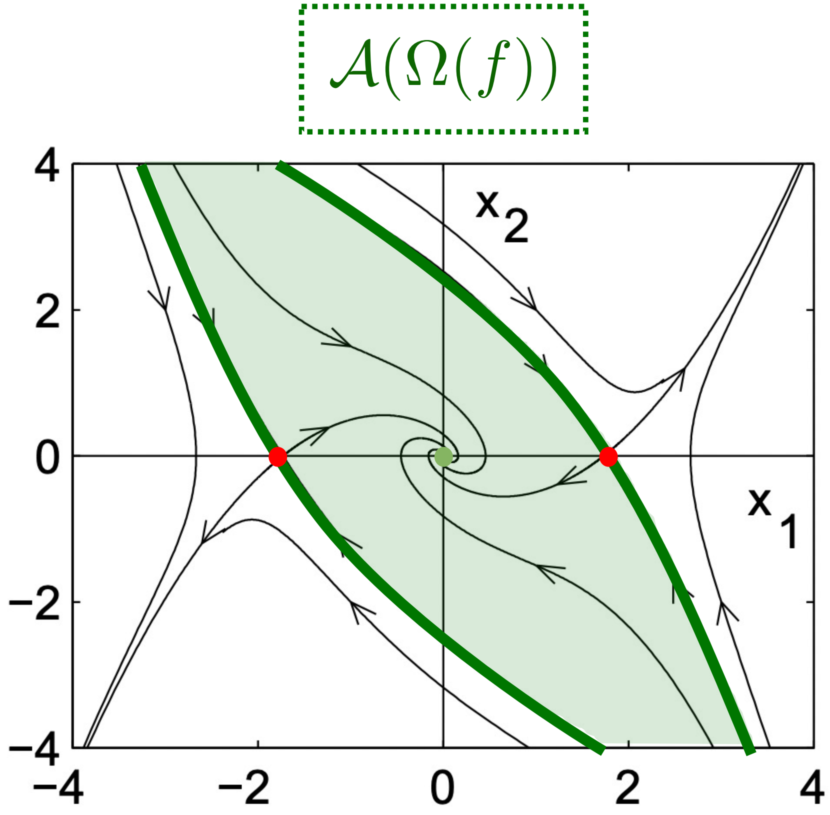

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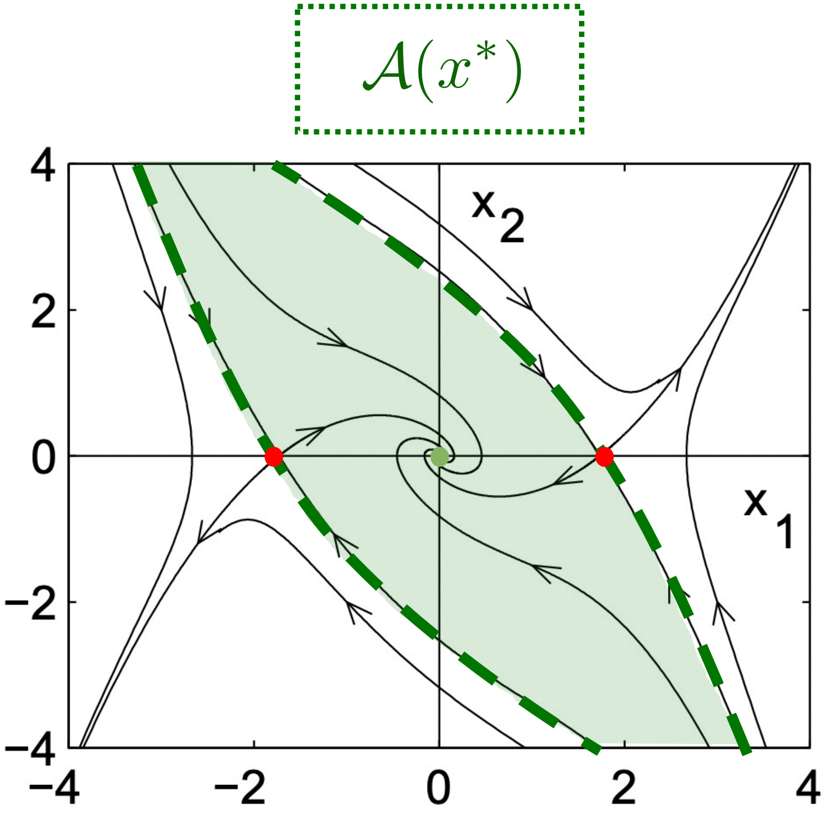
Asymptotically stable equilibrium at $x^* = (0,0)$



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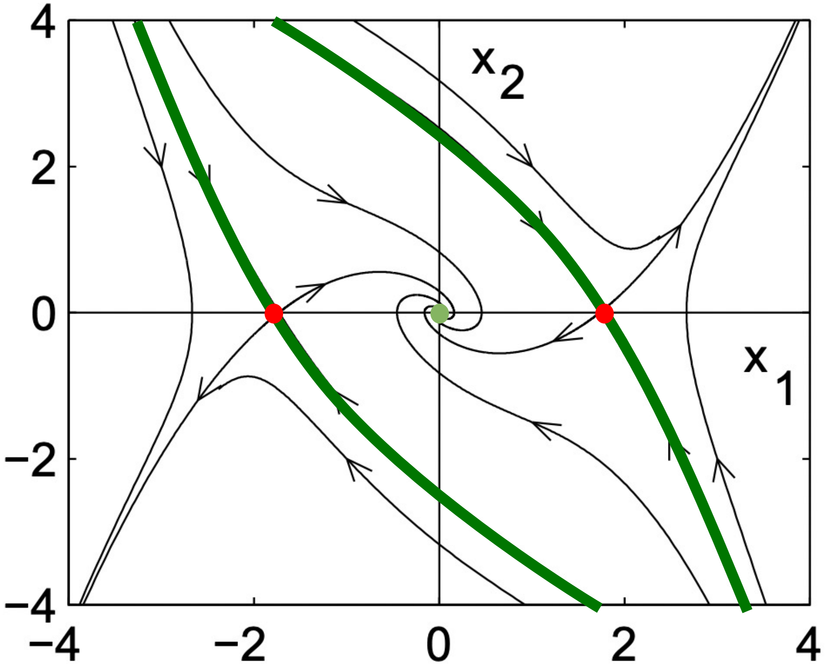
Unstable equilibria $\{(\sqrt{3},0), (-\sqrt{3},0)\}$

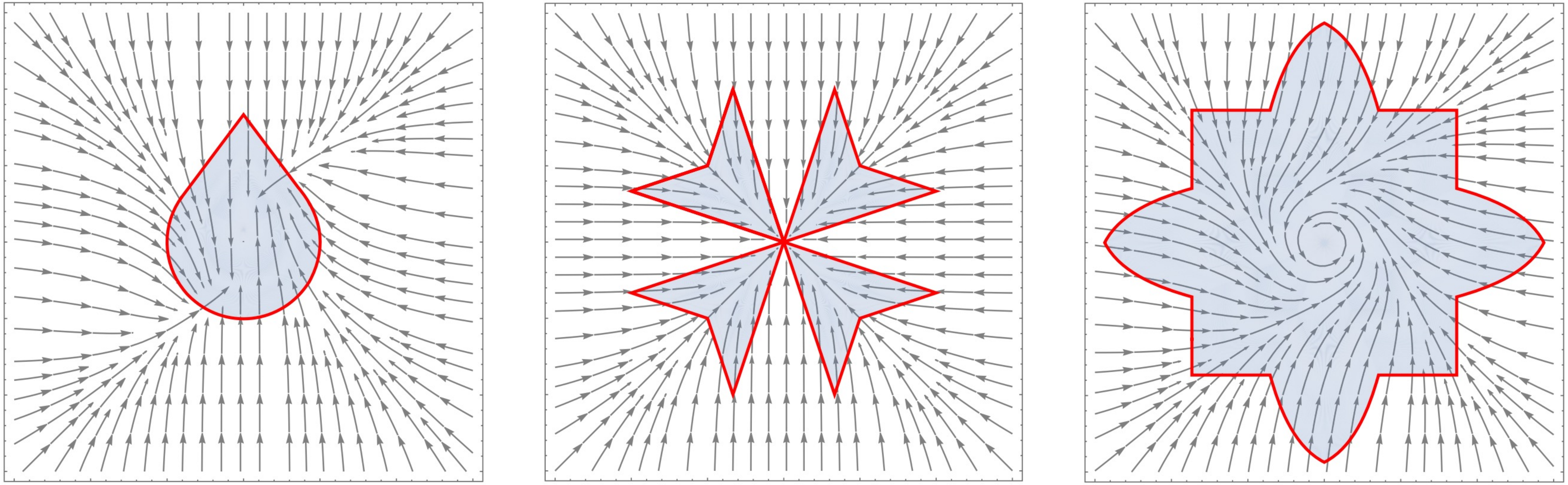
$\mathcal{A}(\{(\sqrt{3},0), (-\sqrt{3},0)\})$



Invariant sets

A set $\mathcal{S} \subseteq \mathbb{R}^d$ is **positively invariant** if and only if: $x_0 \in \mathcal{S} \to \phi(t, x_0) \in \mathcal{S},\ \forall t \geq 0$

Any trajectory starting in the set remains in inside it for all times



Source: K. Ghorbal, K. and A. Sogokon, Characterizing positively invariant sets: Inductive and topological methods. Journal of Symbolic Computation, 2022

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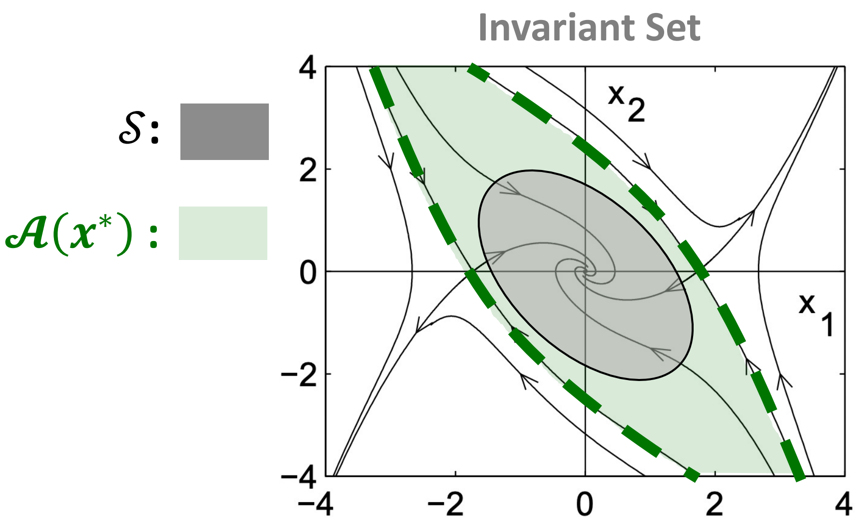
- **Invariant sets approximate regions of attraction**
  Compact invariant set $\mathcal{S}$ containing only $\{x^*\} = \Omega(f) \cap \mathcal{S}$ in the interior must be in the region of attraction $\mathcal{A}(x^*)$



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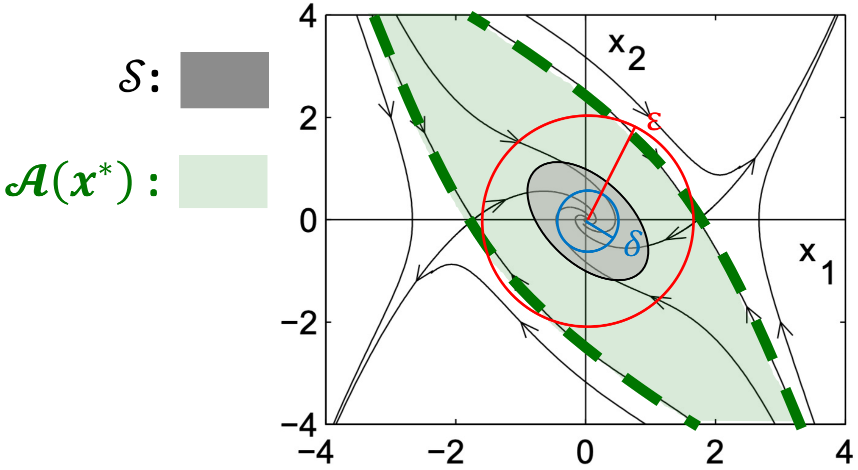
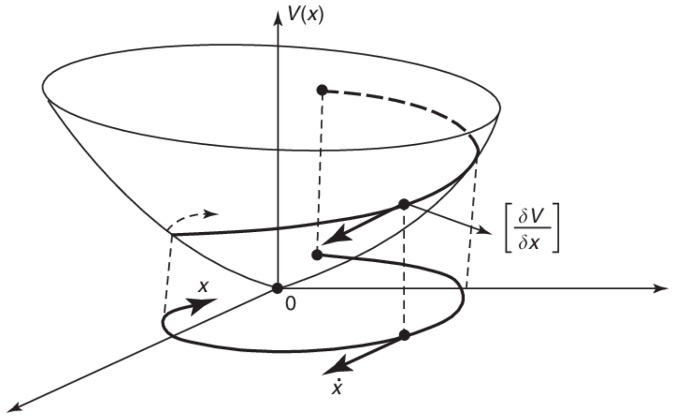
- **Invariant sets guarantee stability**
  Lyapunov stability: solutions starting "close enough" to the equilibrium (within a distance δ) remain "close enough" forever (within a distance ε)
- **Invariant sets further certify asymptotic stability via Lyapunov's direct method**
  Asymptotic stability: solutions that start close enough, remain close enough, and eventually converge to equilibrium.

Invariant Set



Lyapunov Functions



Invariant sets: Challenges

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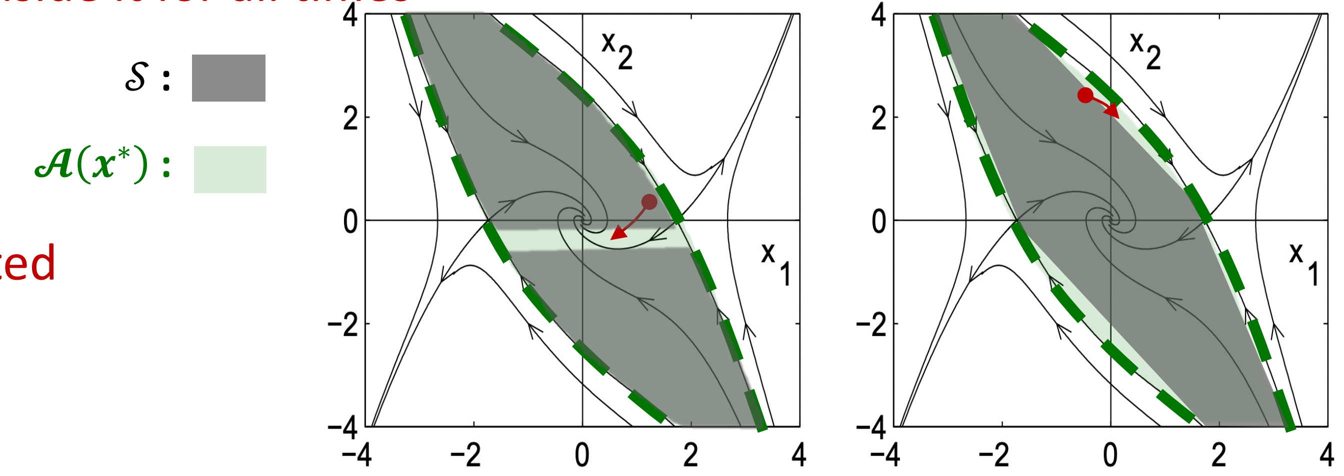
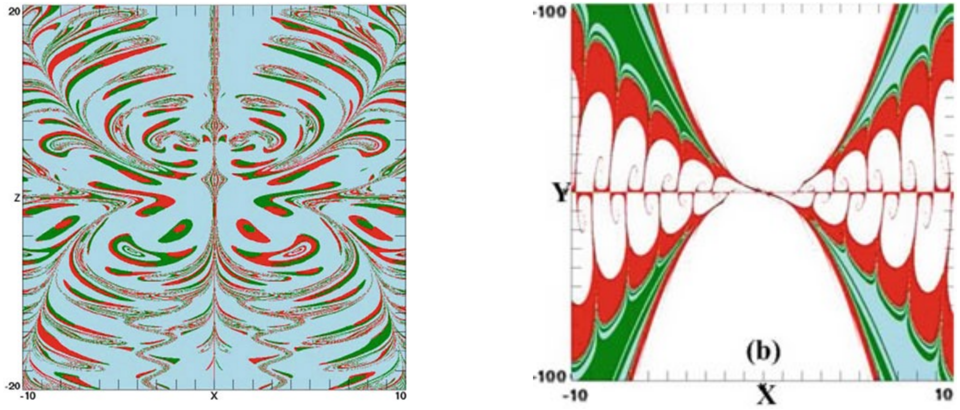


A not invariant trajectory:

- **$\mathcal{S}$ is topologically constrained**
  - If $\mathcal{S} \cap \Omega(f) = \{x^*\}$, then $\mathcal{S}$ is connected
- **$\mathcal{S}$ is geometrically constrained**
  - f should not point outwards for $x \in \partial\mathcal{S}$
- **$\mathcal{S}$ geometry can be wild**
  - $\mathcal{A}(\Omega(f))$ is not necessarily analytic!

Basin of $\mathcal{A}(\Omega(f))$



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- Invariance: Merits and trade-offs
- **Letting things go, and come back: Recurrent sets**
- Analysis using recurrent sets
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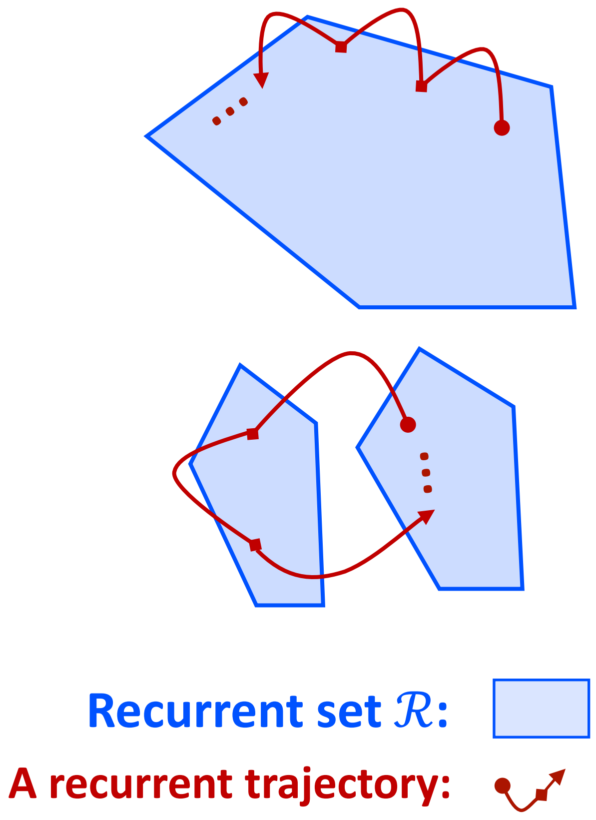
Recurrent sets: Letting things go, and come back

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \geq t$ s.t. $\phi(t', x_0) \in \mathcal{R}$.

Property of Recurrent Sets

- $\mathcal{R}$ need **not** be **connected**
- $\mathcal{R}$ does **not** require f to **point inwards** on all $\partial\mathcal{R}$

> Recurrent sets, while not invariant, guarantee that solutions that start in this set, will come back **infinitely often, forever!**



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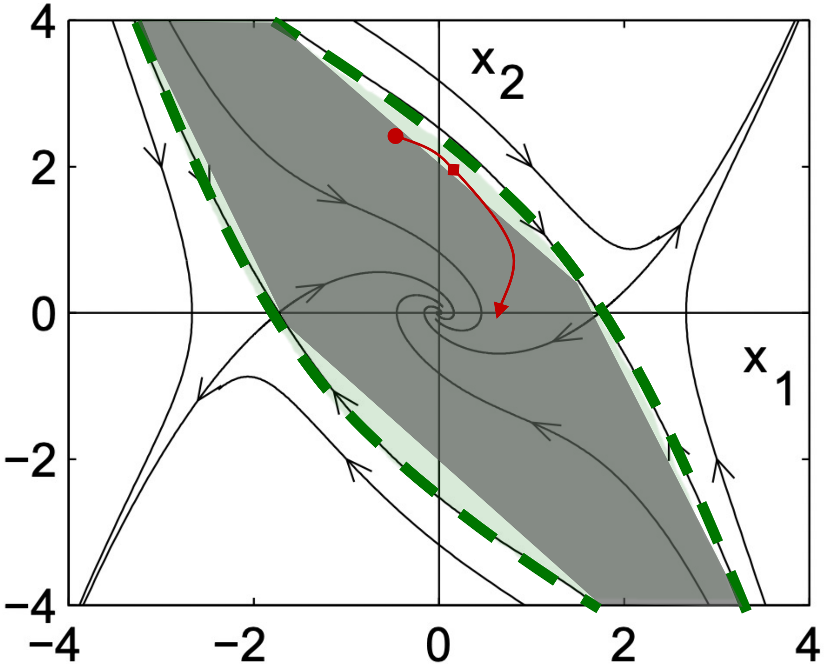
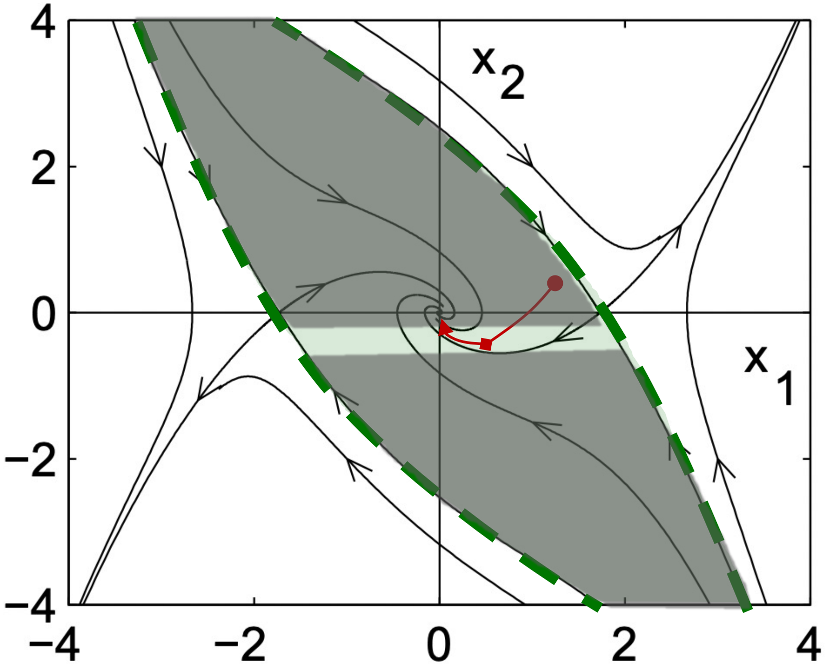
Previous two good inner approximations of $\mathcal{A}(x^*)$ are recurrent sets





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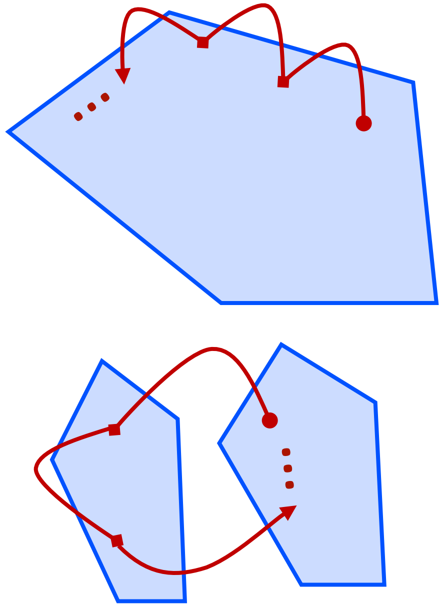

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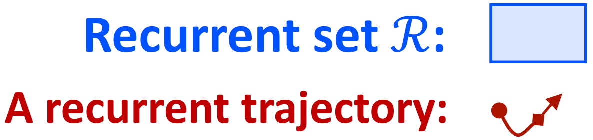

Question: Can we use recurrent sets as a substitute to invariant sets?

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Recurrent sets are subsets of the region of attraction

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \geq t$ s.t. $\phi(t', x_0) \in \mathcal{R}$.

Theorem. *Let $\mathcal{R} \subset \mathbb{R}^d$ be a compact set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$.*

Then:

$\mathcal{R}$ is *invariant* $\Rightarrow$ $\mathcal{R} \cap \Omega(f) \neq \emptyset$, $\mathcal{R} \subset \mathcal{A}(\mathcal{R} \cap \Omega(f))$

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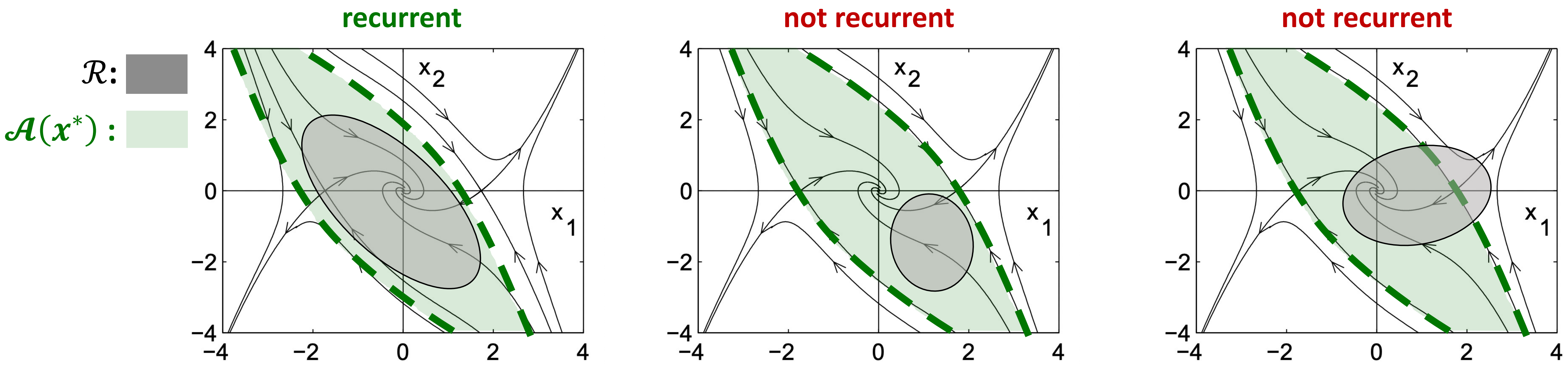
$\mathcal{R}$ is *recurrent* $\Longleftrightarrow$ $\begin{array}{c} \mathcal{R} \cap \Omega(f) \neq \emptyset \\ \mathcal{R} \subset \mathcal{A}(\mathcal{R} \cap \Omega(f)) \end{array}$



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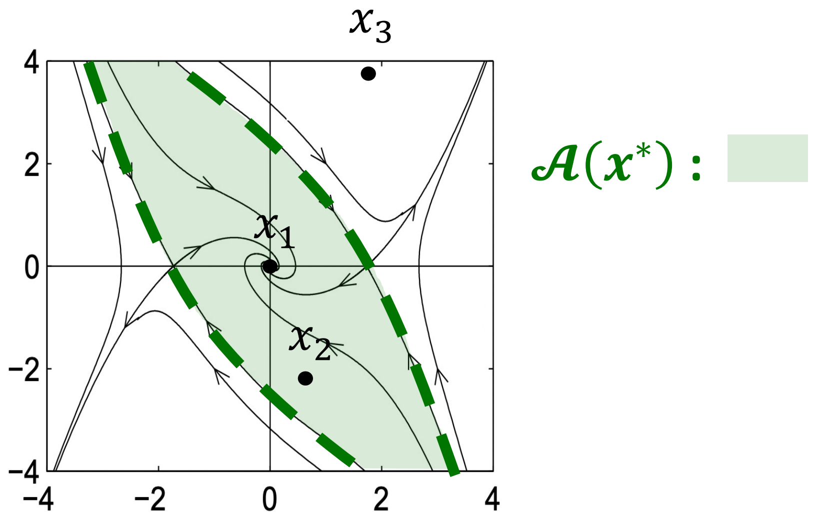
Learning Regions of Attractions via Recurrent Sets

Algorithm: Given h, k, and $\varepsilon > 0$:

- *Build approximation using unions of balls centered at* $x_1, \dots, x_q$, *with* $x_1 = x^*$



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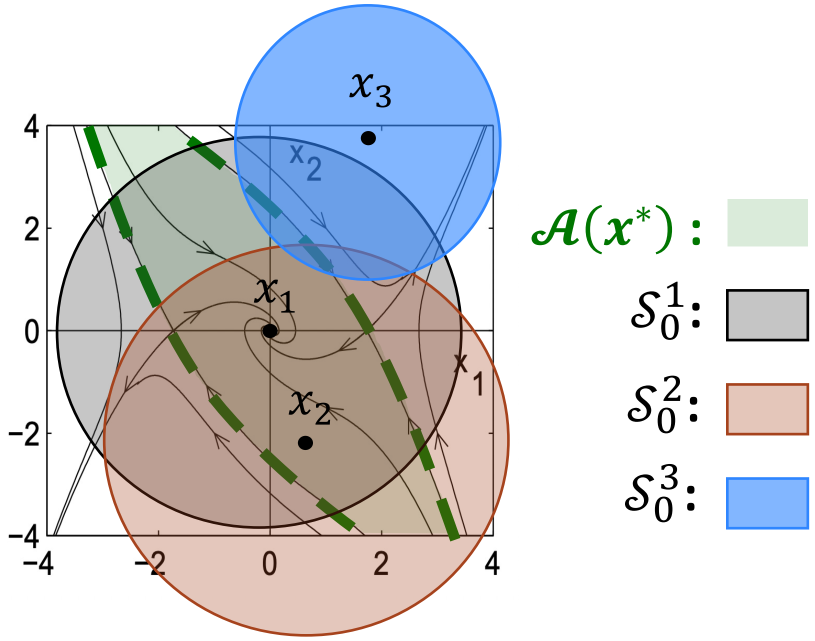

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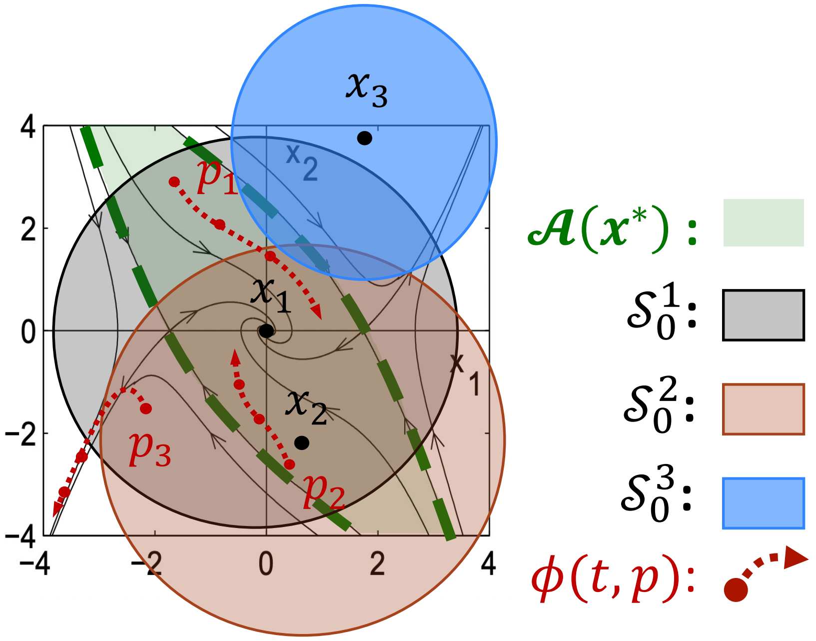
At each iteration l

- *Sample trajectories of* ***duration*** τ *from* $\mathcal{S}_l$ *until* *recurrence is violated (counter-example)*



p_3: counter example

[arXiv 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* **CDC 2022,** journal preprint arXiv:2204.10372.

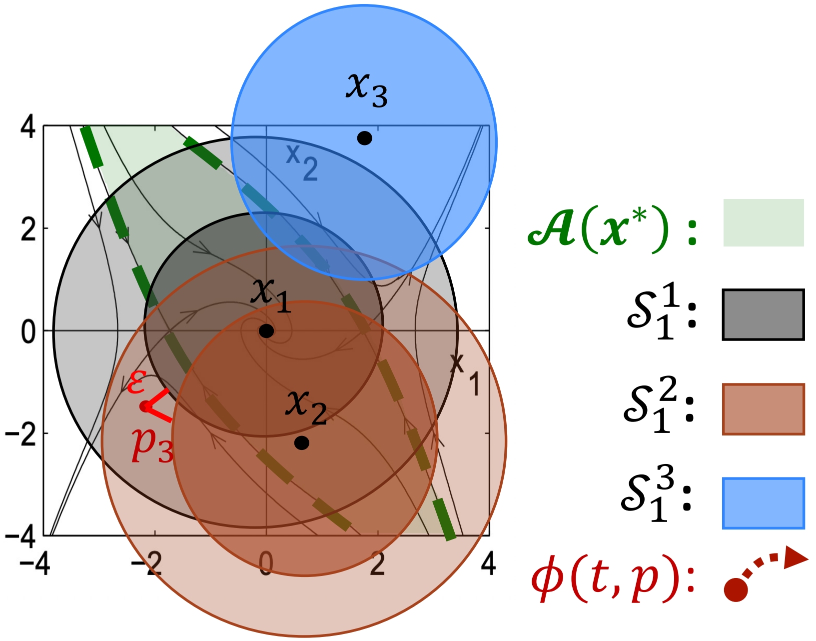
Learning Regions of Attractions via Recurrent Sets

Algorithm: Given h, k, and $\varepsilon > 0$:

- *Build approximation using unions of balls centered at* $x_1, \dots, x_q$, *with* $x_1 = x^*$
- *Initial approximation:* $\mathcal{S}_0 = \cup_{q=1}^{h} \mathcal{S}_0^q$, *where* $\mathcal{S}_0^q = \left\{x \colon \left\| x - x_q \right\| \le b_0^q \right\}$

At each iteration l

- *Sample trajectories of duration* τ *from* $\mathcal{S}_l$ *until recurrence is violated (counter-example)*
- *Update approximation* $\mathcal{S}_{l+1}$ *to exclude counter-example neighborhood:* $p_j + \mathrm{B}_\varepsilon$



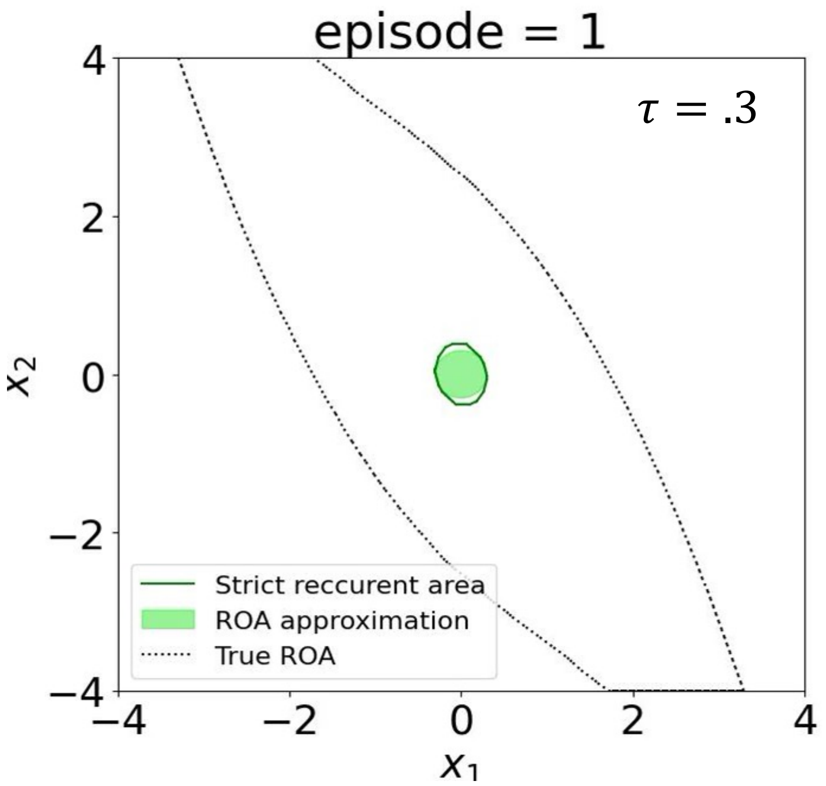
p_3: counter example

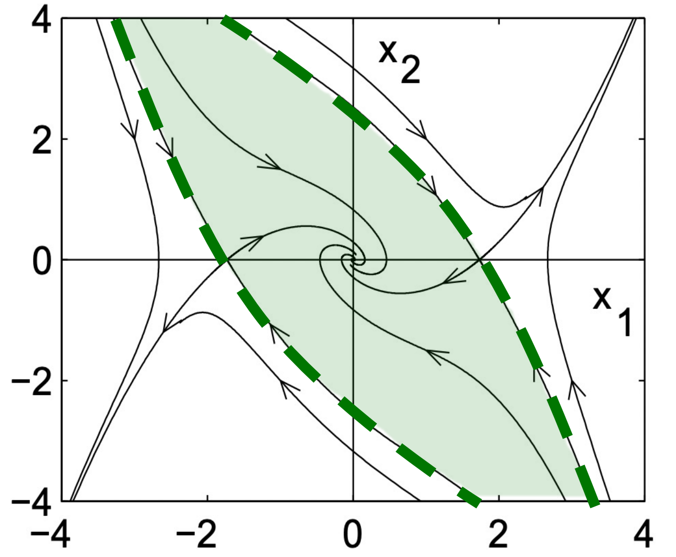
Sample complexity: $m \ge \dfrac{\mathrm{V}(\mathcal{S}_l + \mathrm{B}_\varepsilon)}{\mathrm{V}(\mathrm{B}_\varepsilon)} \log\left(\dfrac{1}{\delta}\right)$

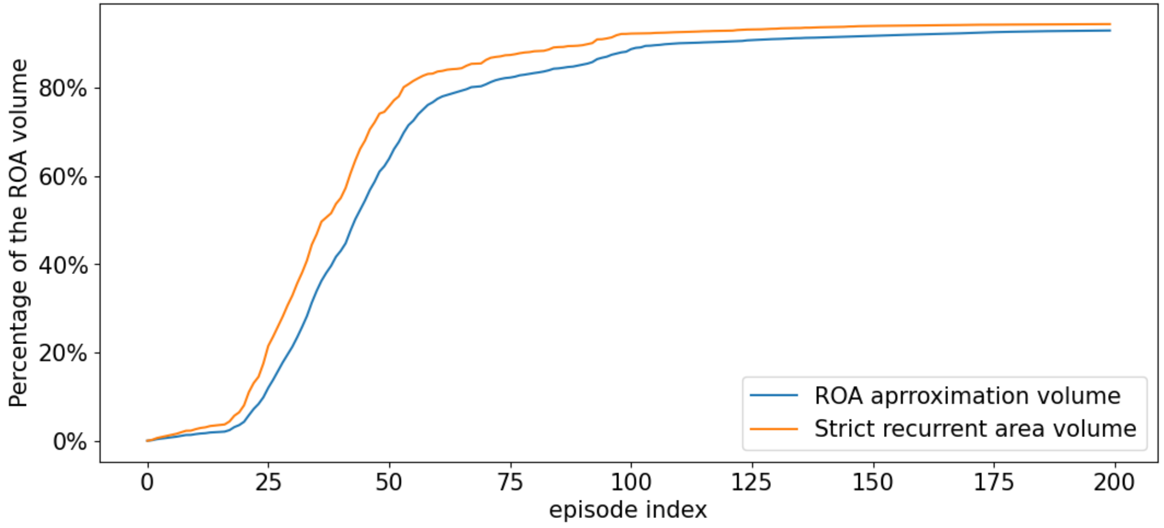
[arXiv 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* **CDC 2022,** journal preprint arXiv:2204.10372.

Example: Progressively Expanding the RoA Approximation

- At Each Episode:
  - **Sample 50** center points (uniformly)
  - **Stopping criteria:** $\delta = 10^{-5}$



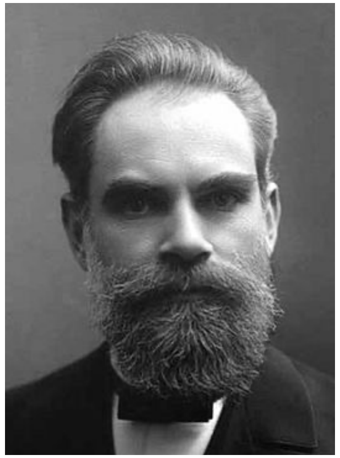




Outline

- Invariance: Merits and trade-offs
- Letting things go, and come back: Recurrent sets
- **Analysis using recurrent sets**
  - Approximating regions of attractions
  - **Stability analysis via non-monotonic Lyapunov functions**

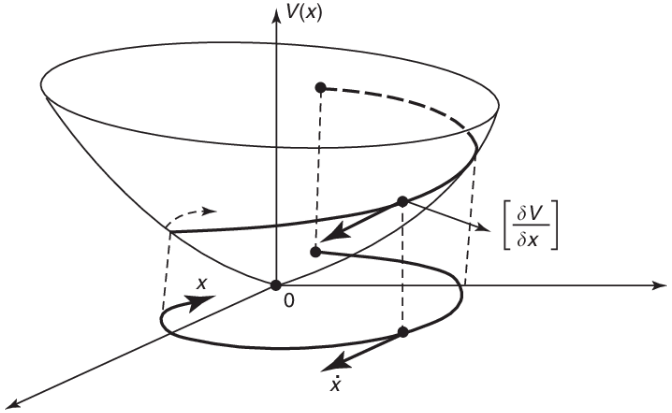
Lyapunov's Direct Method



Key idea: Make sub-level sets invariant to trap trajectories

Theorem [Lyapunov '1892]. *Given* $V: \mathbb{R}^d \to \mathbb{R}_{\geq 0}$, *with* $V(x) > 0, \forall x \in \mathbb{R}^d \backslash \{x^*\}$, *then:*

- $\dot{V} \leq 0 \to x^*$ *stable*
- $\dot{V} < 0 \to x^*$ *as. stable*



Challenge: Couples shape of V and vector field f

- Towards decoupling the $V - f$ geometry
  - Controlling regions where $\dot{V} \geq 0$ **[Karafyllis '09, Liu et al '20]**
  - Higher order conditions: $g(V^{(q)}, \ldots, \dot{V}, V) \leq 0$ **[Butz '69, Gunderson '71, Ahmadi '06, Meigoli '12]**
  - Discretization approach: $V\big(x(T)\big) \leq V(x(0))$ **[Coron et al '94, Aeyels et. al '98, Karafyllis '12]**
  - Multiple Lyapunov Functions: $\{V_j : j \in [k]\}$ **[Ahmadi et al '14]**

A Butz. Higher order derivatives of Lyapunov functions. IEEE Transactions on automatic control, 1969
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Coron, Lionel Rosier. A relation between continuous time-varying and discontinuous feedback stabilization. J. Math. Syst., Estimation, Control, 1994
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Meigoli, Nikravesh. Stability analysis of nonlinear systems using higher order derivatives of Lyapunov function candidates. Systems & Control Letters, 2012
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Ahmadi, Jungers, Parrilo, Roozbehani. Joint spectral radius and path-complete graph Lyapunov functions. SIAM Journal on Control and Optimization, 2014
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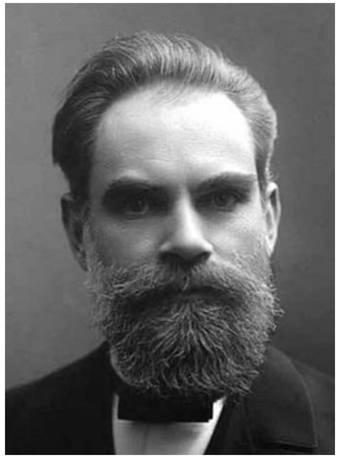
Lyapunov’s Direct Method



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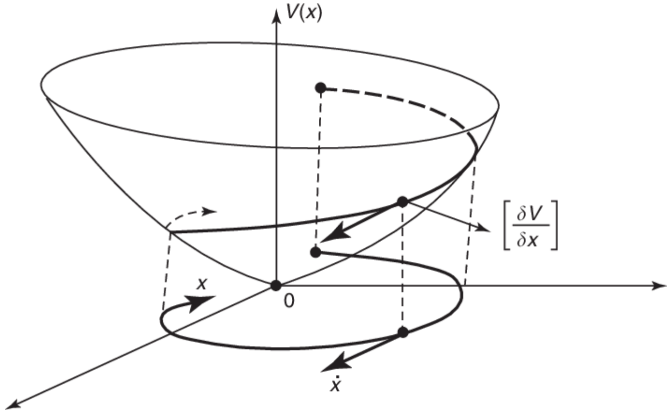

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Question: Can we provide stability conditions based on recurrence?

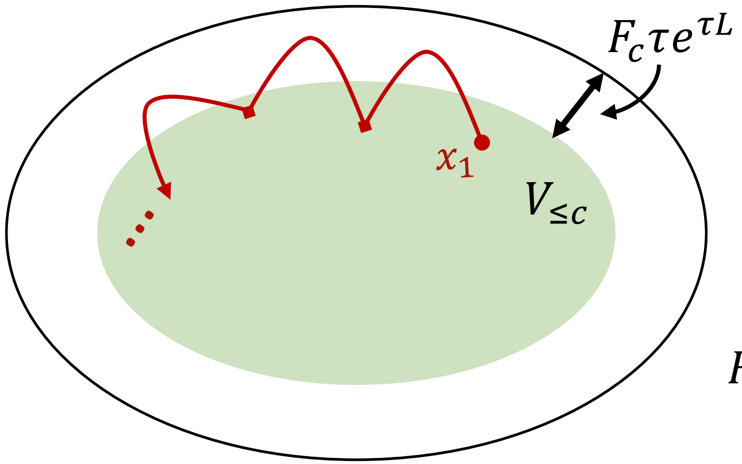
Recurrently Decreasing Lyapunov Functions

A continuous function $V: \mathbb{R}^d \to \mathbb{R}_+$ is a **recurrently non-increasing Lyapunov function** over intervals of length $\boldsymbol{\tau}$ if

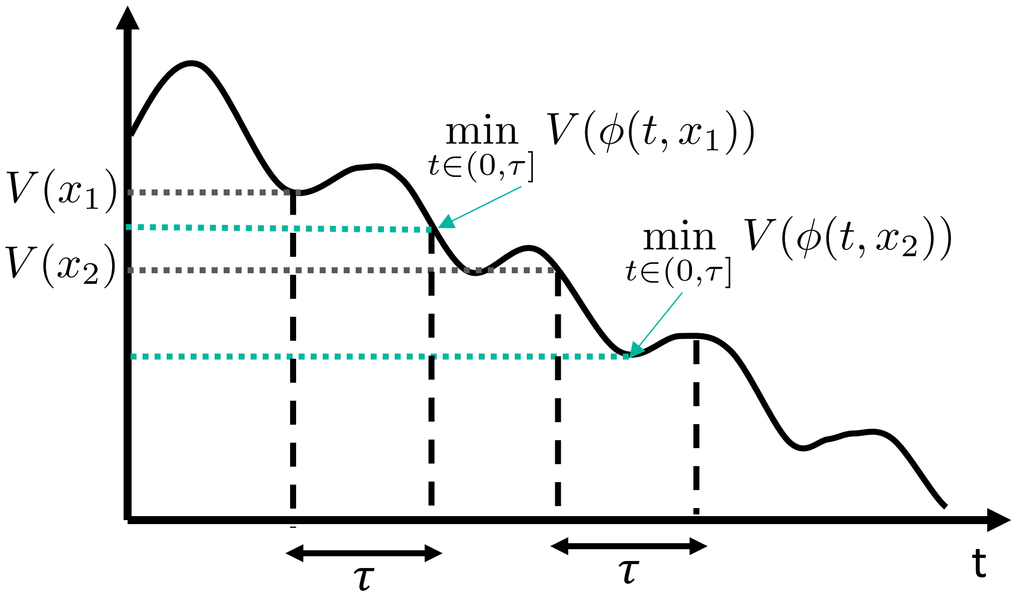
$$\mathcal{L}_f^{(0,\tau]} V(x) := \min_{t\in(0,\tau]} V(\phi(t,x)) - V(x) \leq 0 \quad \forall x \in \mathbb{R}^d$$

Preliminaries:

- Sub-level sets $\{V(x) \leq c\}$ are $\boldsymbol{\tau}$-recurrent sets.
- When f is L-Lipschitz, one can trap trajectories.



$$F_c = \max_{x\in V_{\leq c}} ||f(x)||$$



Recurrently Non-Increasing Lyapunov Functions

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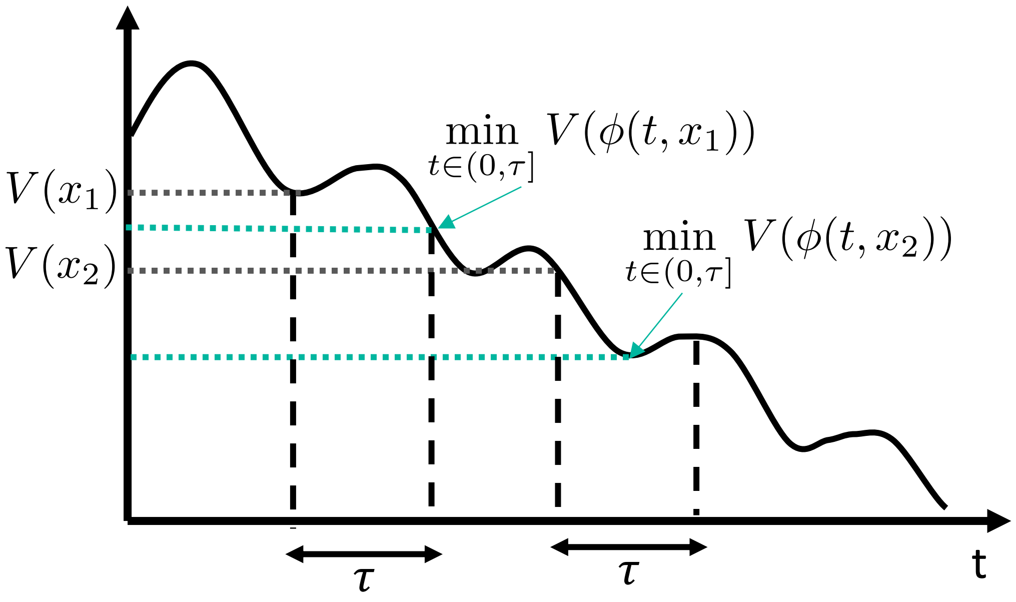
Theorem [CDC 23*]: *Let $V: \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ be a recurrently non-increasing Lyapunov function over intervals of length τ. Let f be L-Lipschitz*

- *Then the equilibrium x^* is stable.*
- *Further, if the* ***inequality is strict****, then x^* is asymptotically stable!*



Siegelmann, Shen, Paganini, M, *A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions,* **submitted CDC 2023**

Exponential Stability Analysis

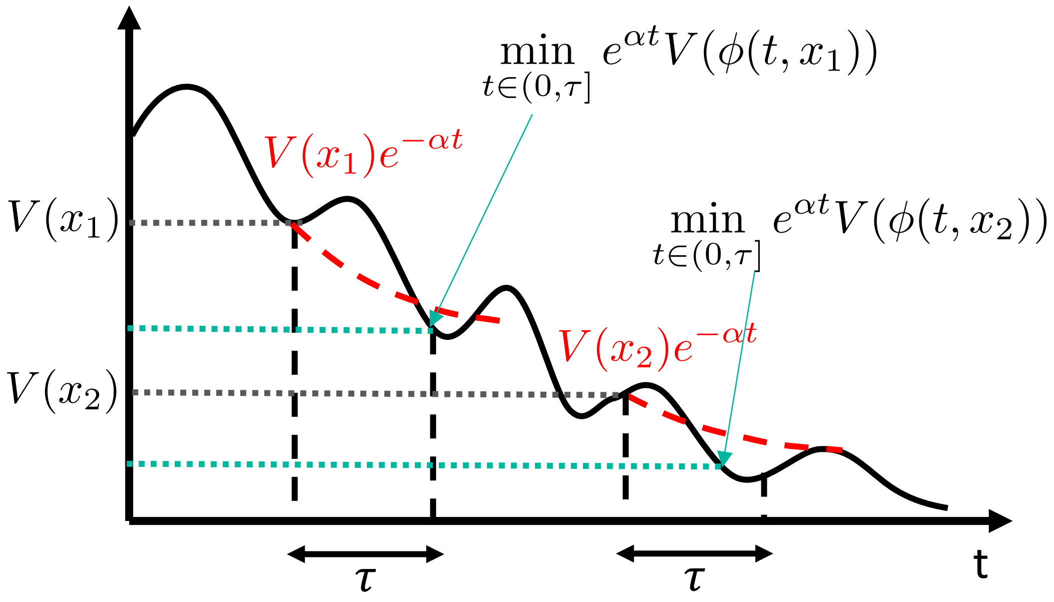
The function $V: \mathbb{R}^d \to \mathbb{R}_+$ is ***α*-exponentially recurrently *τ*-decreasing Lyapunov function** over intervals of length $\boldsymbol{\tau}$ if

$$\mathcal{L}_{f,\alpha}^{(0,\tau]} V(x) := \min_{t\in(0,\tau]} e^{\alpha t} V(\phi(t,x)) - V(x) \leq 0 \quad \forall x \in \mathbb{R}^d$$

Theorem [CDC 23*]: *Let* $V: \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ *satisfy*

$$\alpha_1 ||x - x^*|| \leq V(x) \leq \alpha_2 ||x - x^*||.$$

Then, if V *is α-exponentially recurrently τ-decreasing Lyapunov function, then* x^* *is exponentially stable with rate* α.



Siegelmann, Shen, Paganini, M, *A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions*,
***submitted CDC 2023**

All norms are Lyapunov functions!

Theorem: Assume x^* is globally exponentially stable: $\exists\, K, c > 0$ such that:

$$\left|\left|\phi(t,x) - x^*\right|\right| \leq Ke^{-ct}\left|\left|x_0 - x^*\right|\right|.$$

Then, $V(x) = \left|\left|x - x^*\right|\right|$ *is* α*-exponentially recurrently* τ*-decreasing , i.e.,*

$$\min_{t\in(0,\tau]} e^{\alpha t}\left|\left|\phi(t,x) - x^*\right|\right| - \left|\left|x - x^*\right|\right| \leq 0, \qquad \forall x \in \mathbb{R}^d,$$

whenever $\alpha < c$ *and* $\tau \geq \frac{1}{c-\alpha}\ln K.$

Remarks:

- The rate α must be strictly smaller than the rate of convergence c (giving up optimality).
- *Any norm is a Lyapunov function!*

Question: Is the struggle for its search over?

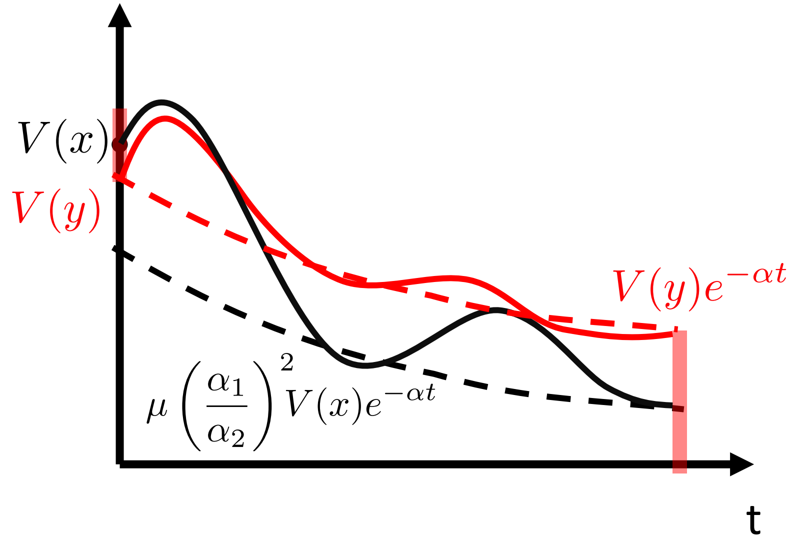
Verification of Exponential Stability

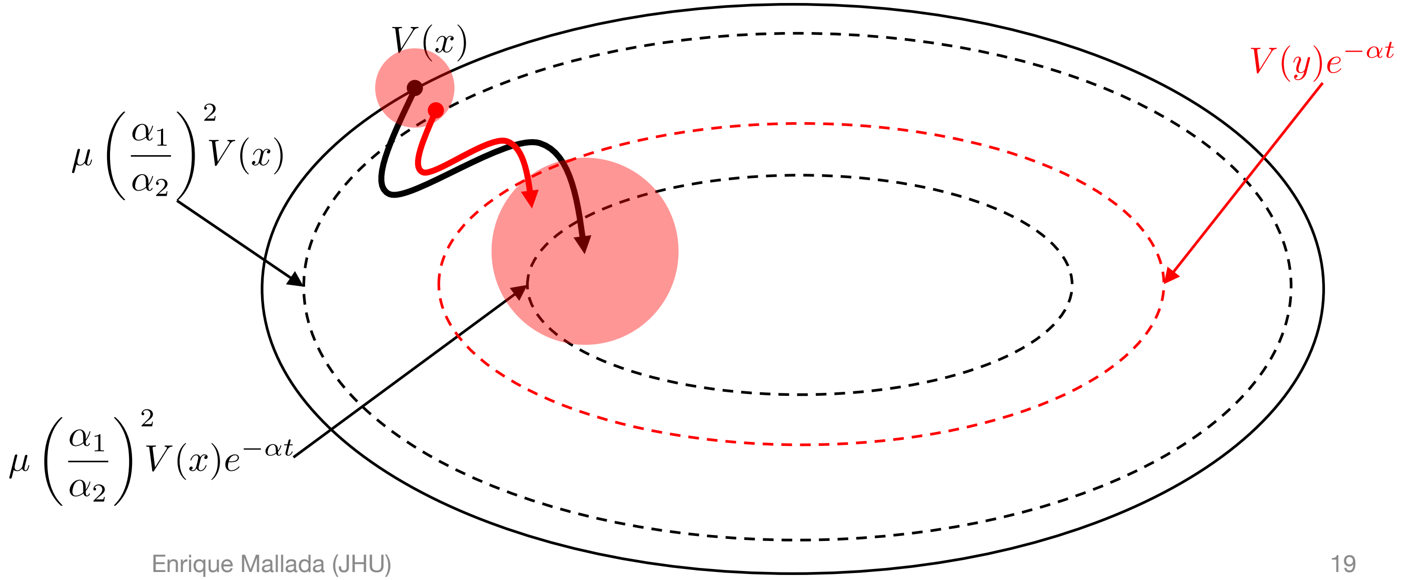
Proposition [CDC 23*]: *Let* $V: \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ *satisfy* $\alpha_1 \|x - x^*\| \leq V(x) \leq \alpha_2 \|x - x^*\|$, *and* $0 < \mu < 1$. *Then, whenever*

$$\min_{t \in (0,\tau]} e^{\alpha t} V(\phi(x,t)) \leq \mu \left(\frac{\alpha_1}{\alpha_2}\right)^2 V(x)$$

for all y *with* $\|y - x\| \leq r := \frac{V(x)}{\alpha_2} g(\mu)$

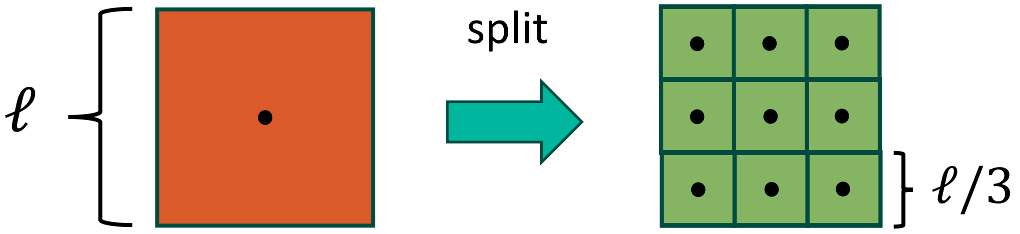
$$\min_{t \in (0,\tau]} e^{\alpha t} V(\phi(y,t)) \leq V(y)$$

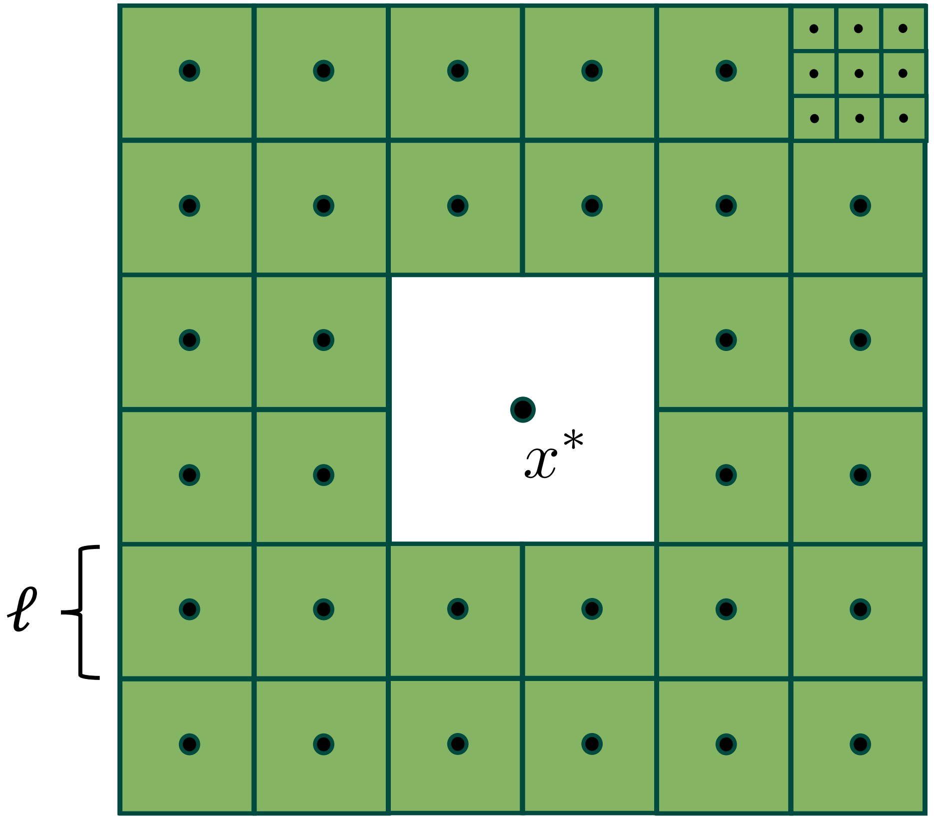




GPU-based Algorithm

- **Basic Algorithm:**
  - Consider $V(x) = ||x - x^*||_\infty$
  - Build a grid of hypercubes surrounding x^*
  - Test the center point and find α s.t. the verified radius is $r \geq \ell/2$
  - Hypercube **not verified**, **split in 3^d** parts
  - Repeat testing of new points





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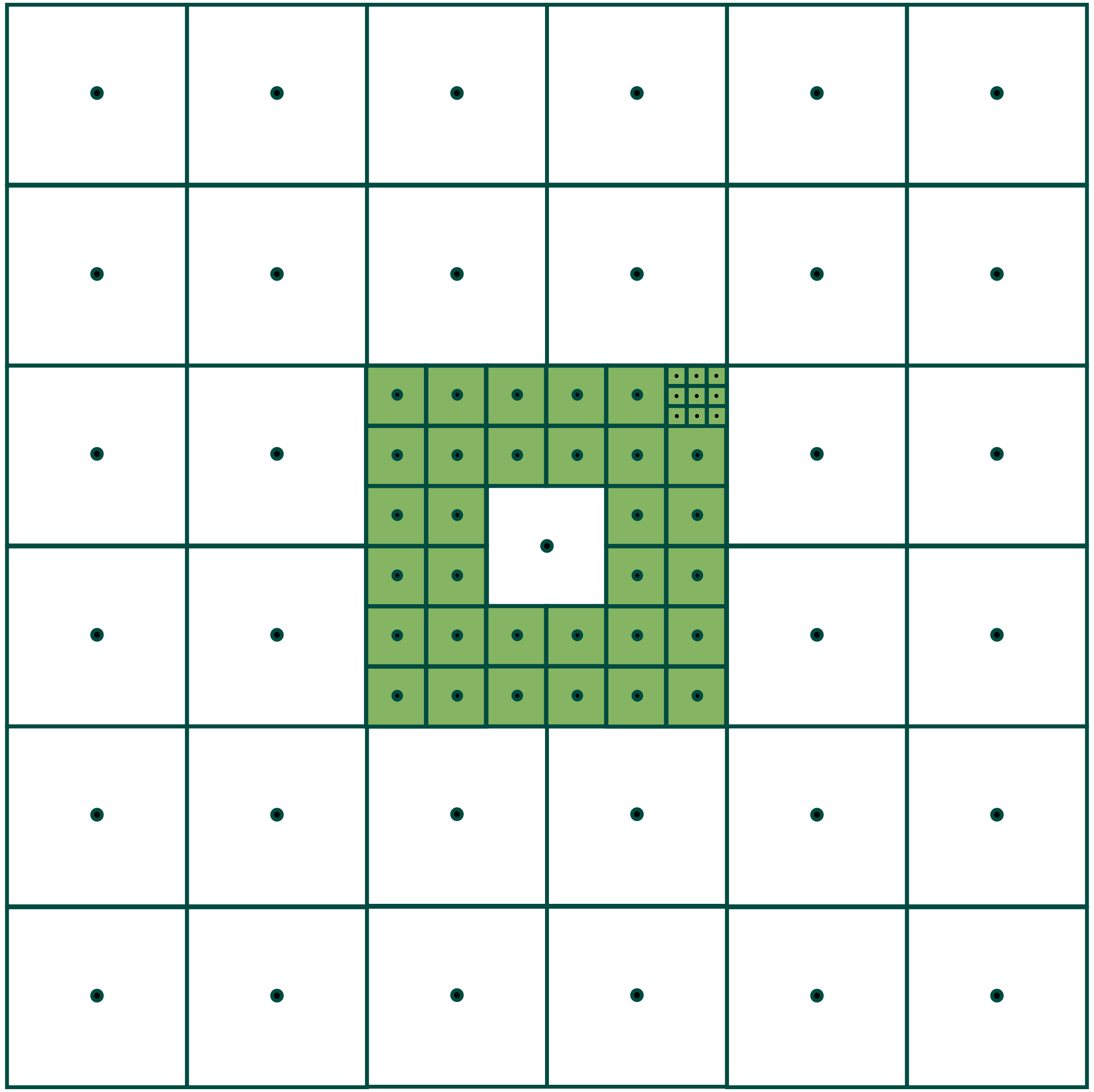
  - Exponentially expand to outer layer
  - Repeat testing in new layer



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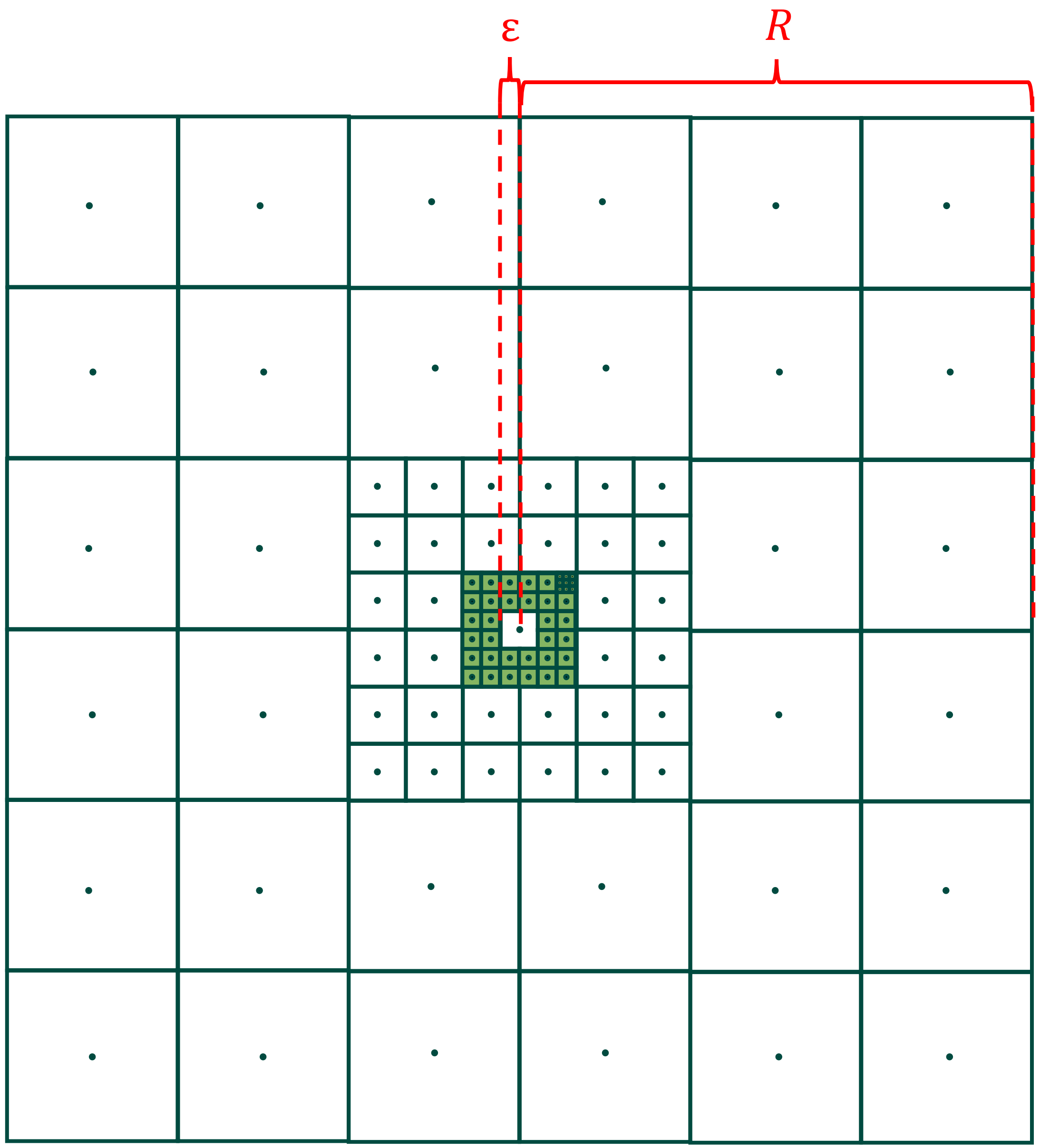
Q: How many samples are needed?

If x^* is λ-exp. stable

$$\mathcal{O}\left(q^{-d} \log\left(\frac{R}{\varepsilon}\right)\right)$$

with $q = \frac{1-Ke^{(\alpha-\lambda)\tau}}{1+e^{(L+\alpha)\tau}}$.



Numerical Illustration

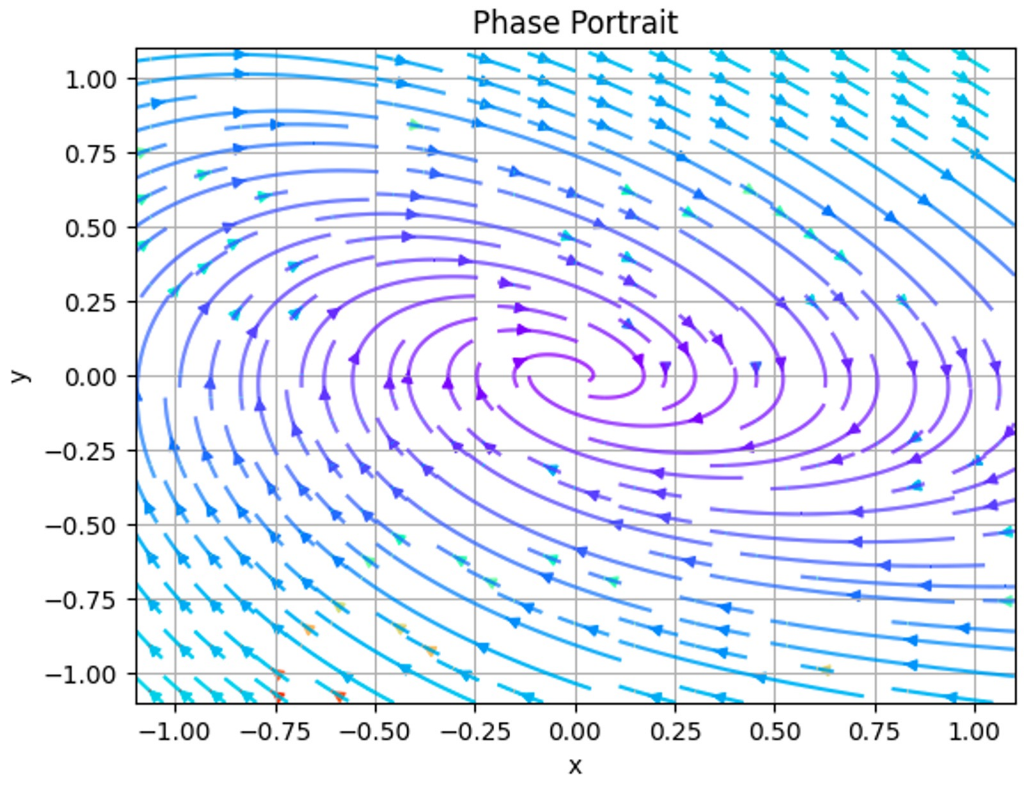
Consider the 2-d non-linear system:

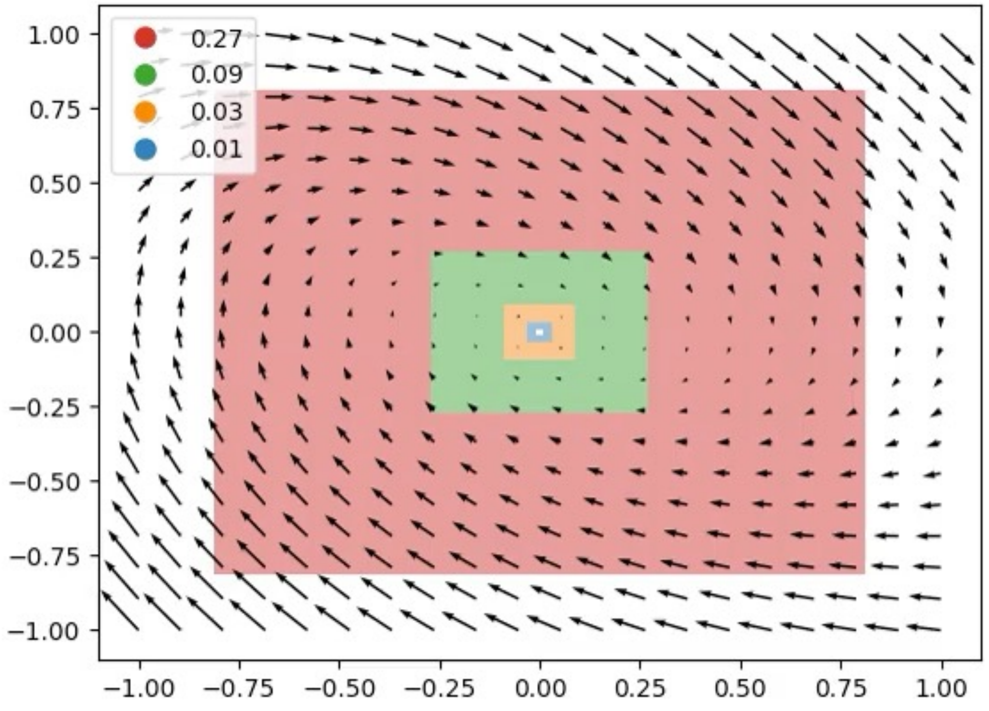
$$\dot{x} = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} x + B \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$$

with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

Parameter	Value
L	1.8
τ	1.5
ℓ	0.01

$\sigma = 0.2$





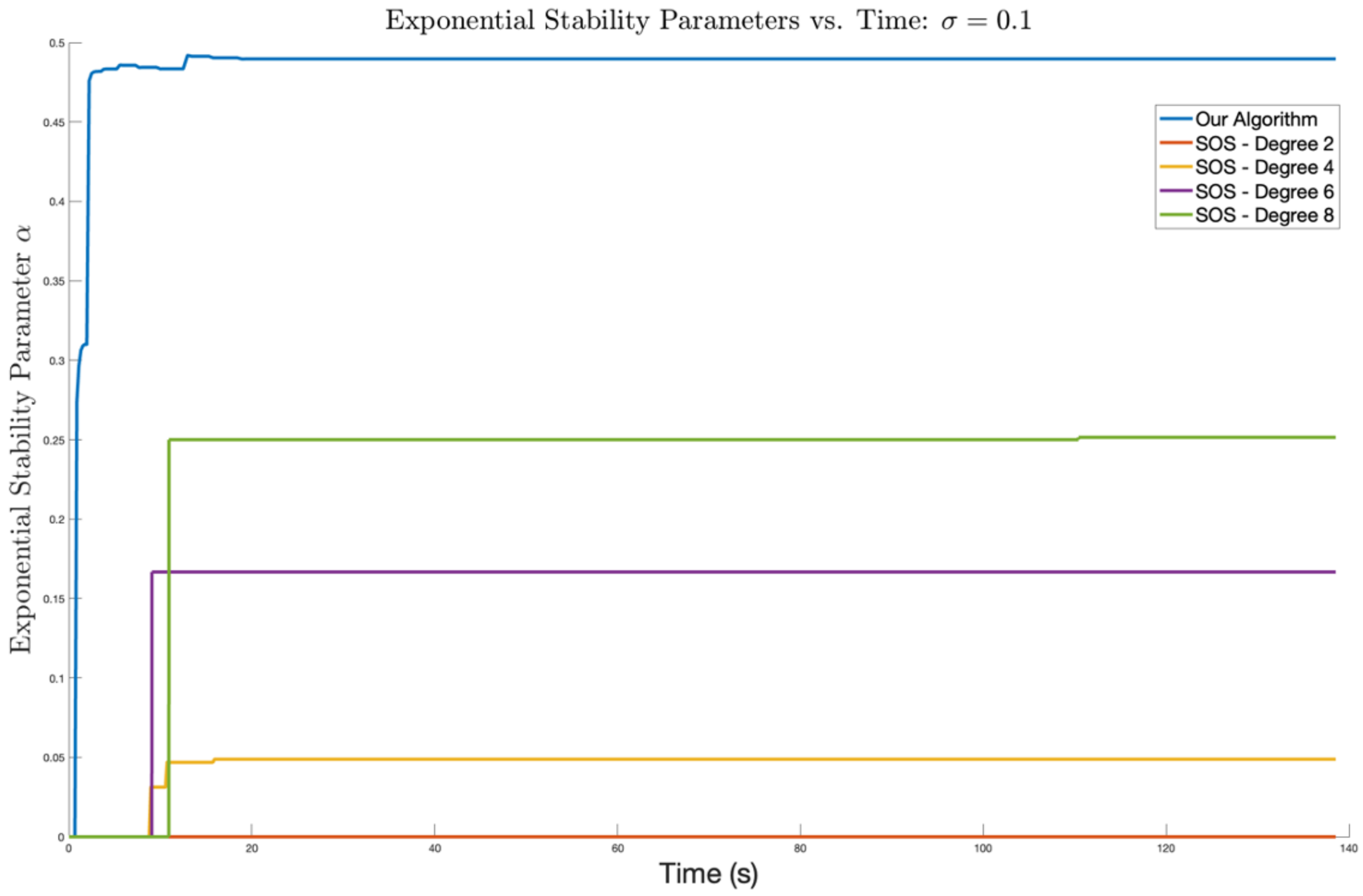
Comparison with SoS

Consider the 2-d non-linear system: $\dot{x} = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} x + B \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$

with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

Parameter	Value
L	1.8
τ	1.5
ℓ	0.01

$\sigma = 0.1$



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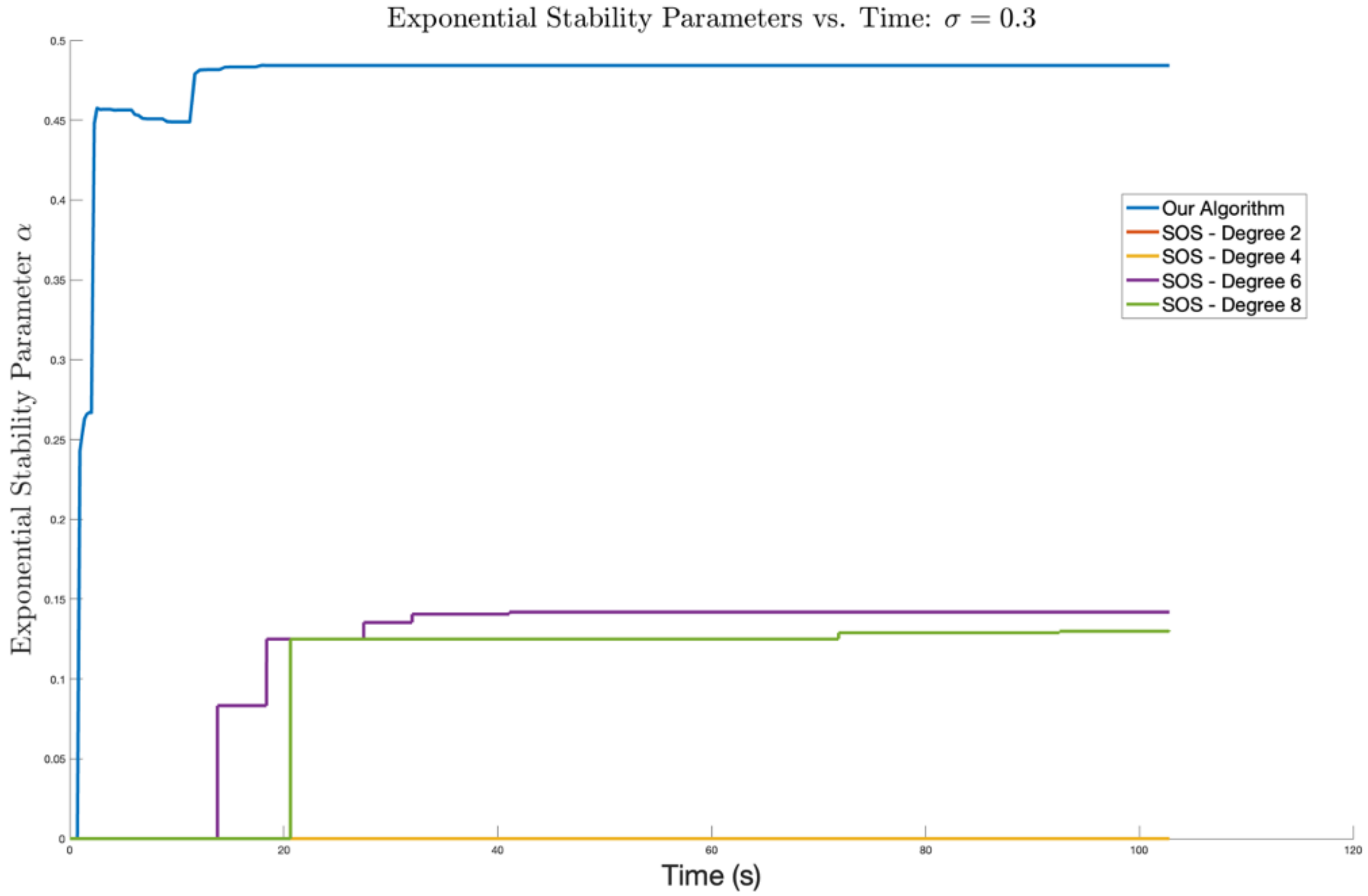
with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

Parameter	Value
L	1.8
τ	1.5
ℓ	0.01

$\sigma = 0.3$



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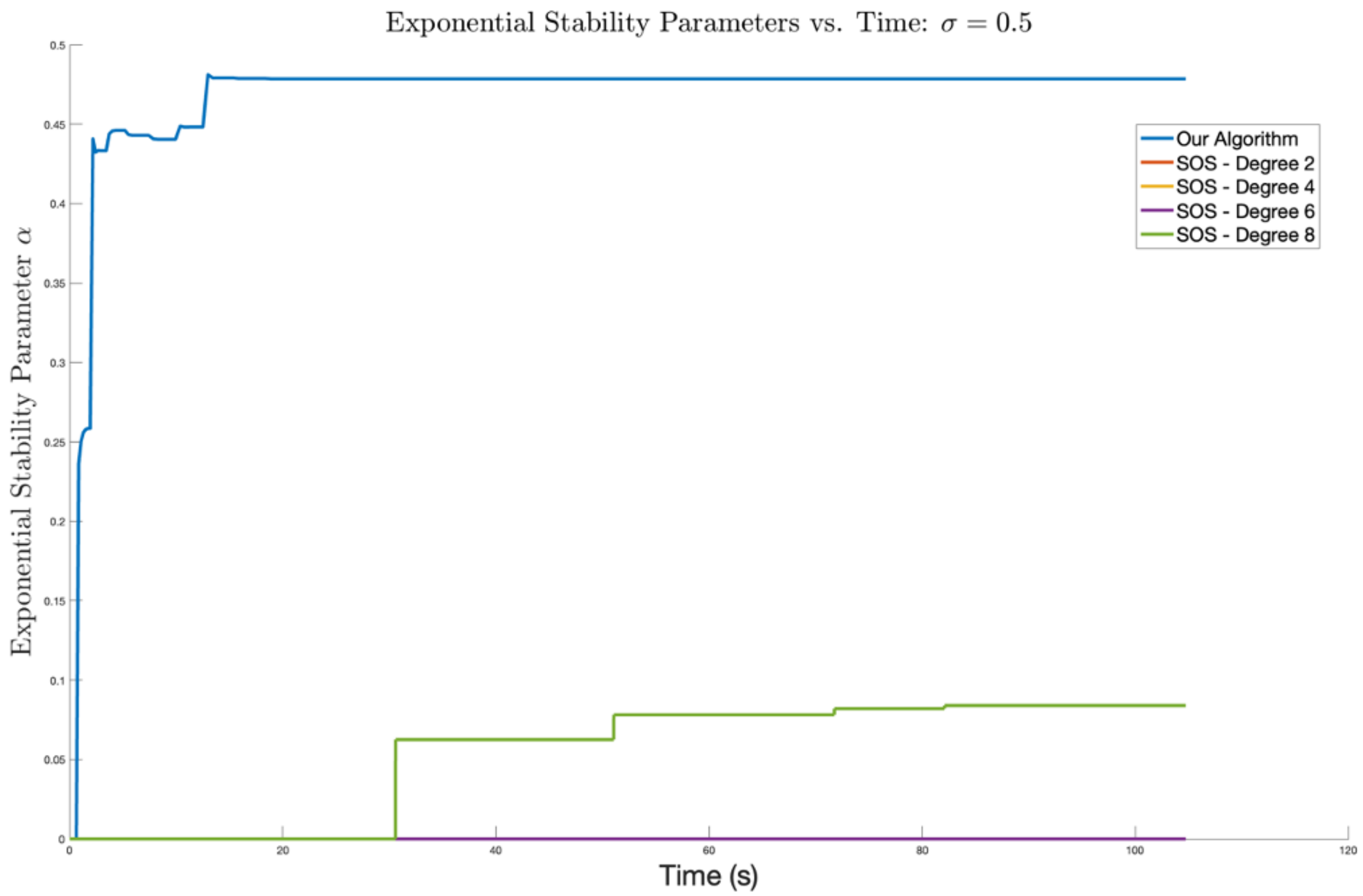
Parameter	Value
L	1.8
τ	1.5
ℓ	0.01

$\sigma = 0.5$



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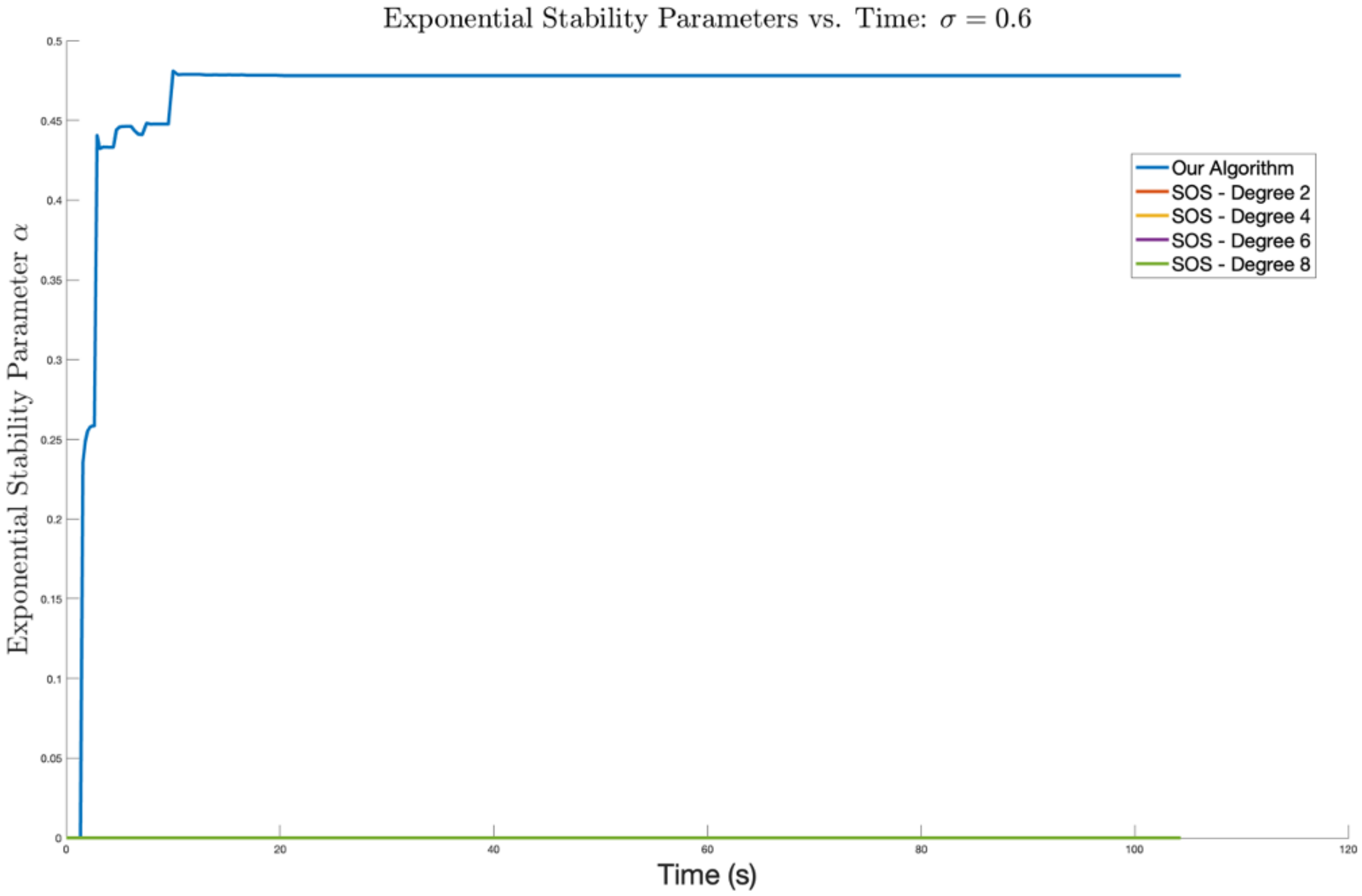
with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

Parameter	Value
L	1.8
τ	1.5
ℓ	0.01

$\sigma = 0.6$



Conclusions and Future work

- **Takeaways**
  - Proposed a **relaxed notion of invariance** known as **recurrence.**
  - Provide **necessary and sufficient conditions** for a recurrent set to be an **inner approximation** of the RoA.
  - Generalized Lyapunov Theory **for recurrently decreasing functions** using recurrent sets
  - From an information theoretical standpoint, **making as set recurrent can be easier than invariant.**

- **Ongoing work**
  - **Recurrent Sets:** Smart choice of multi-points, control recurrent sets, GPU implementation
  - **Lyapunov Functions:** Generalize other Lyapunov notions, Control Lyapunov Functions, Barrier Functions, Control Barrier Functions, Contraction, etc.
  - **Entropy:** Understanding the memory complexity of making a set recurrent and generalizations to other tasks

Thanks!

Related Publications:

[arXiv 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* **CDC 2022, journal preprint arXiv:2204.10372**.
[CDC 23] Siegelmann, Shen, Paganini, M, *A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions*, **CDC 2023**
[HSCC 24] Sibai, M, *Recurrence of nonlinear control systems: Entropy and bit rates,* **HSCC, 2024**

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Model-Free Analysis of Dynamical Systems using Recurrent Sets

- Uses of invariant sets in control theory
- Inner-approximation of regions of attractions
- Stability analysis using non-monotonic Lyapunov functions

Yue Shen

Johns Hopkins University

Maxim Bichuch

University at Buffalo

Model-free Learning of Regions of Attractions via Recurrent Sets

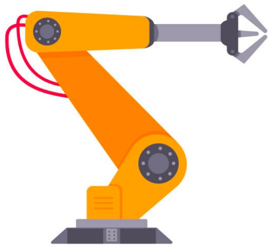
Y Shen, M. Bichuch, and E Mallada, “Model-free Learning of regions of attraction via recurrent sets.” CDC 2022.

Motivation: Estimation of regions of attraction

Having an approximation of the region of attraction allows us to

- **Test the limits of controller designs**
  especially for those based on (possibly linear) approximations of nonlinear systems

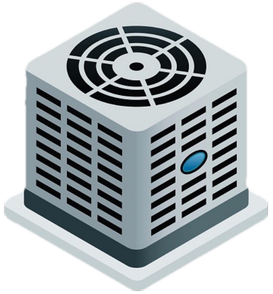
quadcopter



robot arm

. . .

- **Verify safety of certain operating condition**



HVAC system

power grids

. . .

Recall: Problem setup

Continuous time dynamical system: $\dot{x}(t) = f(x(t))$

- Initial condition $x_0 = x(0)$, solution at time t: $\phi(t, x_0)$.
- The ω-limit set of the system: $\Omega(f)$

Region of attraction (ROA) of a set $S \subseteq \Omega(f)$:

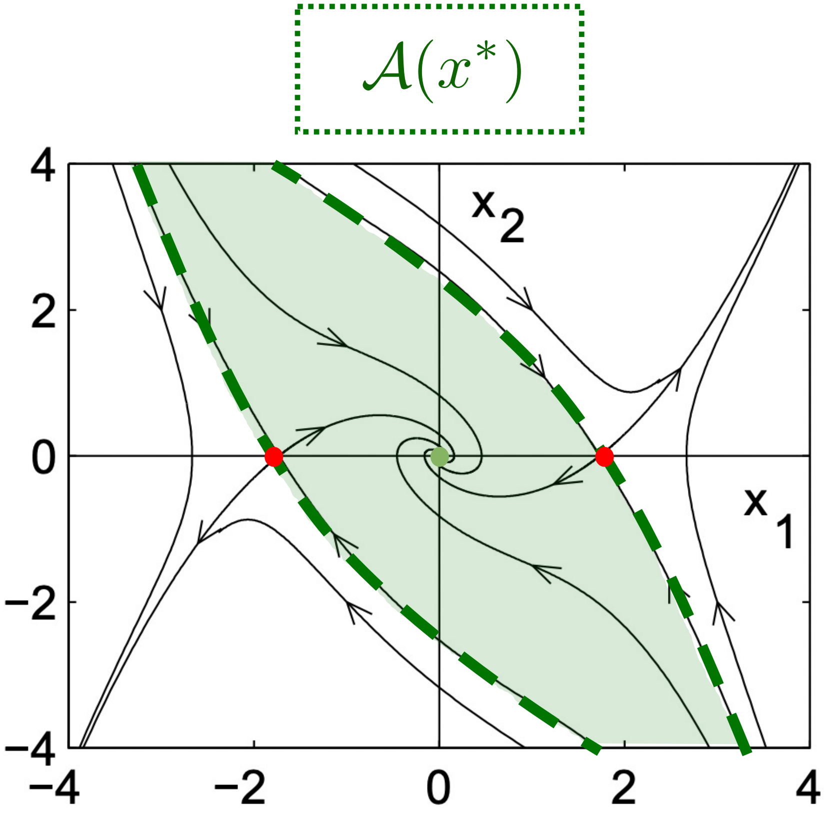
$$\mathcal{A}(S) := \left\{ x \in \mathbb{R}^d \middle| \liminf_{t\to\infty} d(\phi(t,x), S) = 0 \right\}$$

Illustrative Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 + \frac{1}{3}x_1^3 - x_2 \end{bmatrix}$$

$$\Omega(f) = \{(0,0), (-\sqrt{3}, 0), (\sqrt{3}, 0)\}$$

Asymptotically stable equilibrium at $x^* = (0,0)$



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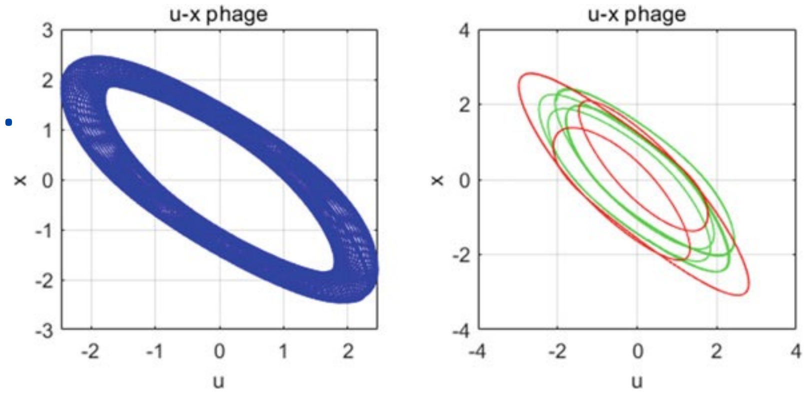
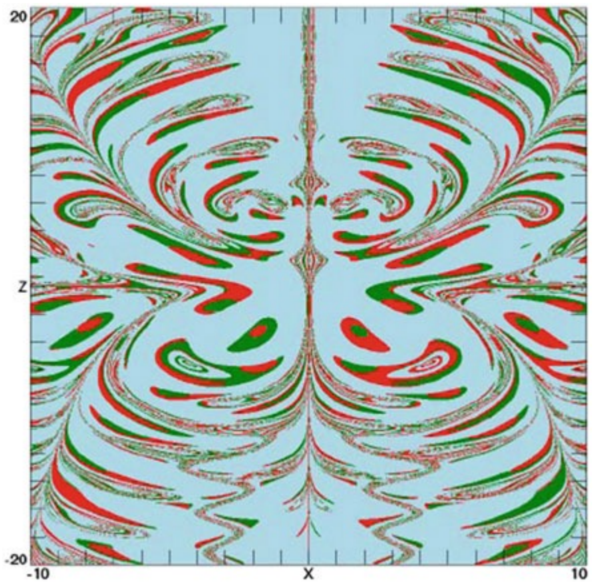
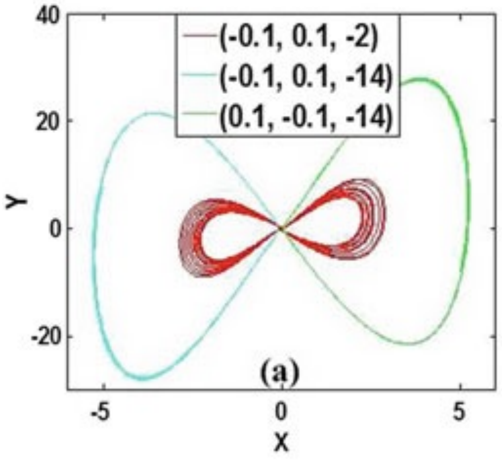
Example III: Limit set $\Omega(f)$



Basin of $\mathcal{A}(\Omega)$



Example II: Limit set $\Omega(f)$



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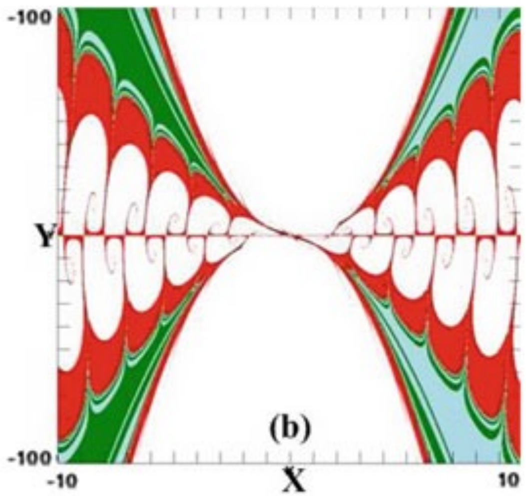


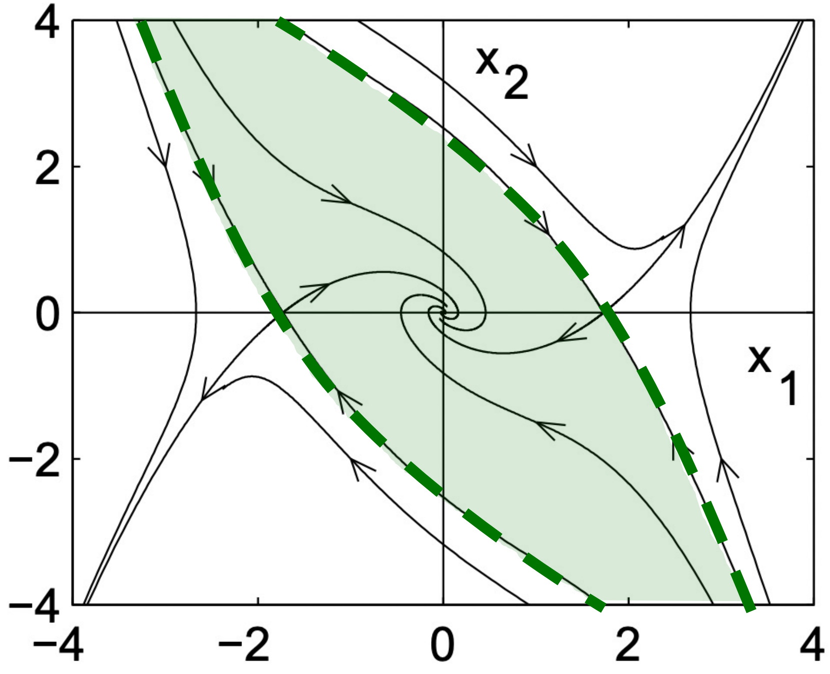
Region of attraction of stable equilibria

Region of attraction (ROA) of a set $S \subseteq \Omega(f)$:

$$\mathcal{A}(S) := \left\{ x_0 \in \mathbb{R}^d \middle| \lim_{t\to\infty} \phi(t, x_0) \in S \right\}$$

Assumption *1. The system $\dot{x}(t) = f(x(t))$ has an asymptotically stable equilibrium at x^*.*

***Remark**. It follows from Assumption 1 that the **positively invariant** ROA $\mathcal{A}(x^*)$ **is an open contractible set** [Sontag, 2013], i.e., the identity map of $\mathcal{A}(x^*)$ to itself is null-homotopic [Munkres, 2000].*



$\mathcal{A}(x^*)$:

E. Sontag. "Mathematical Control Theory: Deterministic Finite Dimensional Systems." Springer 2013

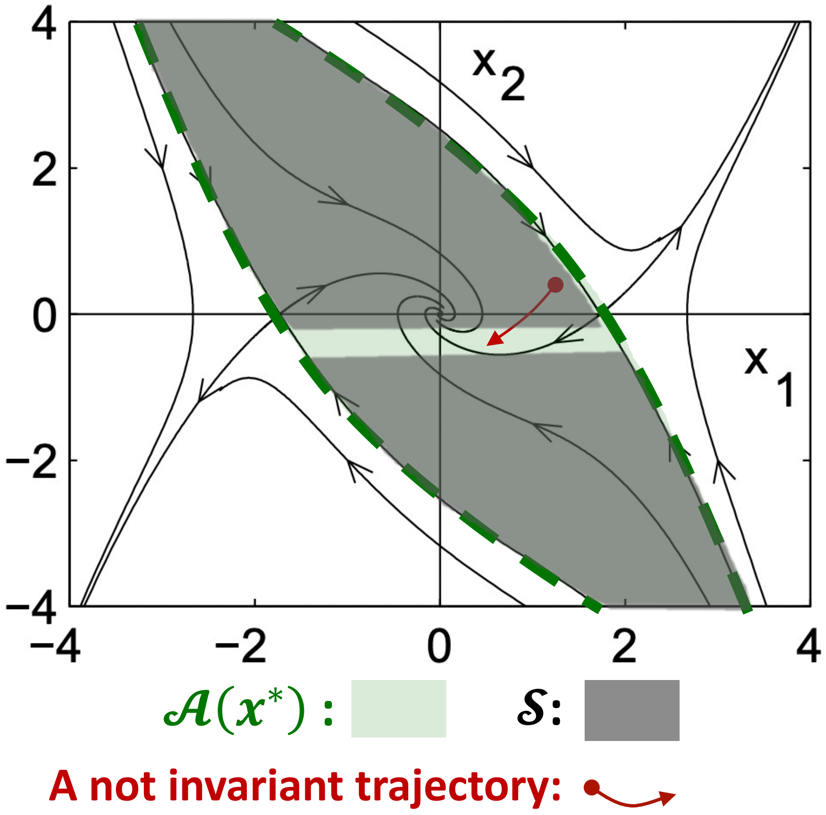
J. R. Munkres. "Topology." Prentice Hall 2000

Challenges of working with invariant set

Approximating ROA $\mathcal{A}(x^*)$ by finding an invariant set $\mathcal{S} \subseteq \mathcal{A}(x^*)$

- $\mathcal{S}$ is topologically constrained
  - If $\mathcal{S} \cap \Omega(f) = \{x^*\}$, then $\mathcal{S}$ is connected

Example 1: $\mathcal{S} \subseteq \mathcal{A}(x^*)$ is not connected, not invariant!

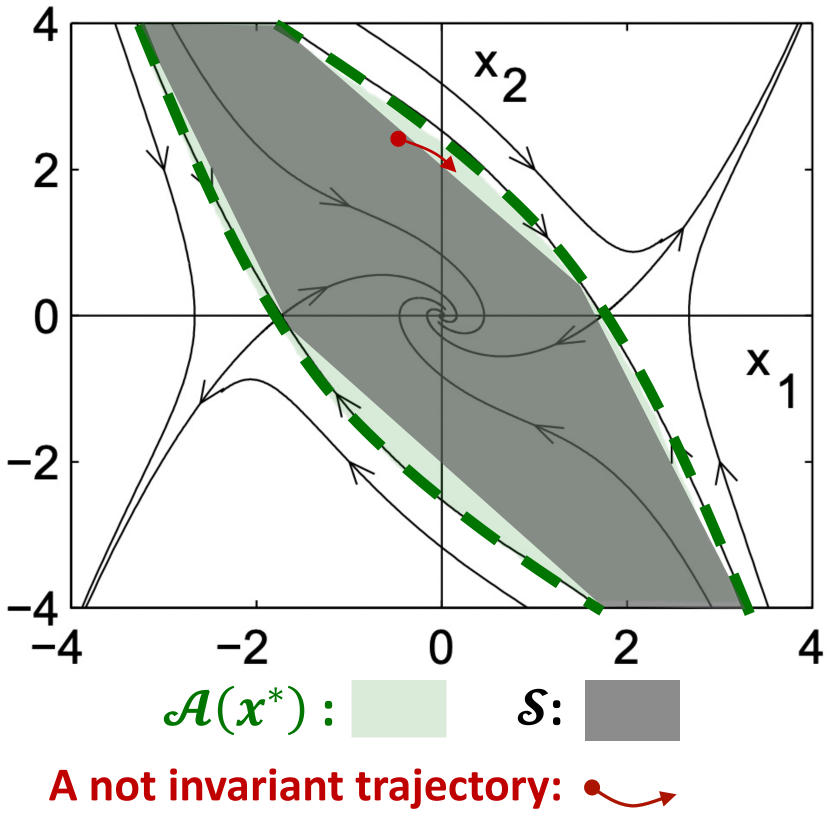


Challenges of working with invariant set

Approximating ROA $\mathcal{A}(x^*)$ by finding an invariant set $\mathcal{S} \subseteq \mathcal{A}(x^*)$

- $\mathcal{S}$ is topologically constrained
  - If $\mathcal{S} \cap \Omega(f) = \{x^*\}$, then $\mathcal{S}$ is connected
- $\mathcal{S}$ is geometrically constrained
  - f should not point outwards for $x \in \partial\mathcal{S}$

Example 2: $\mathcal{S} \subseteq \mathcal{A}(x^*)$, f points outward on $\partial\mathcal{S}$, not invariant



Challenges of working with invariant set

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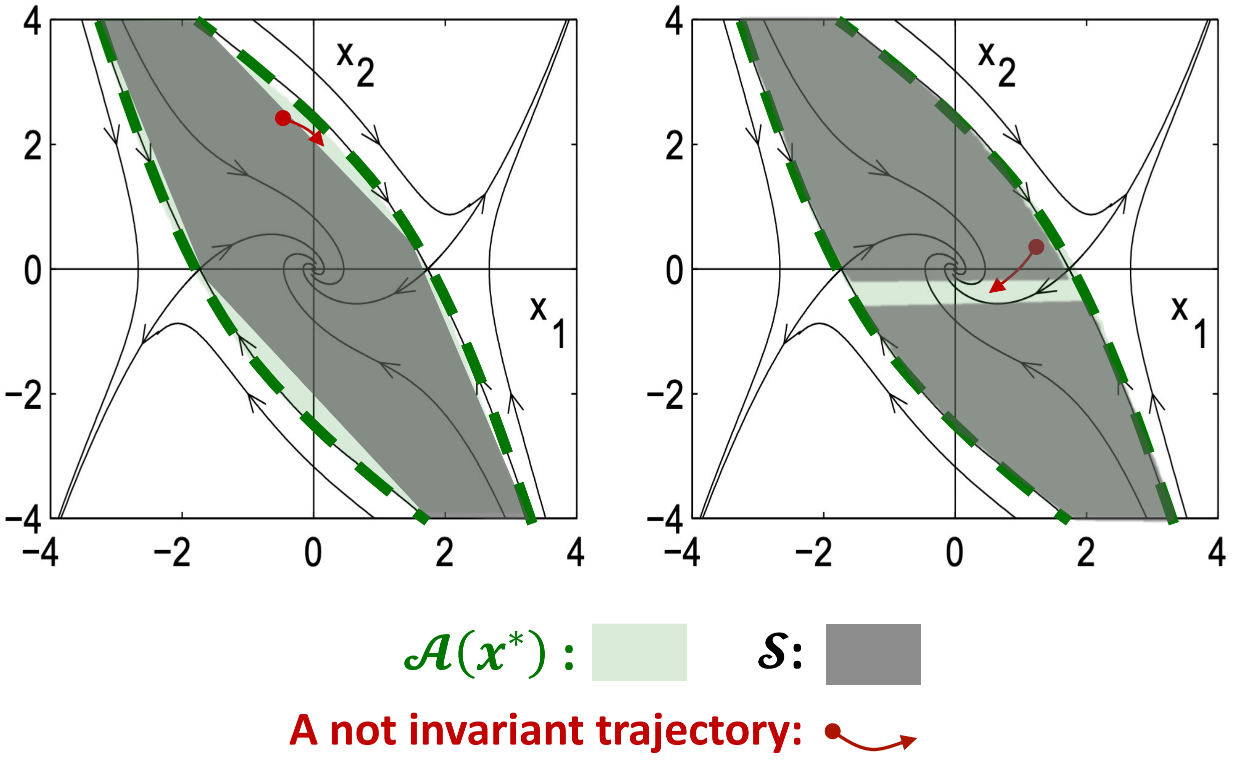
A subset or a superset of an invariant set is not necessarily an invariant set

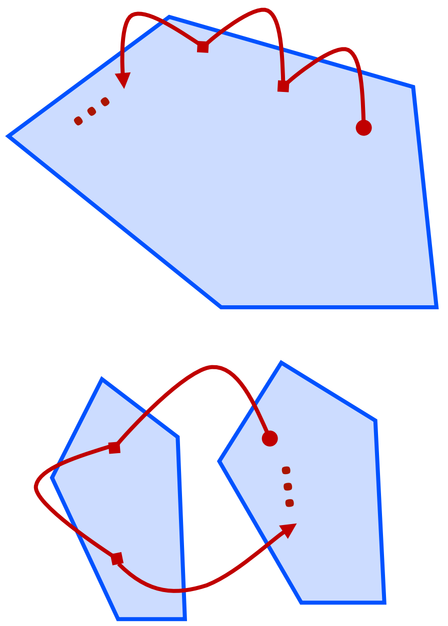


Recurrent sets: Letting things go, and come back

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \geq t$ s.t. $\phi(t', x_0) \in \mathcal{R}$.

Property of Recurrent Sets

- $\mathcal{R}$ need **not** be **connected**
- $\mathcal{R}$ does **not** require f to **point inwards** on all $\partial\mathcal{R}$



Recurrent sets, while not invariant, guarantee that solutions that start in this set, will come back **infinitely often, forever!**

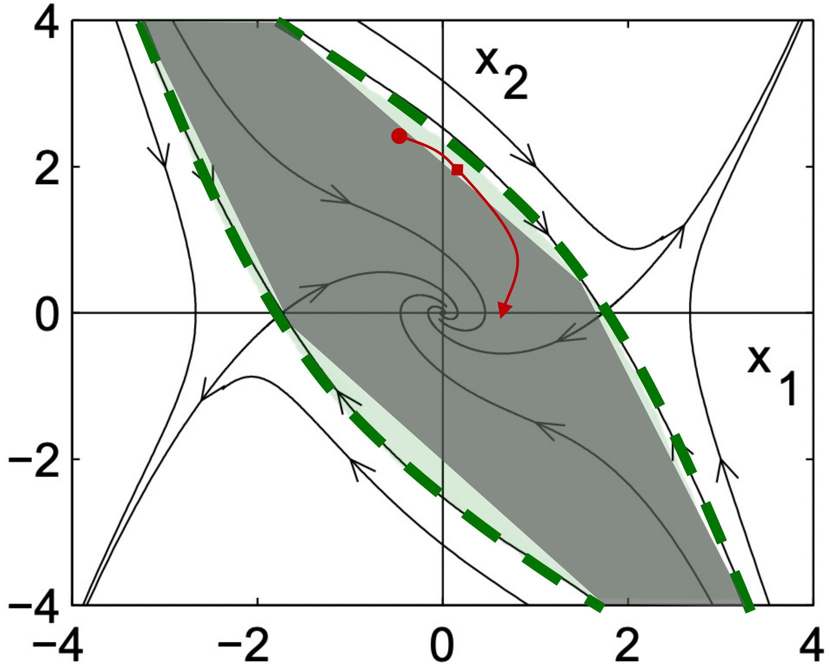
Recurrent set $\mathcal{R}$:

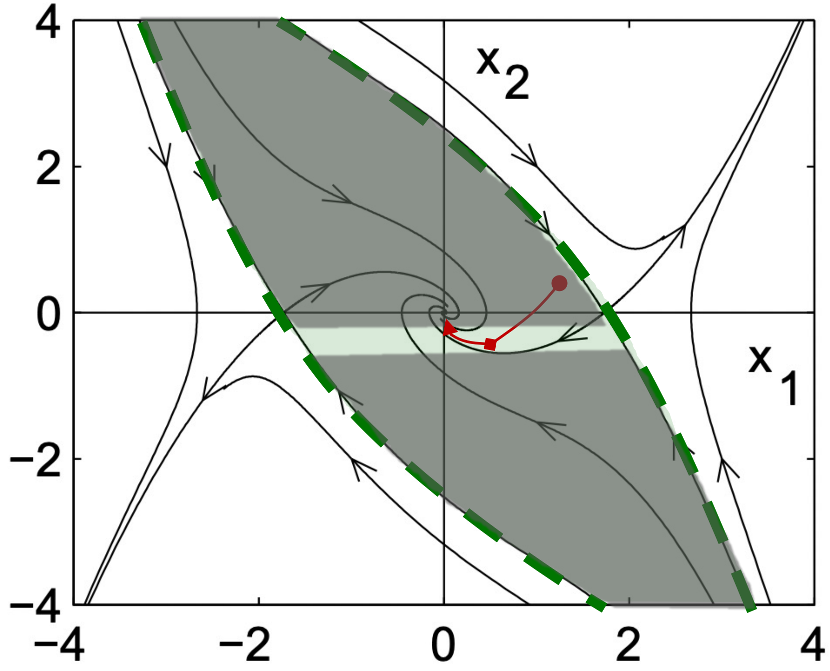
A recurrent trajectory:

Recurrent sets: Letting things go, and come back

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is recurrent if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \geq t$ s.t. $\phi(t', x_0) \in \mathcal{R}$.

Previous two good inner approximations of $\mathcal{A}(x^*)$ are recurrent sets





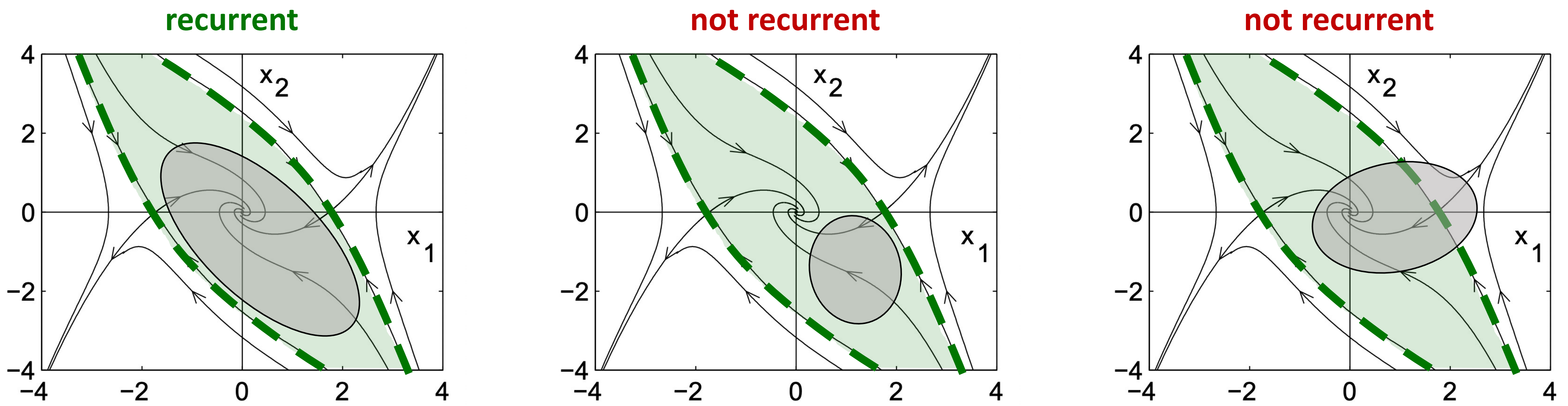
Recurrent sets are subsets of the region of attraction

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \geq t$ s.t. $\phi(t', x_0) \in \mathcal{R}$.

Theorem. *Let $\mathcal{R} \subset \mathbb{R}^d$ be a* <u>*compact*</u> *set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$.*

Then:

$\mathcal{R}$ *is recurrent* ⬌ $\mathcal{R} \cap \Omega(f) \neq \emptyset$, $\mathcal{R} \subset \mathcal{A}(\mathcal{R} \cap \Omega(f))$



$\mathcal{R}$:

$\mathcal{A}(x^*)$:

Recurrent sets are subsets of the region of attraction

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0, \exists t' \geq t$ s.t. $\phi(t', x_0) \in \mathcal{R}$.

***Theorem**. Let $\mathcal{R} \subset \mathbb{R}^d$ be a compact set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$.*

Then:

$\mathcal{R}$ *is recurrent* ⬌ $\mathcal{R} \cap \Omega(f) \neq \emptyset$, $\mathcal{R} \subset \mathcal{A}(\mathcal{R} \cap \Omega(f))$

***Proof**: [Sketch]*

(⟹)
- *If $x_0 \in \mathcal{R}$, the solution $\phi(t, x_0)$ visits $\mathcal{R}$ infinitely often, forever.*
- *We can build a sequence $\{x(t_n)\}_{n=0}^{\infty} \in \mathcal{R}$ with $\lim_{n \to +\infty} t_n = +\infty$*
- *Bolzano-Weierstrass $\implies$ convergent subsequence $x(t_{n_i}) \to \bar{\boldsymbol{x}} \in \Omega(f) \cap \mathcal{R} \neq \emptyset$*
- *$\partial\mathcal{R} \cap \Omega(f) = \emptyset$ + $\mathcal{R}$ recurrent $\implies$ $\phi(t, x_0)$ leaves $\mathcal{R}$ finitely many times*
- *$\mathcal{R}$ is eventually invariant*

(⟸)
- *Trivial*

Recurrent sets are subsets of the region of attraction

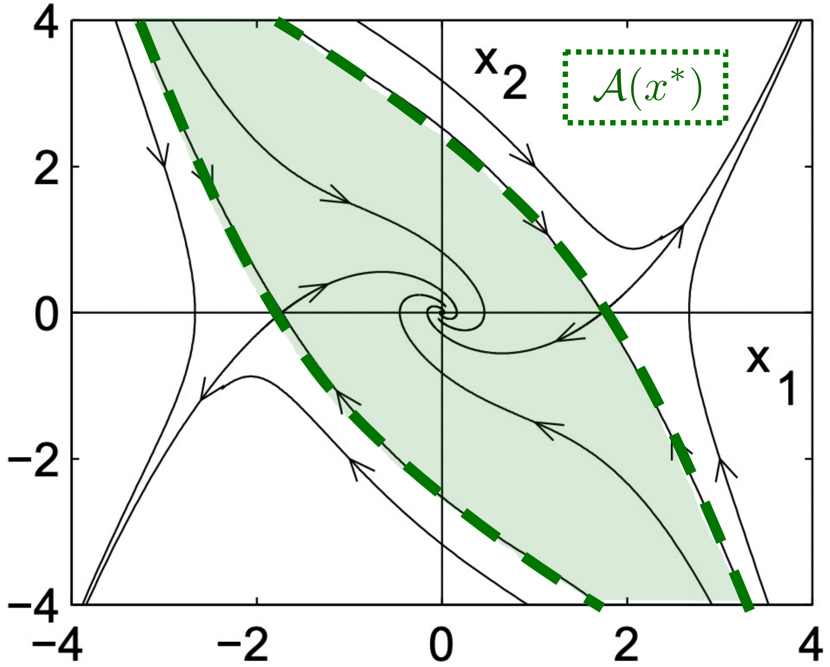
A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \geq t$ s.t. $\phi(t', x_0) \in \mathcal{R}$. **?**

Corollary. *Let Assumption 1 hold, and let $\mathcal{R} \subset \mathbb{R}^d$ be a compact set satisfying:*

$$\partial\mathcal{R} \cap \Omega(f) = \emptyset \text{ and } \mathcal{R} \cap \Omega(f) = \{x^*\}$$

Then:

$\mathcal{R}$ *is recurrent* $\Longleftrightarrow$ $\mathcal{R} \subset \mathcal{A}(x^*)$



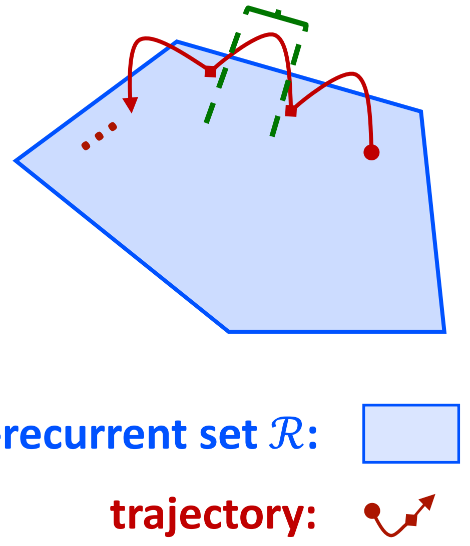
Idea: Use recurrence as a mechanism for finding inner approximations of $\mathcal{A}(x^*)$

Potential Issues:

- We do not know how long it takes to come back!
- We need to adapt results to trajectory samples

τ-recurrent sets

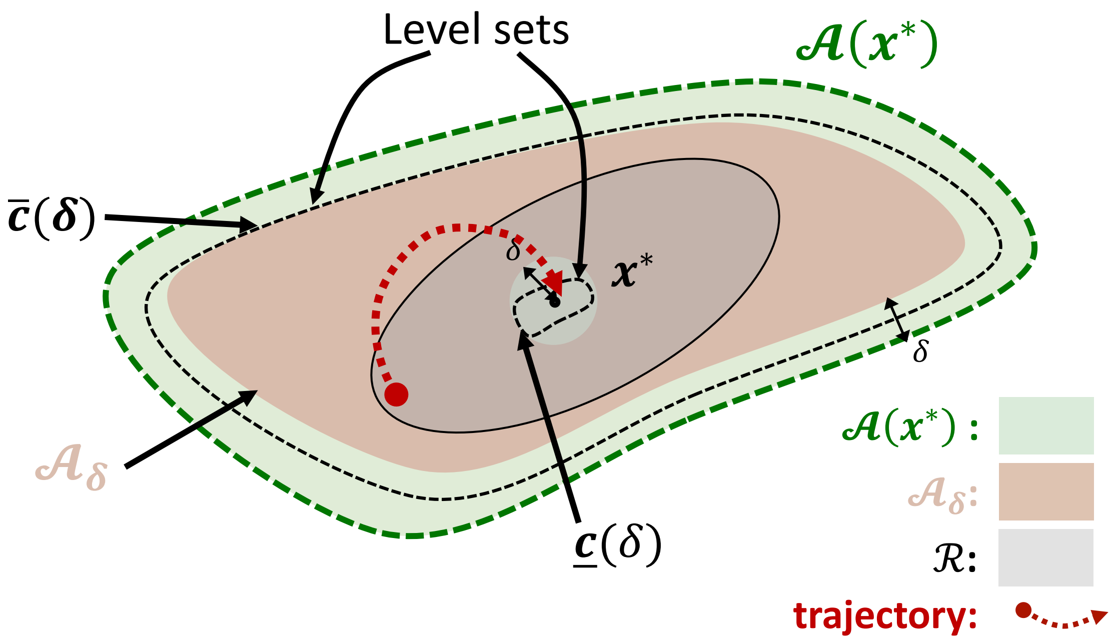
A set $\mathcal{R}$ is **τ-recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists\, t' \in [t, t+\tau]$ such that $\phi(t', x_0) \in \mathcal{R}$



***Theorem**. Under Assumption 1, any compact set $\mathcal{R}$* satisfying:

$$x^* + \mathcal{B}_\delta \subseteq \mathcal{R} \subseteq \mathcal{A}(x^*) \backslash \{\partial\mathcal{A}(x^*) + \text{int}\, \mathcal{B}_\delta\}$$

is τ-recurrent for $\tau \geq \bar{\tau}(\delta) := \frac{\underline{c}(\delta) - \bar{c}(\delta)}{a(\delta)}$.



Recurrent sets are subsets of the region of attraction

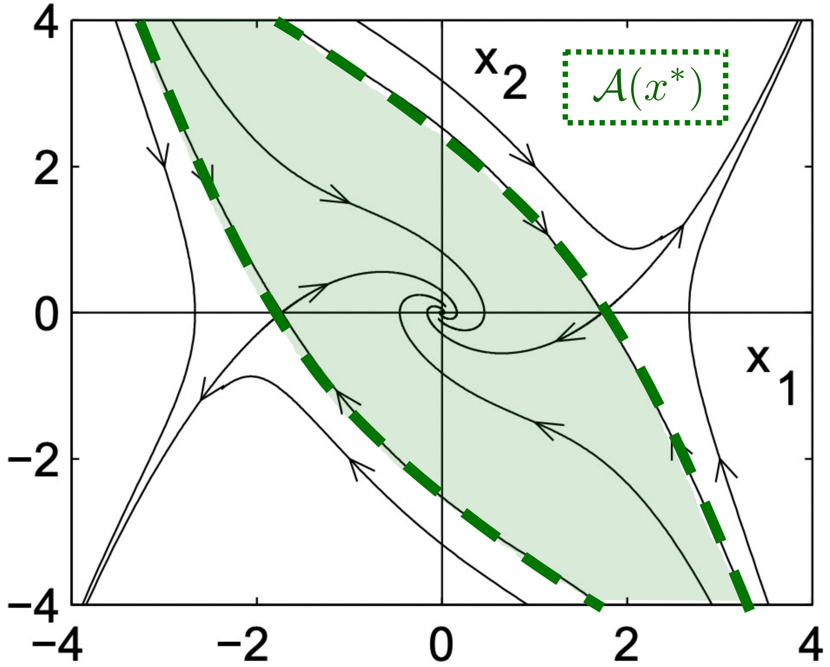
A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for $x_0 \in \mathcal{R}$, for any t $\geq 0 \Rightarrow \exists t' > t,$ s.t. $\phi(t', x_0) \in \mathcal{R}$ **?**

***Corollary**. Let Assumption 1 hold, and let $\mathcal{R} \subset \mathbb{R}^d$ be a <u>compact</u> set satisfying:*

$$\partial\mathcal{R} \cap \Omega(f) = \emptyset \text{ and } \mathcal{R} \cap \Omega(f) = \{x^*\}$$

Then:

$\mathcal{R}$ is recurrent ⬌ $\mathcal{R} \subset \mathcal{A}(x^*)$



Idea: Use recurrence as a mechanism for finding inner approximations of $\mathcal{A}(x^*)$

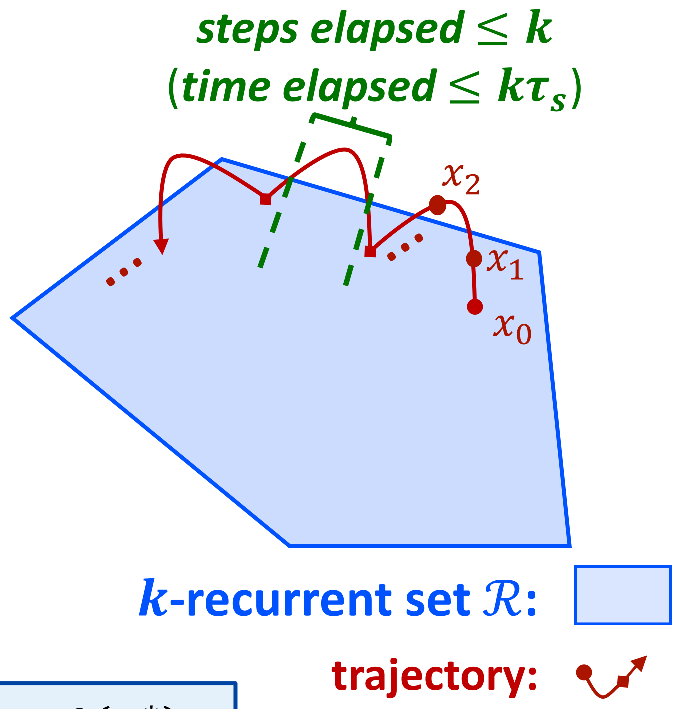
Potential Issues:

- We do not know how long it takes to come back! ✔
- We need to adapt results to trajectory samples

Learning recurrent sets from k-length trajectory samples

- Consider **finite length** trajectories:
  $$x_n = \phi(n\tau_s, x_0), \qquad x_0 \in \mathbb{R}^d, n \in \mathbb{N},$$
  where $\tau_s > 0$ is the sampling period.

- A set $\mathcal{R} \subseteq \mathbb{R}^d$ is k-**recurrent** if whenever $x_0 \in \mathcal{R}$, then $\exists\, n \in \{1, \dots, k\}$ s.t. $x_n \in \mathcal{R}$



Sufficiency:

$\mathcal{R}$ is k-recurrent ➡ $\mathcal{R}$ is τ-recurrent with $\tau = k\tau_s$ ✚ $\mathcal{R}$ is compact, $\mathcal{R} \cap \Omega(f) = \{x^*\}$, $x^* \in \text{int}\,\mathcal{R}$ ➡ $\mathcal{R} \subset \mathcal{A}(x^*)$

Necessity:

Theorem *3. Under Assumption 1, any compact set* $\mathcal{R}$ satisfying:
$$\mathcal{B}_\delta + x^* \subseteq \mathcal{R} \subseteq \mathcal{A}(x^*) \backslash \{\partial\mathcal{A}(x^*) + \text{int}\,\mathcal{B}_\delta\}$$
is k*-recurrent for any* $k > \bar{k} := \bar{\tau}(\delta)/\tau_s$.

Recurrent sets are subsets of the region of attraction

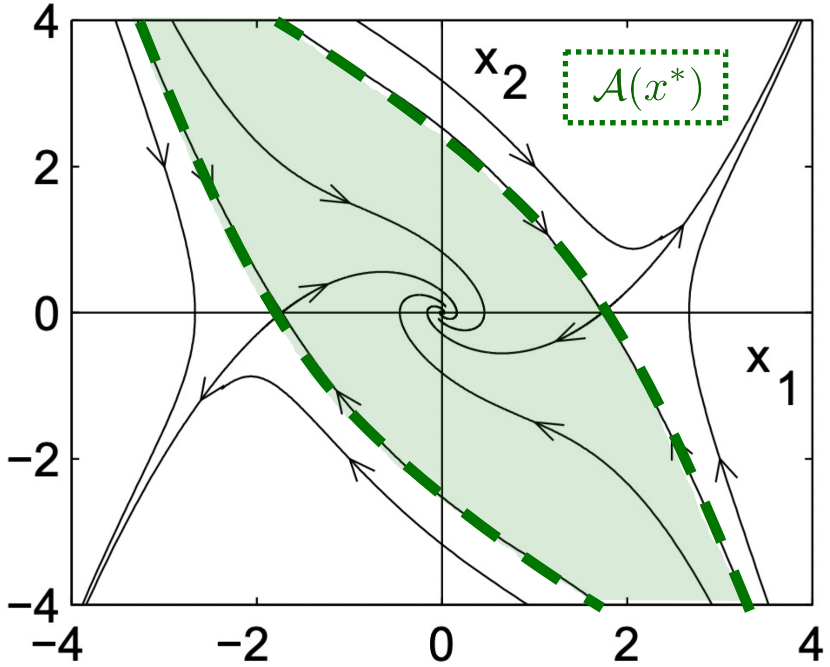
A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for $x_0 \in \mathcal{R}$, $\phi(t, x_0) \notin \mathcal{R} \Rightarrow \exists t' > t$, s.t. $\phi(t', x_0) \in \mathcal{R}$ **?**

Corollary. *Let Assumption 1 hold, and let $\mathcal{R} \subset \mathbb{R}^d$ be a compact set satisfying:*

$$\partial\mathcal{R} \cap \Omega(f) = \emptyset \text{ and } \mathcal{R} \cap \Omega(f) = \{x^*\}$$

Then:

$\mathcal{R}$ *is recurrent* ⬌ $\mathcal{R} \subset \mathcal{A}(x^*)$



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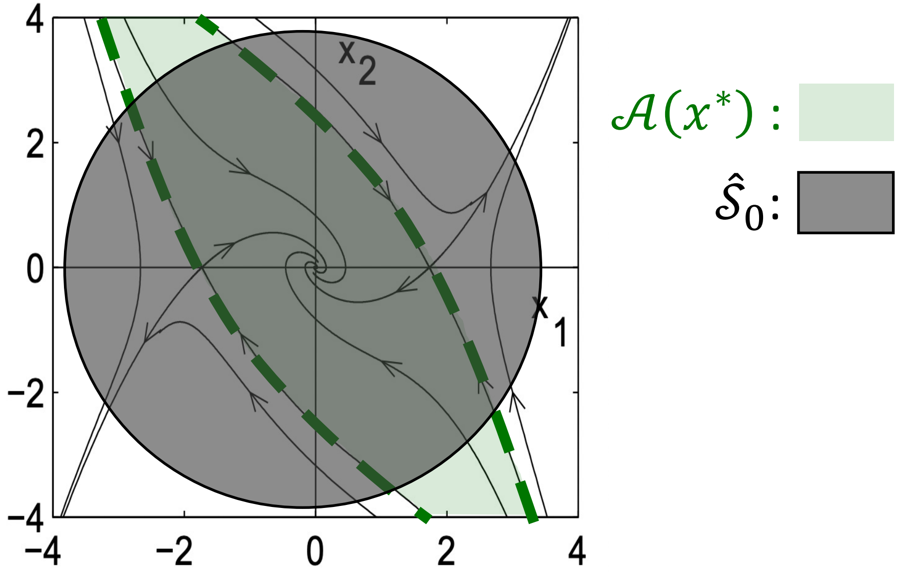
Sphere approximations of RoA

Algorithm: Given k and $\varepsilon > 0$:

At each iteration l

- *Sample trajectories of length k from the sphere $\hat{\mathcal{S}}_l$ until recurrence is violated (counter-example)*



$$\hat{\mathcal{S}}_l := \{x \mid \|x - x^*\|_2 \leq b_l\}$$

Sphere approximations of RoA

Algorithm: Given k and $\varepsilon > 0$:

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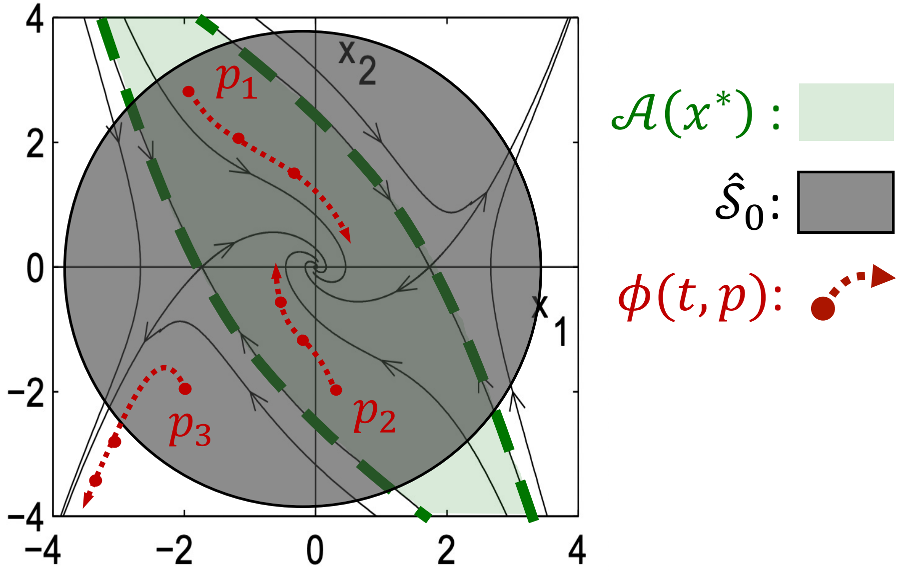
- *Sample trajectories of length k from the sphere $\hat{\mathcal{S}}_t$ until recurrence is violated (counter-example)*

$l = 0$



p_3: counter example

$$\hat{\mathcal{S}}_l := \{x \,|\, \|x - x^*\|_2 \leq b_l\}$$

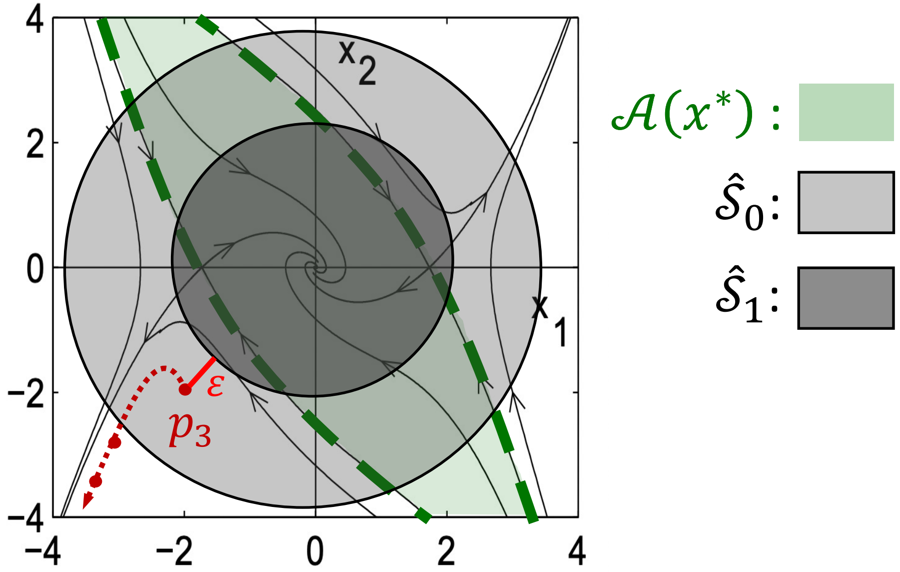
Sphere approximations of RoA

Algorithm: Given k and $\varepsilon > 0$:

At each iteration l

- *Sample trajectories of length k from the sphere $\hat{\mathcal{S}}_t$ until recurrence is violated (counter-example)*
- *Update sphere $\hat{\mathcal{S}}_{l+1}$ to exclude counter example point p_j*

$$l = 0$$



p_3: counter example

$$\hat{\mathcal{S}}_l \coloneqq \{x \,|\, \|x - x^*\|_2 \leq b_l\}$$

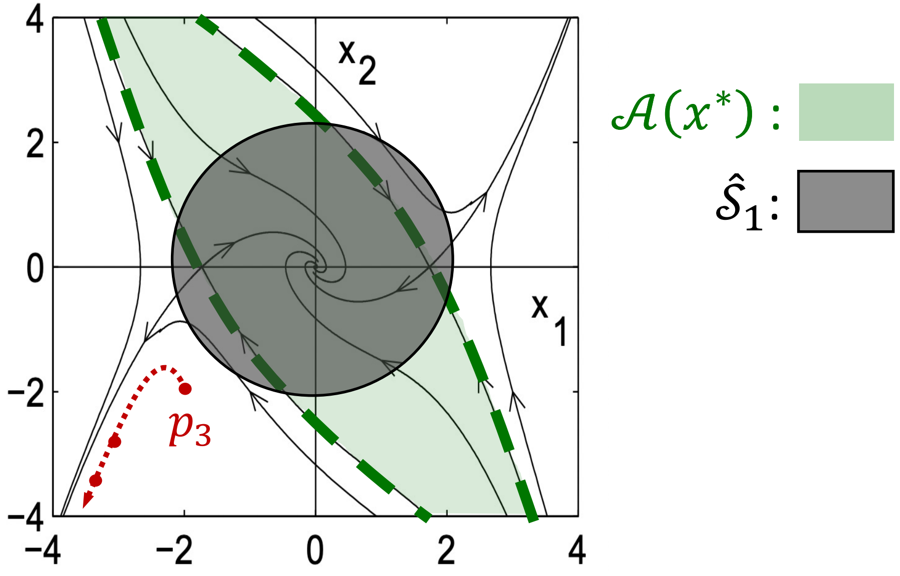
Sphere approximations of RoA

Algorithm: Given k and $\varepsilon > 0$:

At each iteration l

- *Sample trajectories of length k from the sphere $\hat{\mathcal{S}}_t$ until recurrence is violated (counter-example)*
- *Update sphere $\hat{\mathcal{S}}_{l+1}$ to exclude counter example point p_j, and start again*

$l = 1$



p_3: counter example

Sample complexity:

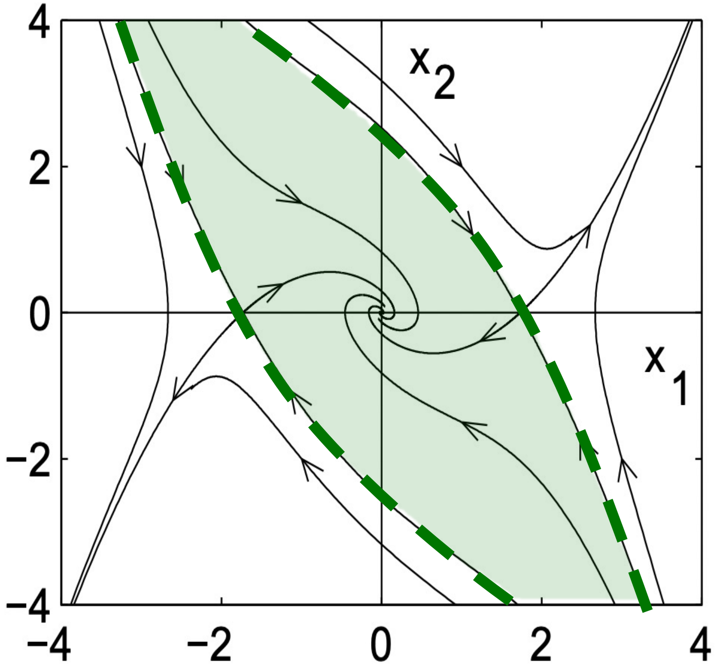
$$m \geq \frac{\mathrm{V}\left(\hat{\mathcal{S}}_l + \mathrm{B}_\eta\right)}{\mathrm{V}(\mathrm{B}_\eta)} \log\left(\frac{1}{\rho}\right)$$

failure probability

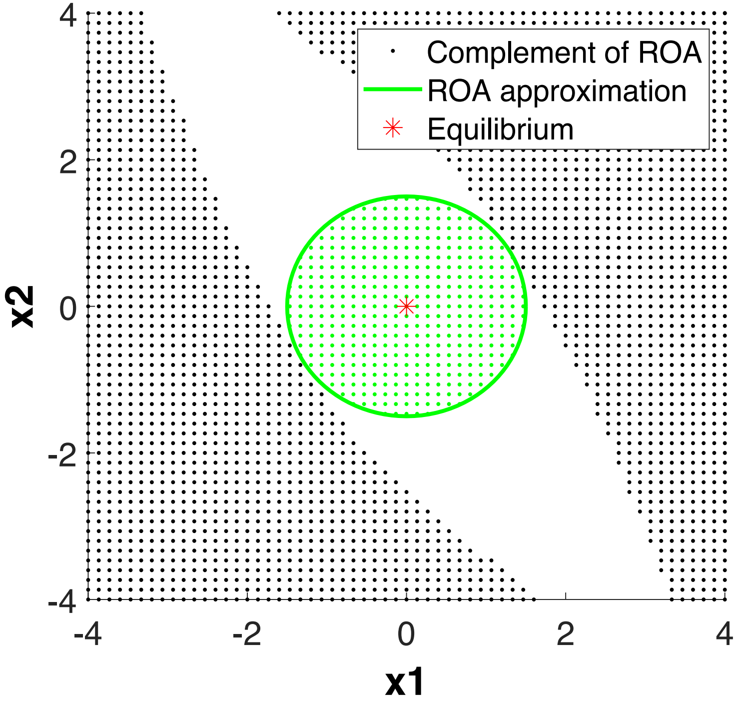
$\hat{\mathcal{S}}_l \coloneqq \{x \mid \|x - x^*\|_2 \leq b_l\}$

* requires stricter notion of η-strict τ-recurrence

Algorithm Result - Sphere Approximations

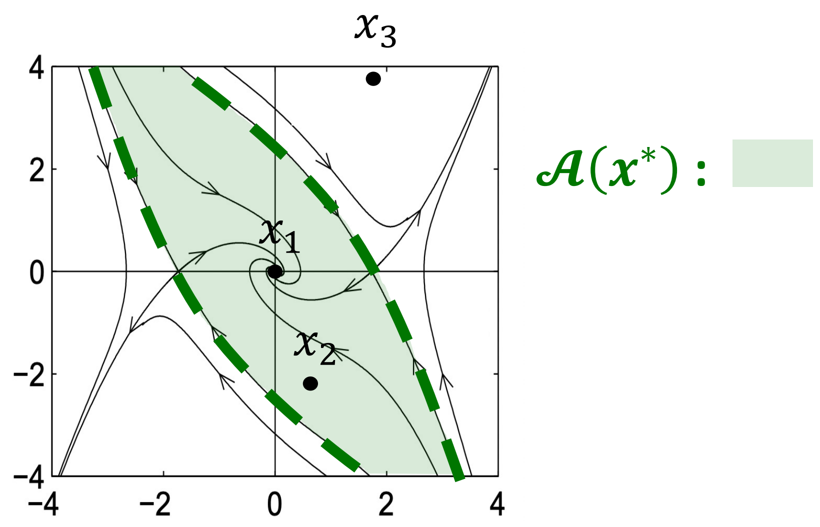


$\mathcal{A}(0)$:



Multi-center approximation

- **Consider $\boldsymbol{h} \in \mathbb{N}^+$ center points $\boldsymbol{x_q}$ indexed by $\boldsymbol{q} \in \{\mathbf{1}, \dots, \boldsymbol{h}\}$.**
  - Let the first center point $x_1 = x^* = 0$
  - Additional center point $x_2, \dots, x_h$ can be designed chosen uniformly.



Multi-center approximation

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  - Let the first center point $x_1 = x^* = 0$
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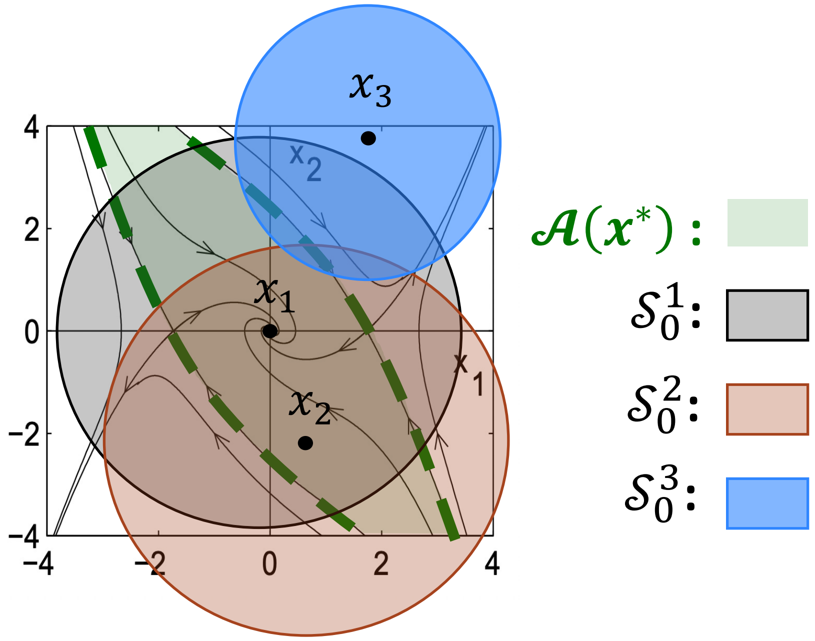
- **Respectively define approximations centered at each x_q**
  - $\mathcal{S}_l^q := \{x | \|x - x_q\|_2 \leq b_q^l\}$



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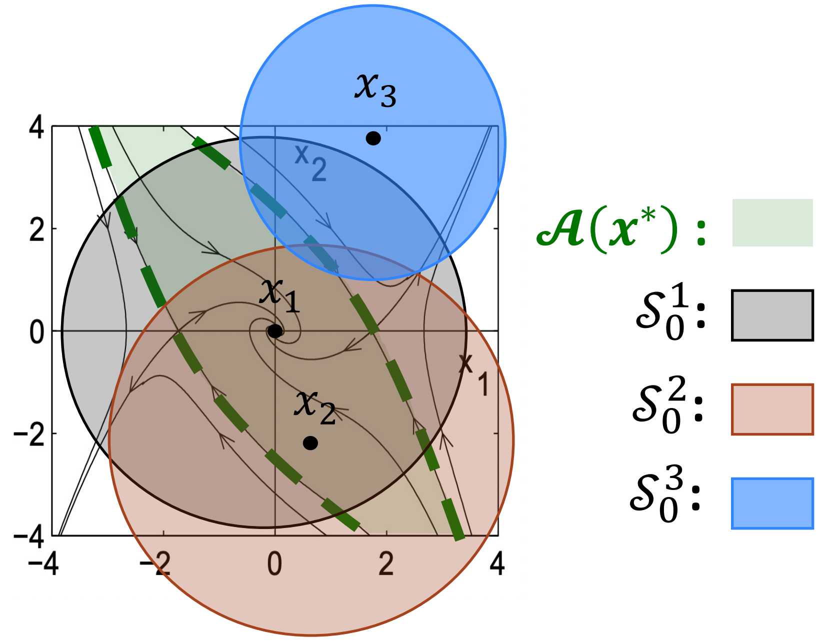
  - $\mathcal{S}_l^q := \{x | \|x - x_q\|_2 \leq b_q^l\}$

- **Multi-center approximation given by** $\hat{\mathcal{S}}_l = \cup_{q=1}^{h} \mathcal{S}_l^q$



Multi-center approximation

- **Consider $\boldsymbol{h} \in \mathbb{N}^+$ center points $\boldsymbol{x_q}$ indexed by $\boldsymbol{q} \in \{\mathbf{1}, \dots, \boldsymbol{h}\}$.**
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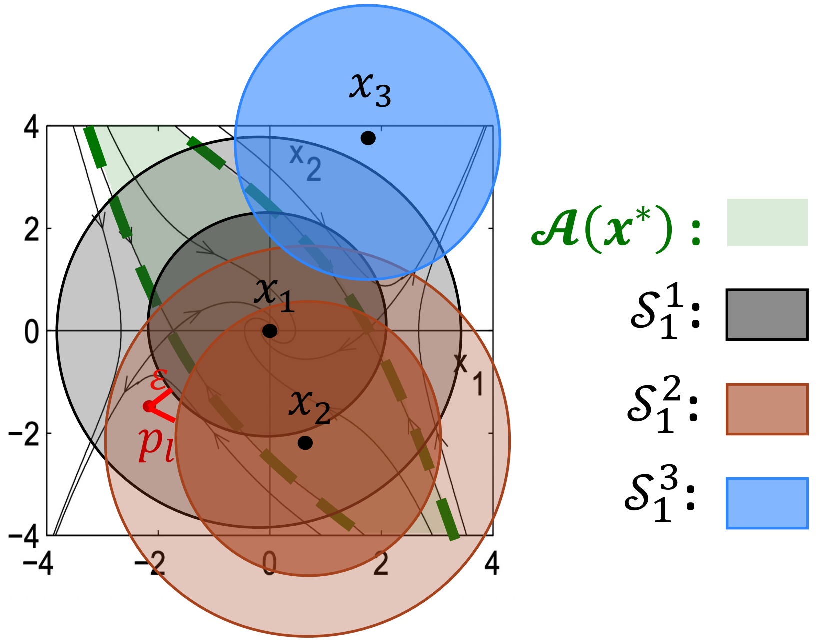
  - $\mathcal{S}_l^q := \{x \mid \|x - x_q\|_2 \le b_q^l\}$

- **Multi-center approximation given by** $\hat{\mathcal{S}}_l = \cup_{q=1}^{h} \mathcal{S}_l^q$

- **If $\boldsymbol{p_l}$ is a counter-example w.r.t** $\hat{\mathcal{S}}_l$
  - We shrink every $\hat{\mathcal{S}}_q^l$ satisfying $p_l \in \hat{\mathcal{S}}_q^l$
  - For the rest approximations, we simply let $\hat{\mathcal{S}}_q^{l+1} = \hat{\mathcal{S}}_q^l$

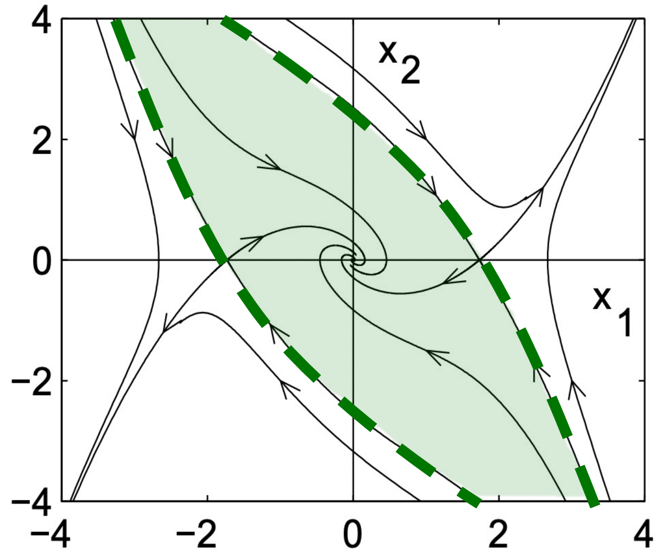
Sample complexity: $m \ge \frac{\mathrm{V}(\hat{\mathcal{S}}_l + \mathrm{B}_\eta)}{\mathrm{V}(\mathrm{B}_\eta)} \log\left(\frac{1}{\rho}\right)$

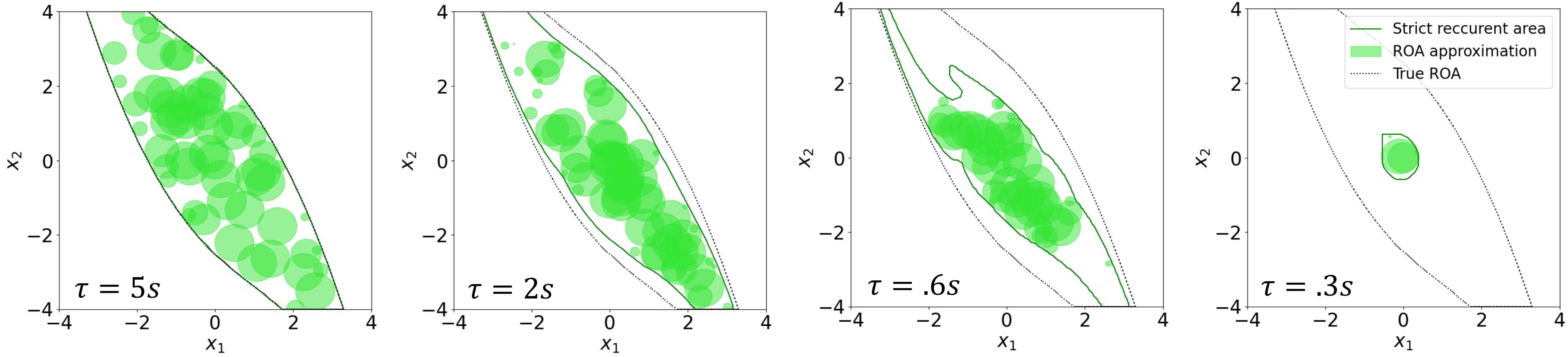


Numerical illustrations

- **Run:** 200 center points sampled (uniformly)
- **Stopping criteria:** $\rho = 10^{-5}$

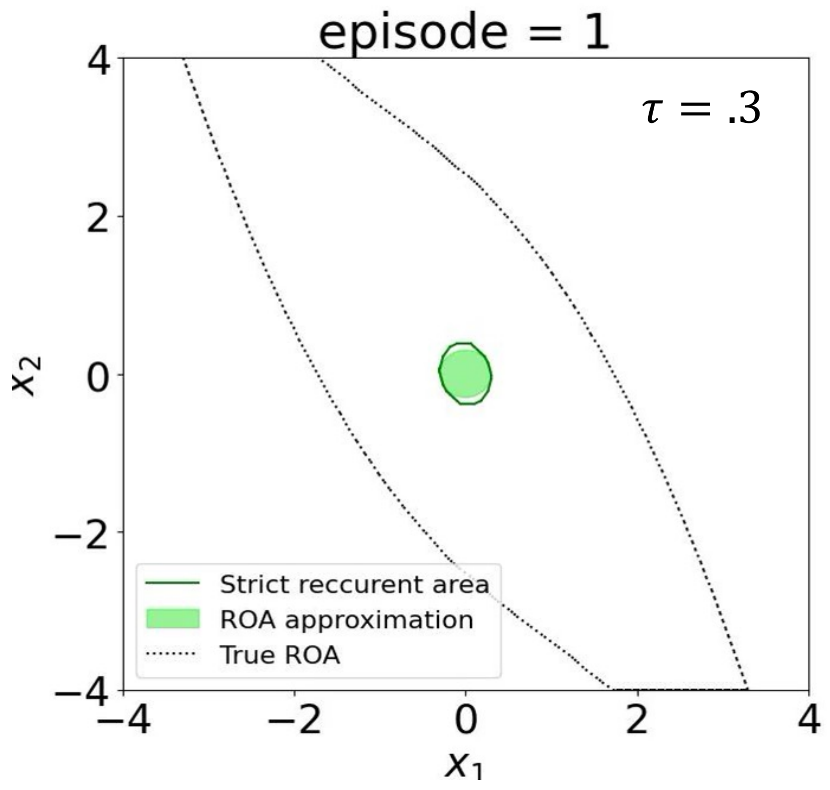
τ (s)	Running time	Volume %
5	57.7	72.0%
2	55.8	51.2%
.6	47.1	31.2%
.3	28.7	3.24%

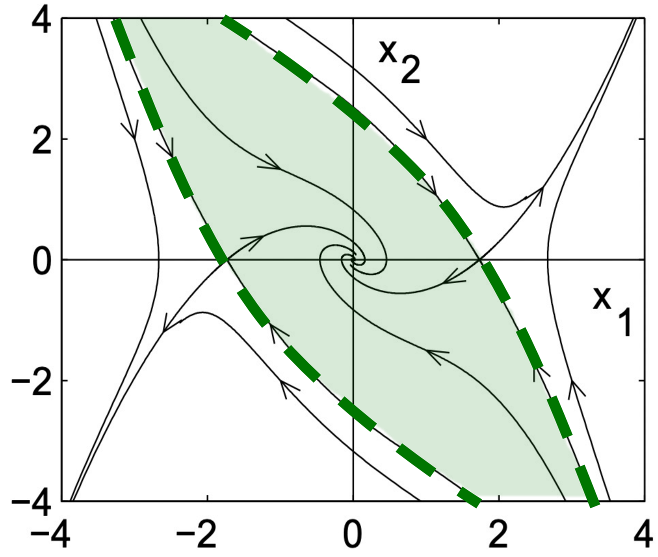


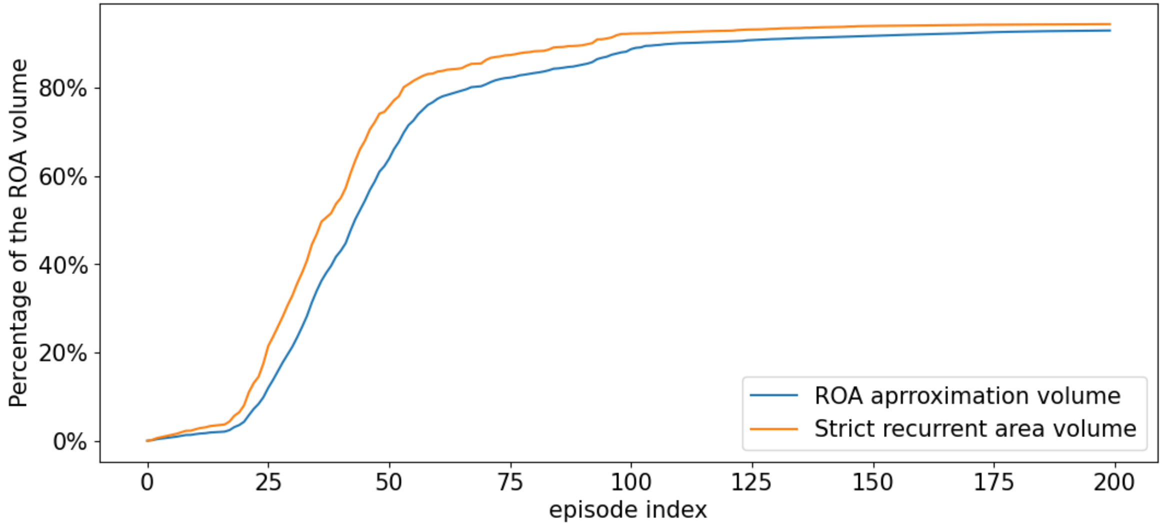


Example: Progressively Expanding the RoA

- At Each Episode:
  - **Sample 50** center points (uniformly)
  - **Stopping criteria:** $\delta = 10^{-5}$

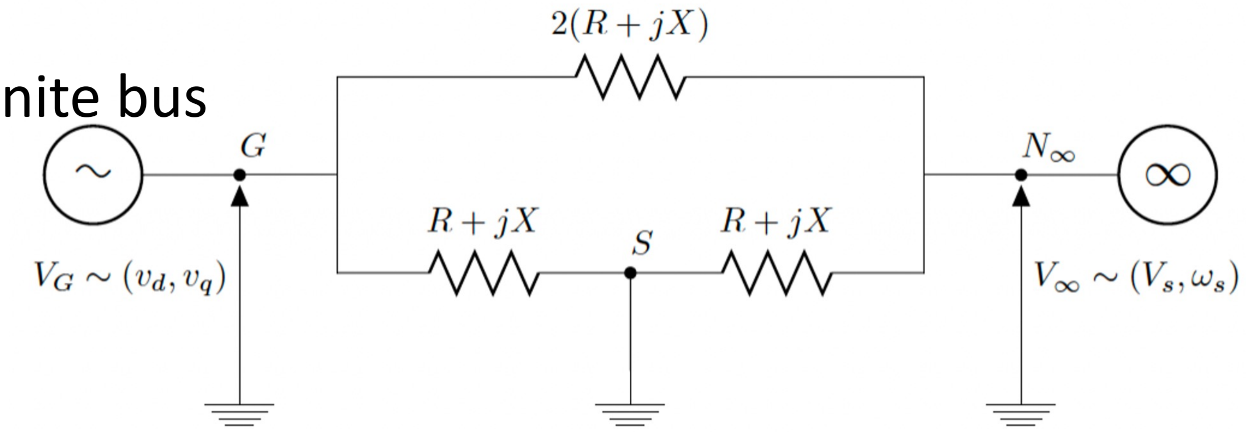






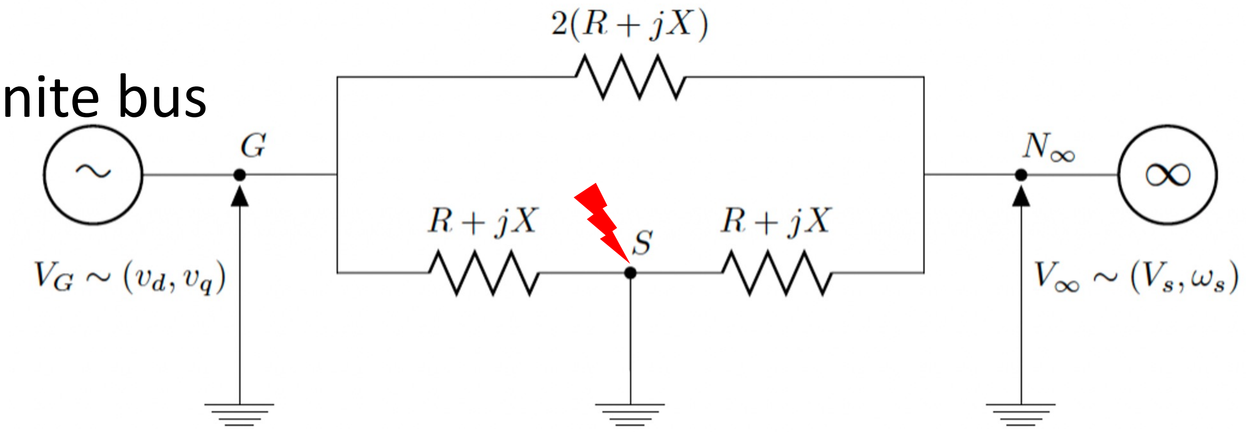
Transient Stability Analysis

- Synchronous machine connected to infinite bus



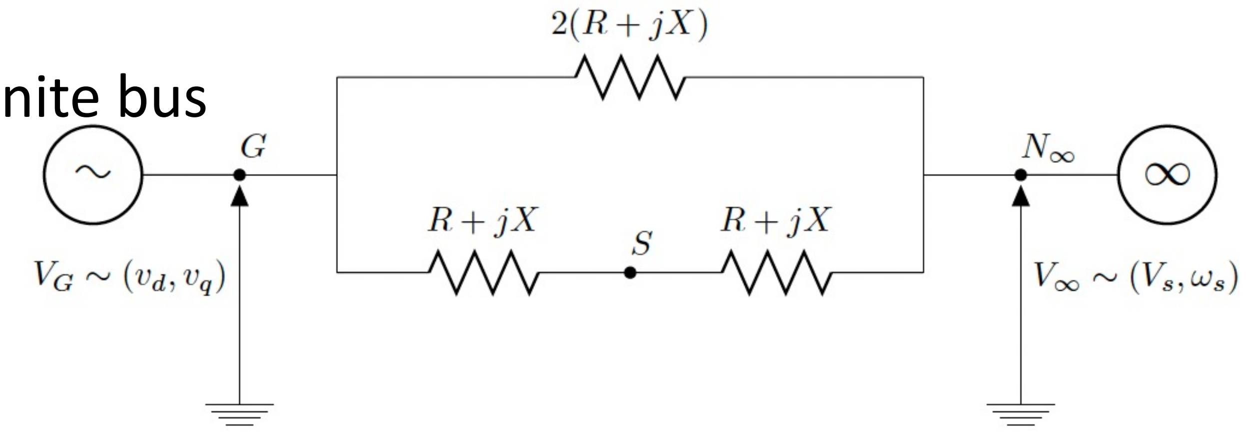
Transient Stability Analysis

- Synchronous machine connected to infinite bus
- t_1 lower line is short-circuited



Transient Stability Analysis

- Synchronous machine connected to infinite bus
- t_1 lower line is short-circuited
- t_2 fault is cleared



Transient Stability Analysis

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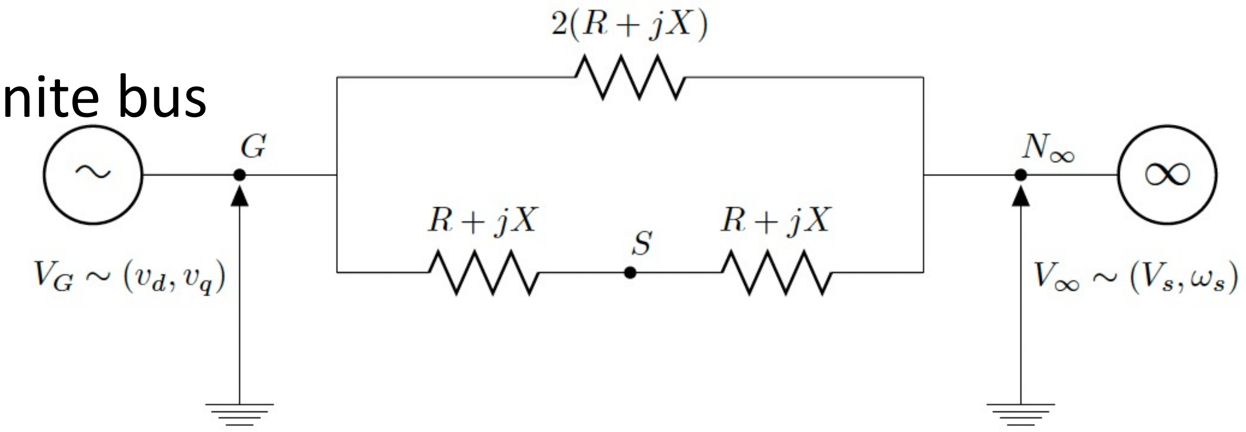


$$\frac{d\delta}{dt} = \omega - \omega_s$$

$$2H\frac{d\omega}{dt} = P_m - (v_d i_d + v_q i_q + e i_d^2 + r i_q^2)$$

$$T'_{d_0}\frac{de'_q}{dt} = -e'_q - (x_d - x'_d)i_d + E_{fd}$$

$$T_a\frac{dE_{fd}}{dt} = -E_{fd} + K_a(V_{ref} - V_t)$$

$$T_g\frac{dP_m}{dt} = -P_m + P_{ref} + K_g(\omega_{ref} - \omega)$$

$$i_q = \frac{(X - x'_d)V_s\sin(\delta) - (R+r)(V_s\cos(\delta) - e'_q)}{(R+r)^2 + (X + x'_d)(X + x_q)}$$

$$i_d = \frac{X - x_q}{R + r}i_q - \frac{1}{R+r}V_s\sin(\delta)$$

$$v_d = x_q i_q - r - i_d$$

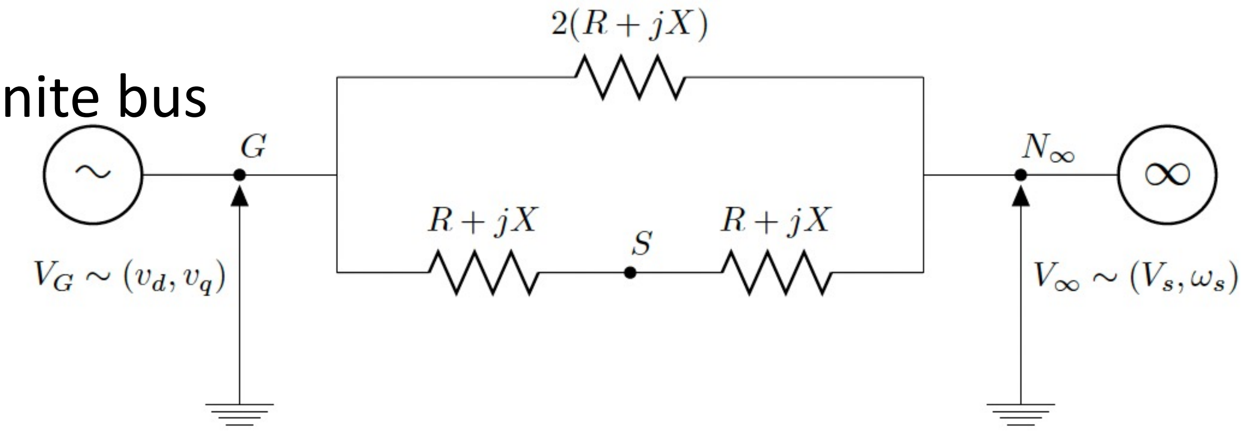
$$v_q = R i_q + X i_d + V_s\cos(\delta)$$

$$V_t = \sqrt{v_d^2 + v_q^2}$$

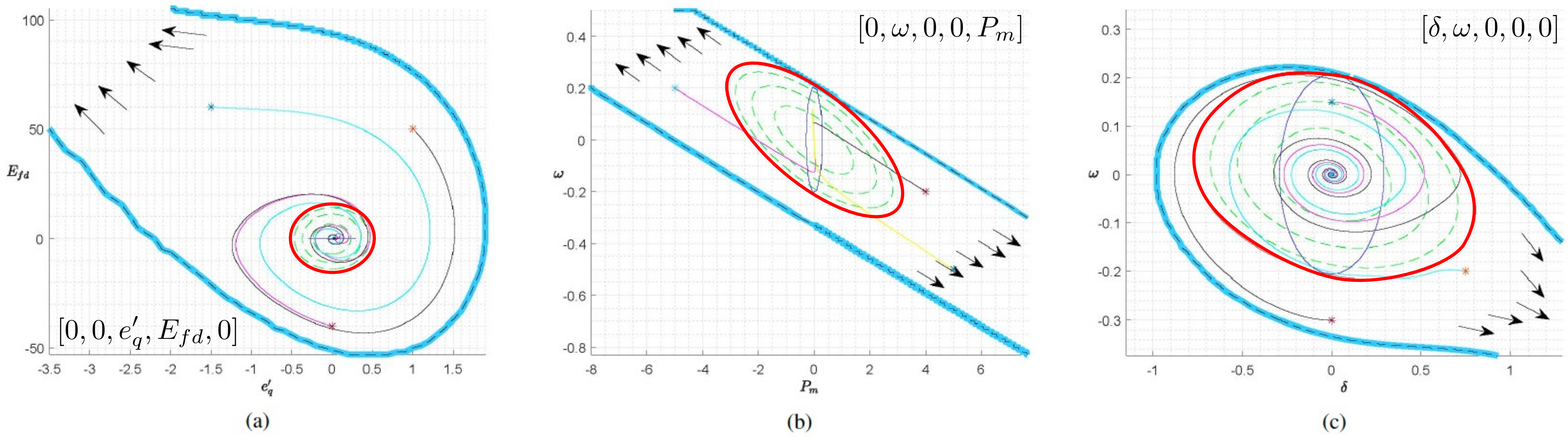
$T'_{d_0} = 9.67$	$x_d = 2.38$	$x'_d = 0.336$	$x_q = 1.21$
$H = 3$	$r = 0.002$	$\omega_s = \omega_{ref} = 1$	$R = 0.01$
$X = 1.185$	$V_s = 1$	$T_a = 1$	$K_a = 70$
$V_{ref} = 1$	$T_g = 0.4$	$K_g = 0.5$	$P_{ref} = 0.7$

Transient Stability Analysis

- Synchronous machine connected to infinite bus
- t_1 lower line is short-circuited
- t_2 fault is cleared



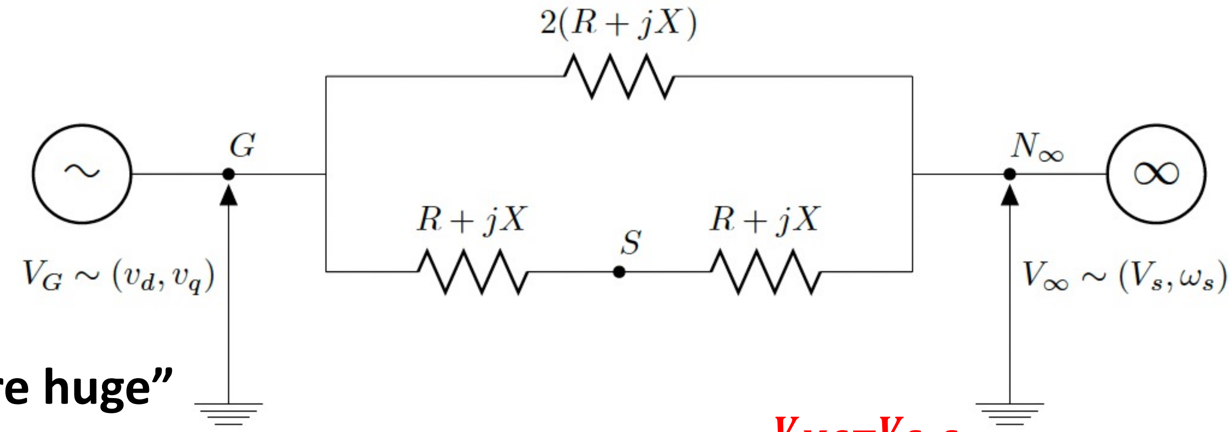
SoS approx. in red (2d-sections)



M. Tacchi et al "Power system transient stability analysis using SoS programming" Power System Computation Conference (PSCC) 2018

Transient Stability Analysis

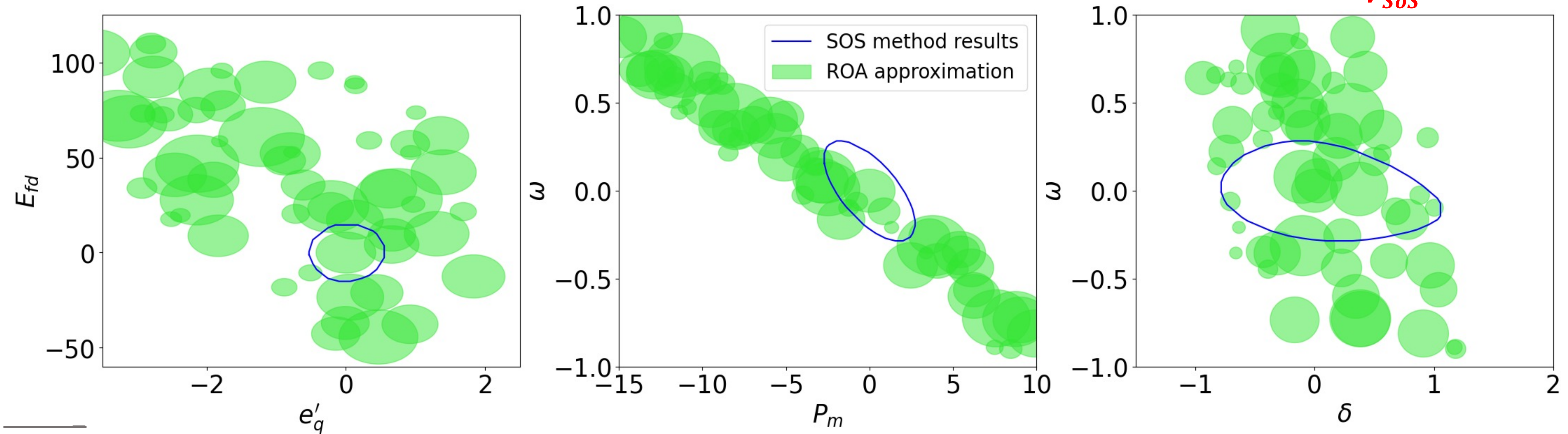
- **Algorithm parameters:**
  - **Centers:** 1000 per episode
  - **Failure prob.:** $\rho = 10^{-5}$
  - **Time constant:** $\tau = 100$ s



SoS in **blue**: [Tacchi 18] **vol = 0.05%,** run time **"they are huge"**

Multi-center in **green**: **vol = 0.23%, 1** episode, run time **3 min**

Percent vol. gain: $\frac{V_{MC} - V_{SoS}}{V_{SoS}} = 360\%$



M. Tacchi et al *Power system transient stability analysis using SoS programming,* Power System Computation Conference (PSCC) 2018

Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* Control and Decision Conference (**CDC) 2022**

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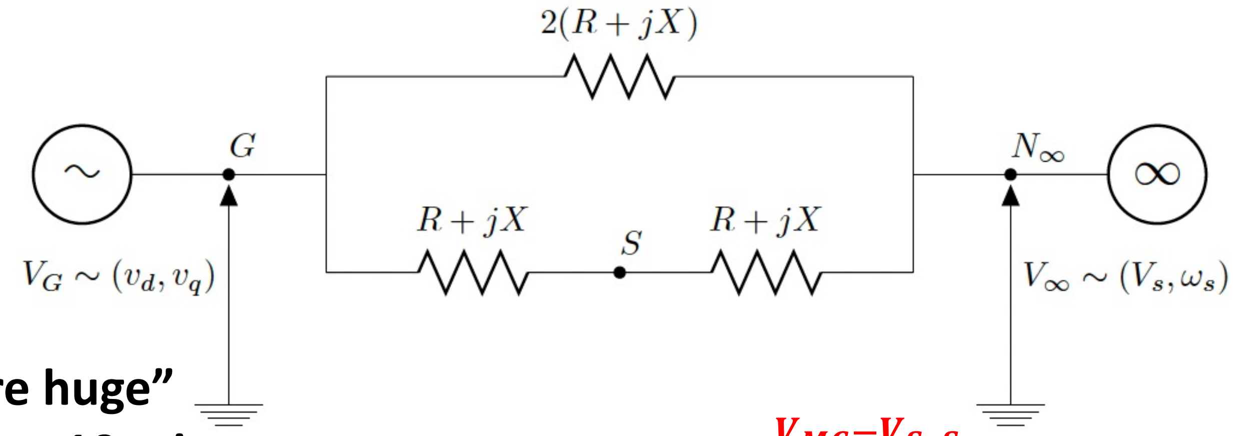
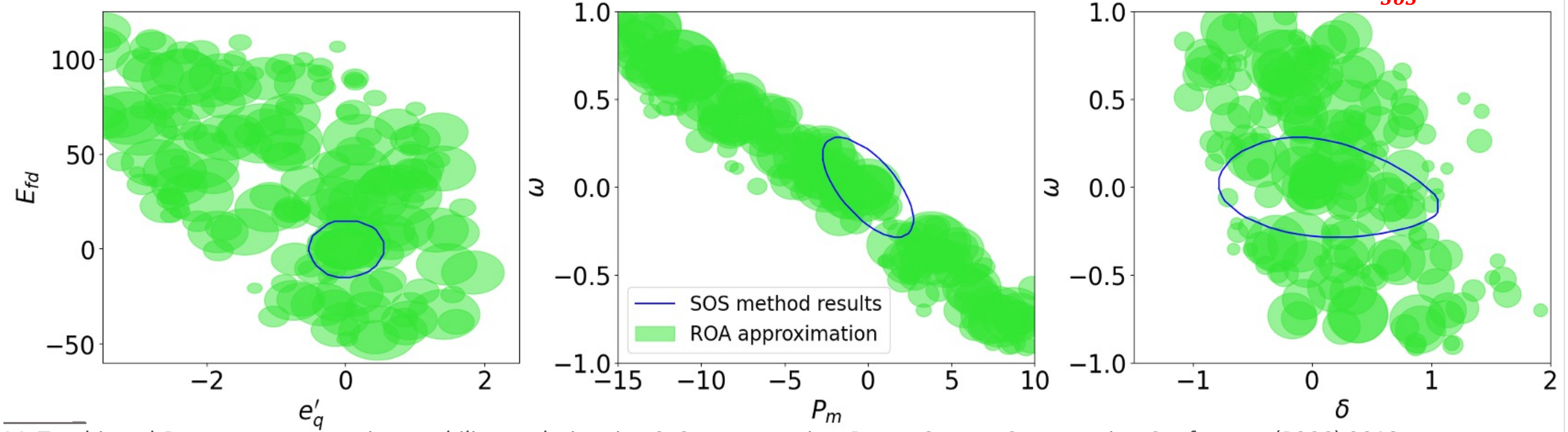


SoS in **blue**: [Tacchi 18] **vol = 0.05%,** run time **"they are huge"**
Multi-center in **green**: **vol = 0.45%, 3** episodes, run time **10 min**

Percent vol. gain: $\frac{V_{MC} - V_{SoS}}{V_{SoS}} = 800\%$



M. Tacchi et al *Power system transient stability analysis using SoS programming,* Power System Computation Conference (PSCC) 2018
Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* Control and Decision Conference (**CDC) 2022**

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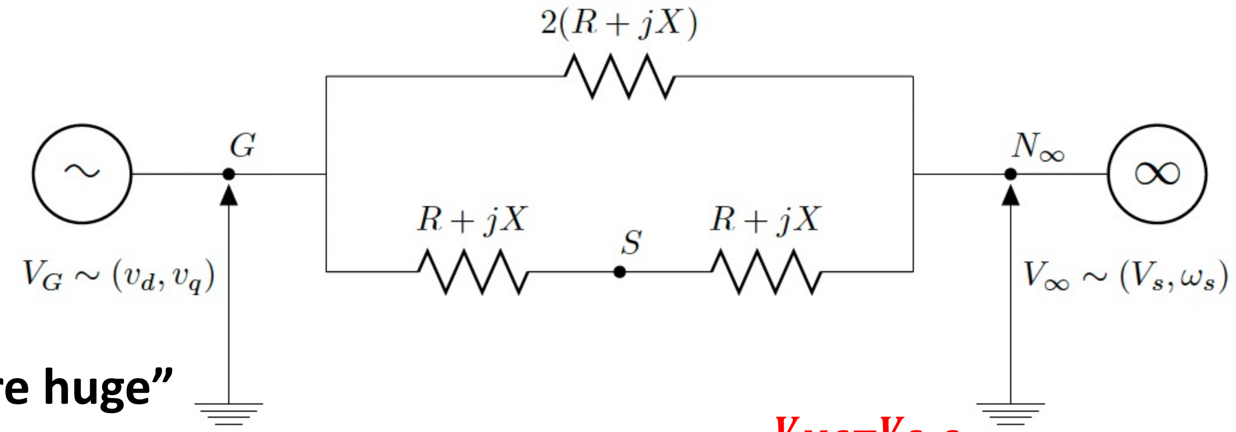
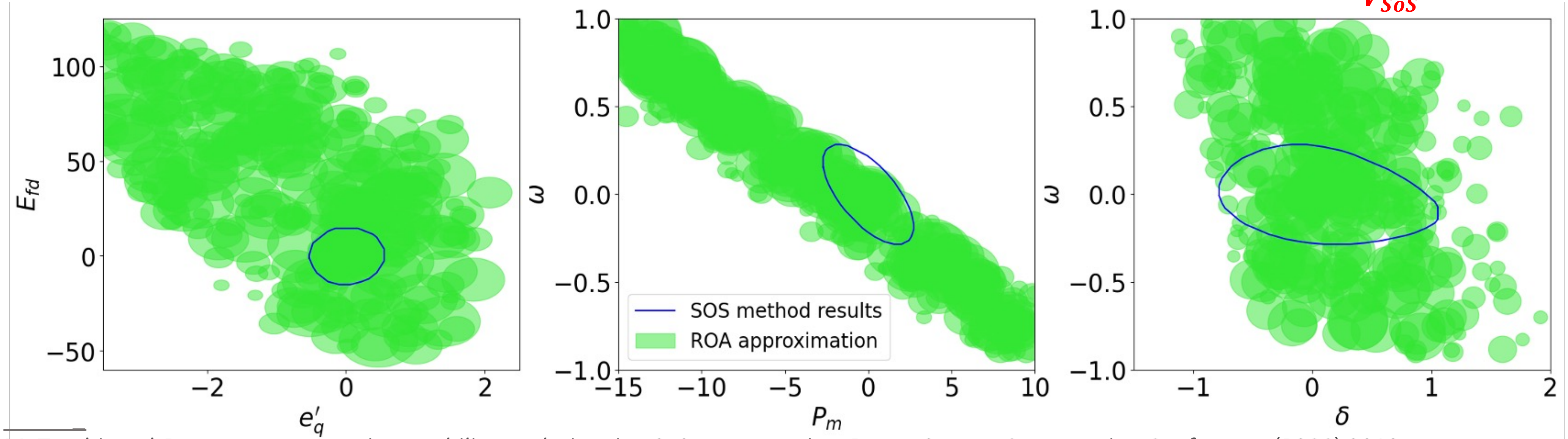


SoS in **blue**: [Tacchi 18] **vol = 0.05%,** run time **"they are huge"**

Multi-center in **green**: **vol = 0.74%, 5** episode, run time **17.5 min**

Percent vol. gain: $\frac{V_{MC} - V_{SoS}}{V_{SoS}} = $ **1380%**



M. Tacchi et al *Power system transient stability analysis using SoS programming,* Power System Computation Conference (PSCC) 2018

Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* Control and Decision Conference (**CDC) 2022**

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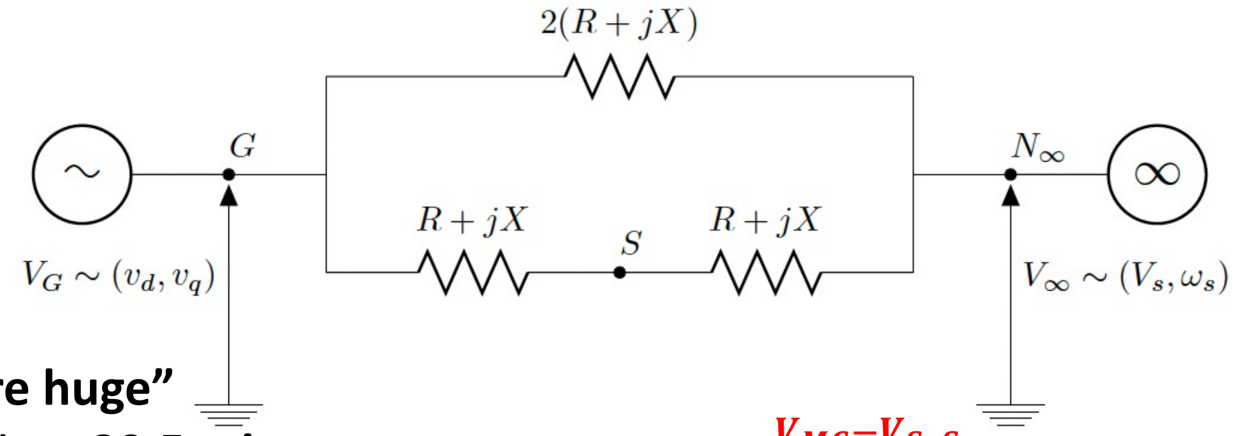
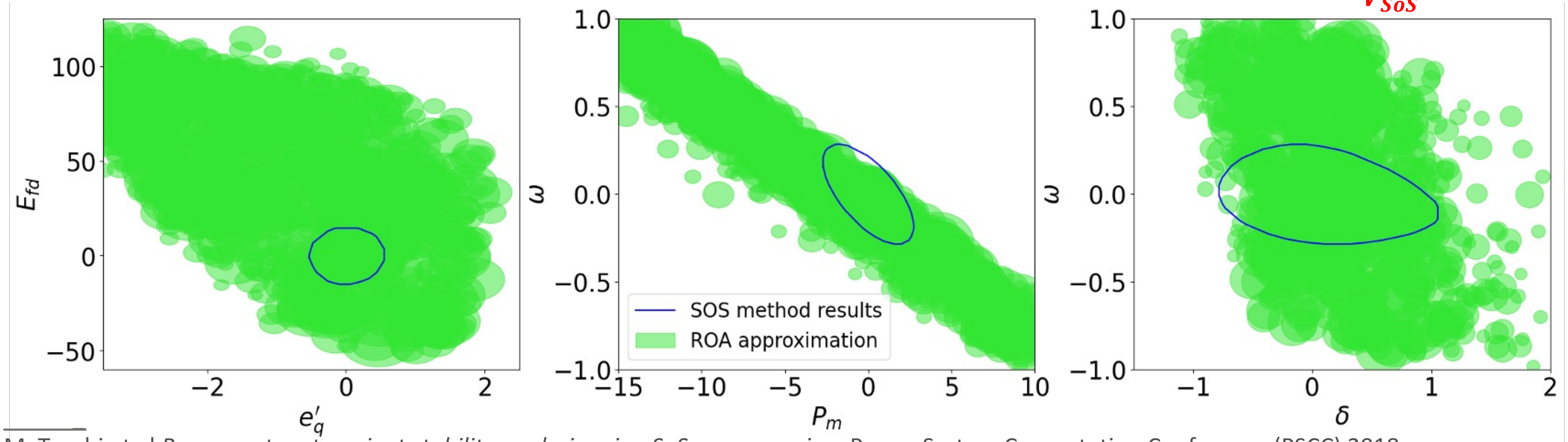


SoS in **blue**: [Tacchi 18] **vol = 0.05%,** run time **"they are huge"**

Multi-center in **green**: **vol = 1.56%, 10** episodes, run time **39.5 min**

Percent vol. gain: $\frac{V_{MC} - V_{SoS}}{V_{SoS}} = $ **3020%**



M. Tacchi et al *Power system transient stability analysis using SoS programming,* Power System Computation Conference (PSCC) 2018

Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* Control and Decision Conference (**CDC) 2022**

Model-Free Analysis of Dynamical Systems using Recurrent Sets

- Uses of invariant sets in control theory
- Inner-approximation of regions of attractions
- Stability analysis using non-monotonic Lyapunov functions

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Fernando Paganini
Universidad ORT Uruguay

Recurrently Non-Increasing Lyapunov Functions

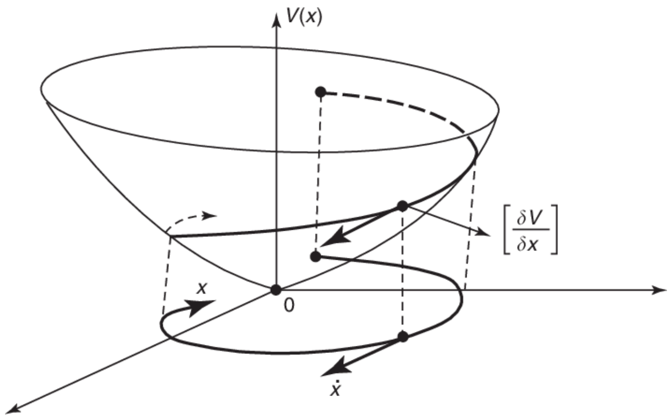
R. Siegelmann, Y. Shen, F. Paganini, and E. Mallada, “A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions”, submitted CDC 2023

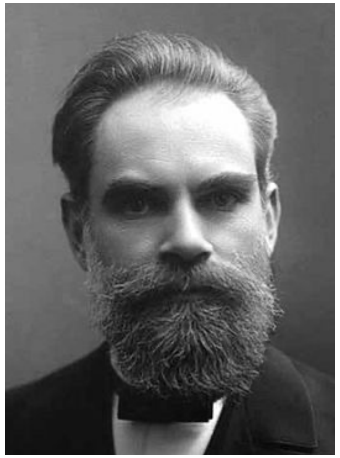
Lyapunov's Direct Method

Key idea: Make sub-level sets invariant to trap trajectories

Theorem [Lyapunov '1892]*. Given $V: \mathbb{R}^d \to \mathbb{R}_{\geq 0}$, with $V(x) > 0, \forall x \in \mathbb{R}^d \backslash \{x^*\}$, then:*

- $\dot{V} \leq 0 \to x^*$ *stable*
- $\dot{V} < 0 \to x^*$ *as. stable*





Challenge: Couples shape of V and vector field f

- Towards decoupling the $V - f$ geometry
  - Controlling regions where $\dot{V} \geq 0$ **[Karafyllis '09, Liu et al '20]**
  - Higher order conditions: $g(V^{(q)}, \ldots, \dot{V}, V) \leq 0$ **[Butz '69, Gunderson '71, Ahmadi '06, Meigoli '12]**
  - Discretization approach: $V\big(x(T)\big) \leq V(x(0))$ **[Coron et al '94, Aeyels et. al '98, Karafyllis '12]**
  - Multiple Lyapunov Functions: $\{V_j : j \in [k]\}$ **[Ahmadi et al '14]**

A Butz. Higher order derivatives of Lyapunov functions. IEEE Transactions on automatic control, 1969
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Coron, Lionel Rosier. A relation between continuous time-varying and discontinuous feedback stabilization. J. Math. Syst., Estimation, Control, 1994
Aeyels, Peuteman. A new asymptotic stability criterion for nonlinear time-variant differential equations. IEEE Transactions on automatic control, 1998
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Meigoli, Nikravesh. Stability analysis of nonlinear systems using higher order derivatives of Lyapunov function candidates. Systems & Control Letters, 2012
Karafyllis. Can we prove stability by using a positive definite function with non sign-definite derivative? IMA Journal of Mathematical Control and Information, 2012
Ahmadi, RM Jungers, PA Parrilo, M Roozbehani. Joint spectral radius and path-complete graph Lyapunov functions. SIAM Journal on Control and Optimization, 2014
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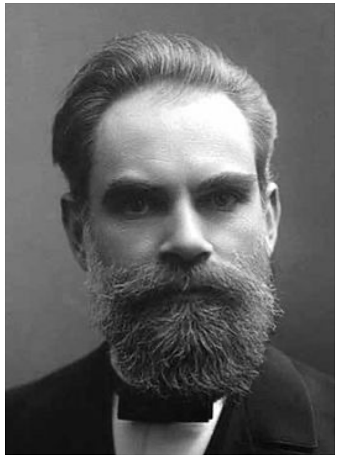
Lyapunov's Direct Method



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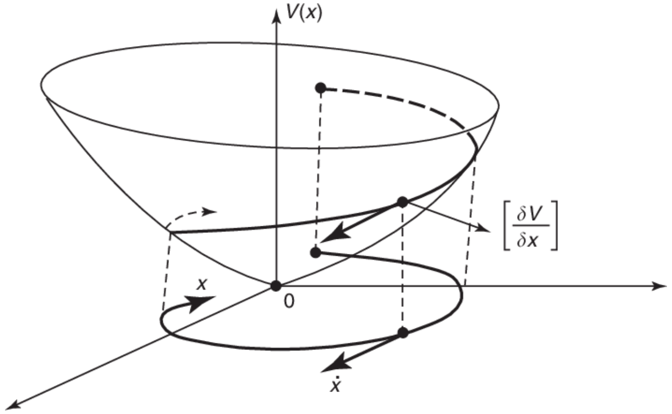

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Question: Can we provide stability conditions based on recurrence?

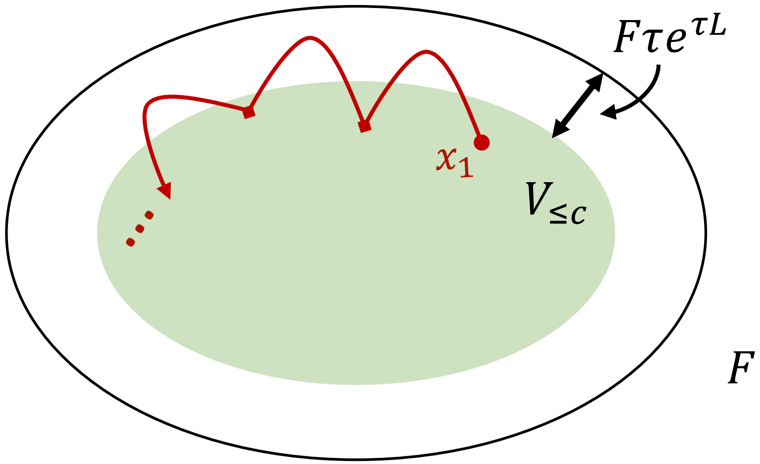
Recurrently Decreasing Lyapunov Functions

A continuous function $V: \mathbb{R}^d \to \mathbb{R}_+$ is a **recurrently non-increasing Lyapunov function** over intervals of length $\boldsymbol{\tau}$ if

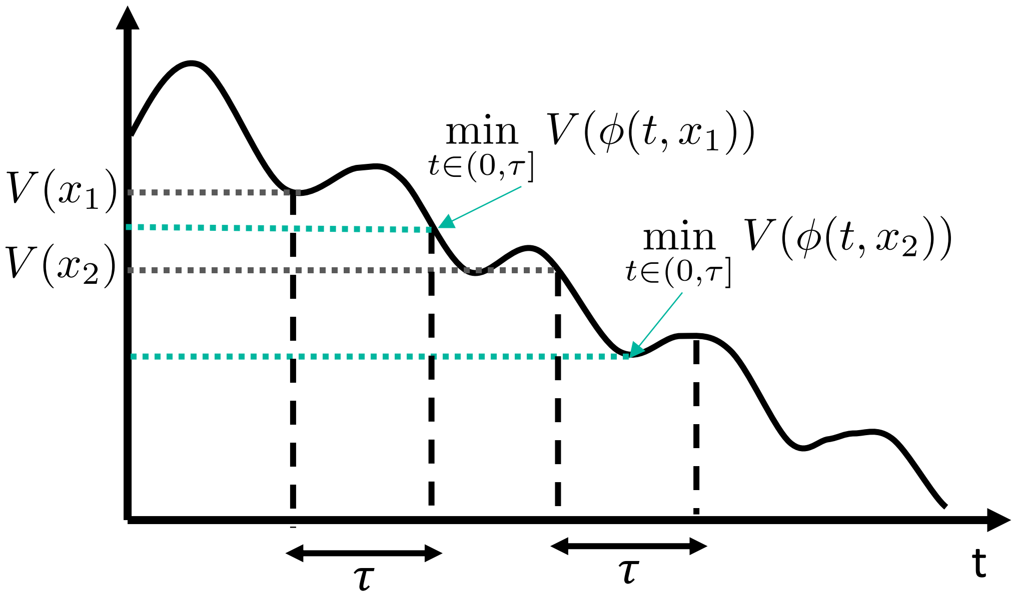
$$\mathcal{L}_f^{(0,\tau]} V(x) := \min_{t \in (0,\tau]} V(\phi(t,x)) - V(x) \leq 0 \quad \forall x \in \mathbb{R}^d$$

Preliminaries:

- Sub-level sets $\{V(x) \leq c\}$ are $\boldsymbol{\tau}$-recurrent sets.
- When f is L-Lipschitz, one can trap trajectories.



$F = \max_{x \in S} ||f(x)||$



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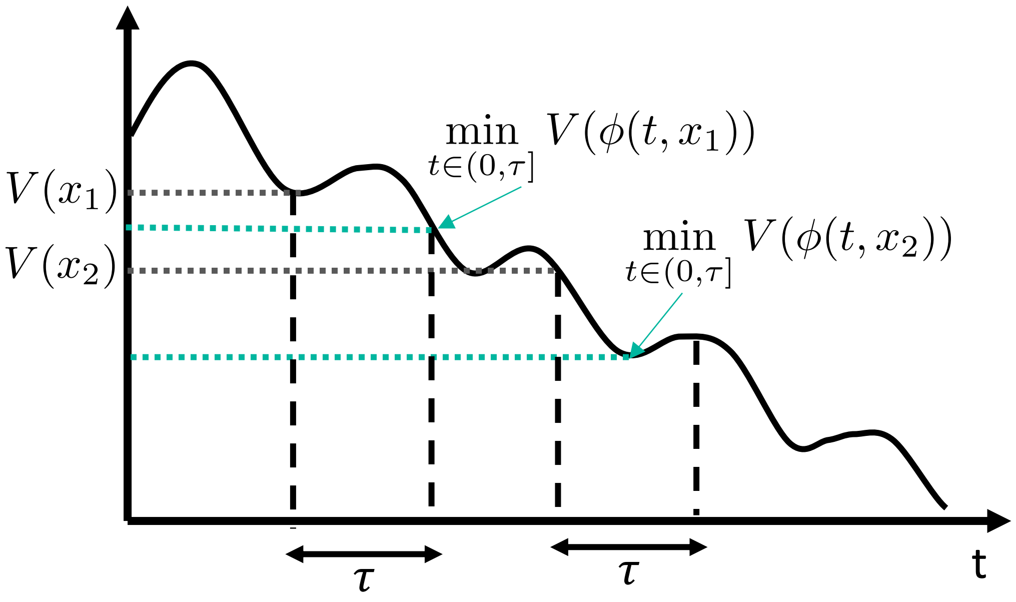
Theorem [CDC 23*]: *Let $V\colon \mathbb{R}^d \to \mathbb{R}_{\ge 0}$ be a recurrently non-increasing Lyapunov function over intervals of length τ. Let f be L-Lipschitz*

- *Then the equilibrium x^* is stable.*
- *Further, if the* ***inequality is strict****, then x^* is asymptotically stable!*



Siegelmann, Shen, Paganini, M, *A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions,* **submitted CDC 2023**

Exponential Stability Analysis

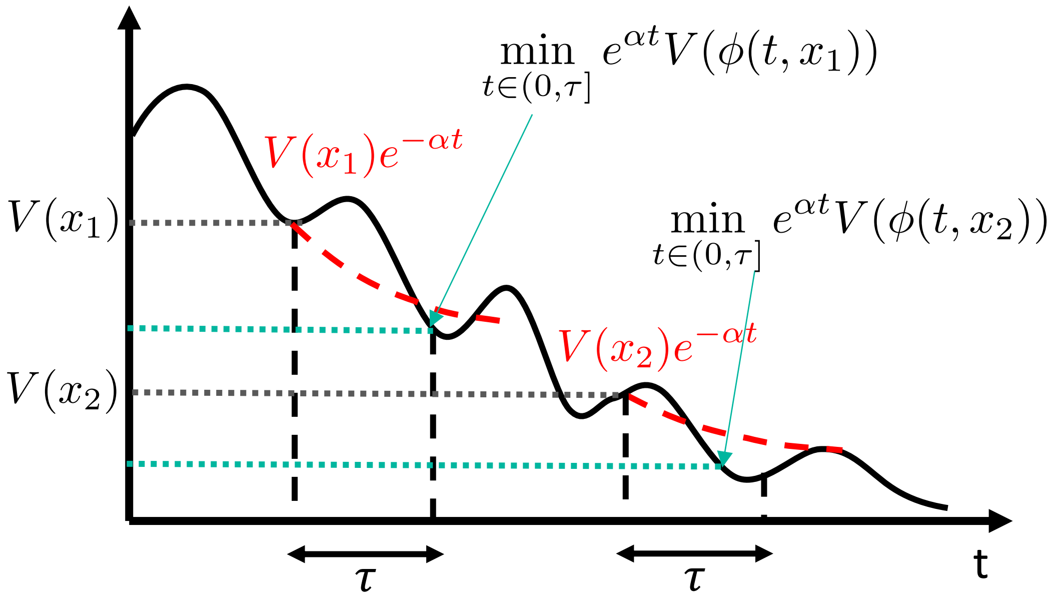
The function $V: \mathbb{R}^d \to \mathbb{R}_+$ is ***α*-exponentially recurrently non-increasing Lyapunov function** over intervals of length $\boldsymbol{\tau}$ if

$$\mathcal{L}_{f,\alpha}^{(0,\tau]} V(x) := \min_{t \in (0,\tau]} e^{\alpha t} V(\phi(t,x)) - V(x) \leq 0 \quad \forall x \in \mathbb{R}^d$$

Theorem [CDC 23*]: *Let* $V: \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ *satisfy*

$$\alpha_1 \left|\left| x - x^* \right|\right| \leq V(x) \leq \alpha_2 \left|\left| x - x^* \right|\right|.$$

Then, if V is α-exponentially recurrently τ-decreasing Lyapunov function, then x^* is exponentially stable with rate α.



Siegelmann, Shen, Paganini, M, *A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions,*
***submitted CDC 2023**

All norms are Lyapunov functions!

Theorem: Assume x^* is globally exponentially stable: $\exists\, K, c > 0$ such that:

$$\left|\left|\phi(t,x) - x^*\right|\right| \leq Ke^{-ct}\left|\left|x_0 - x^*\right|\right|.$$

Then, $V(x) = \left|\left|x - x^*\right|\right|$ *is* α*-exponentially recurrently* τ*-decreasing , i.e.,*

$$\min_{t\in(0,\tau]} e^{\alpha t}\left|\left|\phi(t,x) - x^*\right|\right| - \left|\left|x - x^*\right|\right| \leq 0, \qquad \forall x \in \mathbb{R}^d,$$

whenever $\alpha < c$ *and* $\tau \geq \frac{1}{c-\alpha}\ln K.$

Remarks:

- The rate α must be strictly smaller than the rate of convergence c (giving up optimality).
- *Any norm is a Lyapunov function!*

Question: Is the struggle for its search over?

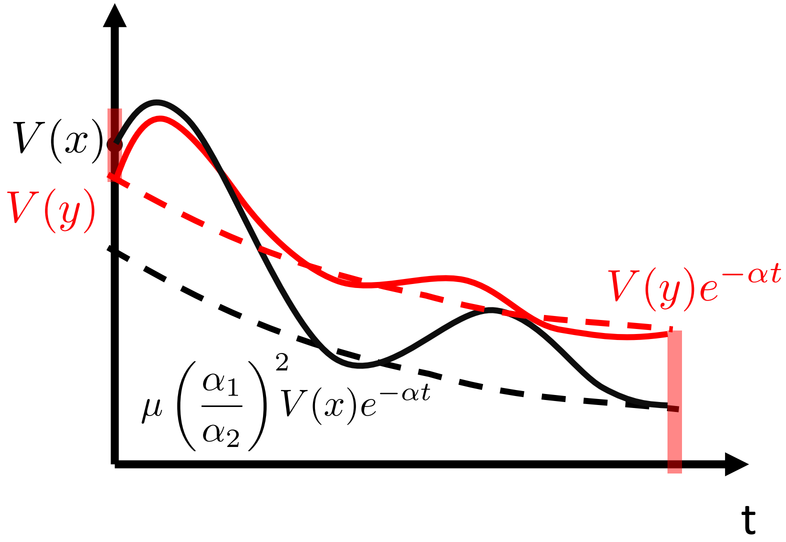
Verification of Exponential Stability

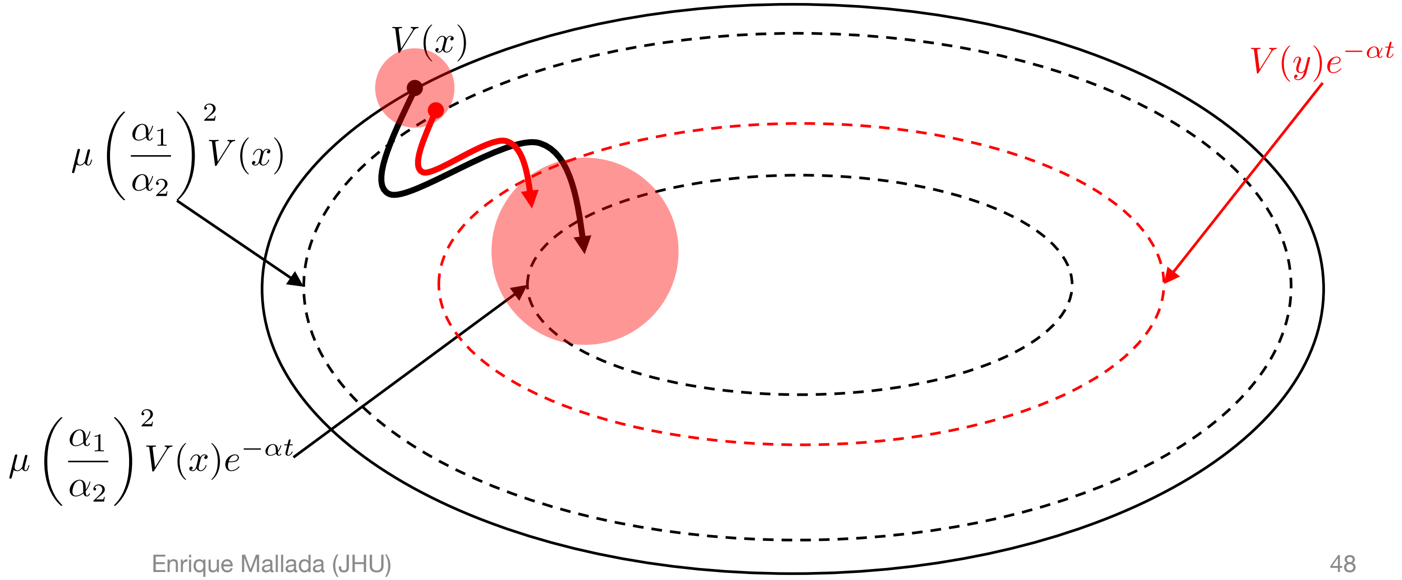
Proposition [CDC 23*]: *Let* $V: \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ *satisfy* $\alpha_1 \|x - x^*\| \leq V(x) \leq \alpha_2 \|x - x^*\|$, *and* $0 < \mu < 1$. *Then, whenever*

$$\min_{t \in (0,\tau]} e^{\alpha t} V(\phi(x,t)) \leq \mu \left(\frac{\alpha_1}{\alpha_2}\right)^2 V(x)$$

for all y *with* $\|y - x\| \leq r \coloneqq \frac{V(x)}{\alpha_2} g(\mu)$

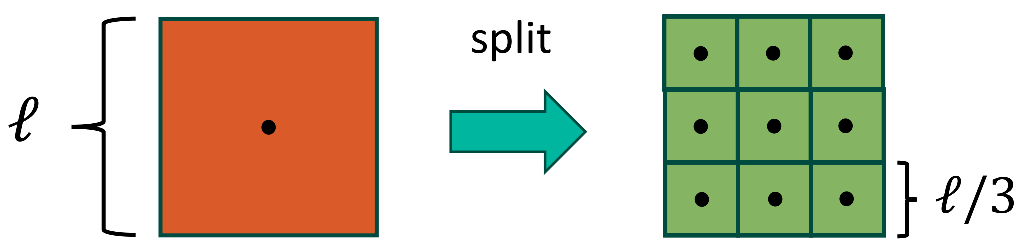
$$\min_{t \in (0,\tau]} e^{\alpha t} V(\phi(y,t)) \leq V(y)$$

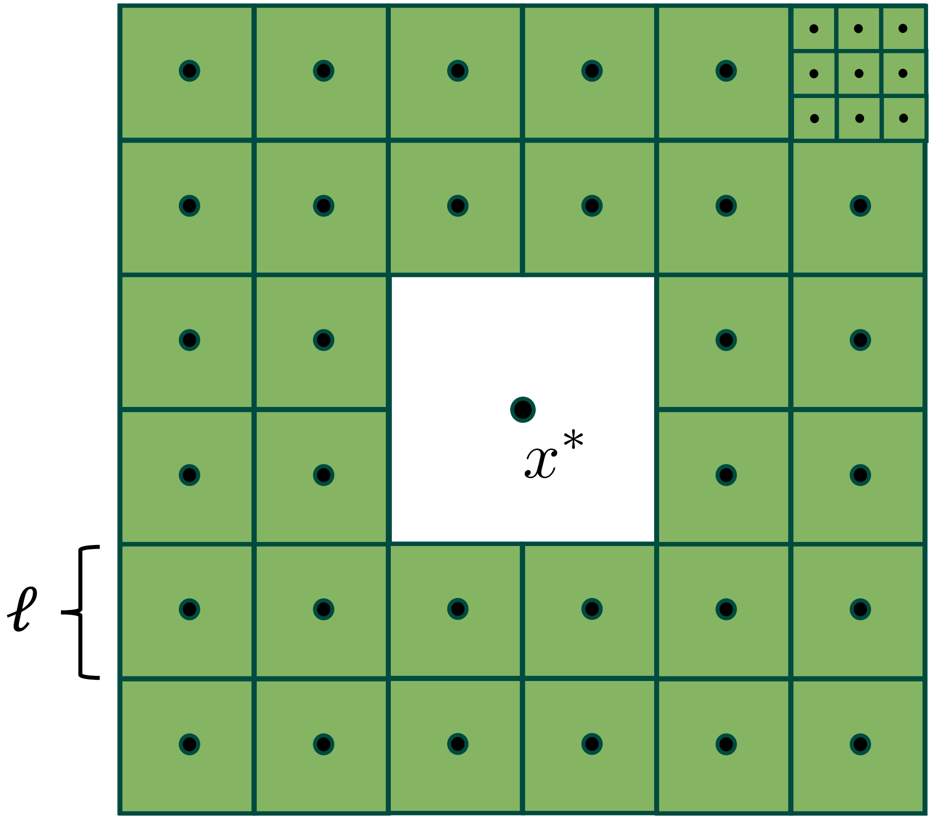




GPU-based Algorithm

- **Basic Algorithm:**
  - Consider $V(x) = ||x - x^*||_\infty$
  - Build a grid of hypercubes surrounding x^*
  - Test the center point and find κ s.t. the verified radius is $r \geq \ell/2$
  - If one hypercube is **not verified**, **split in $\mathbf{3^d}$** parts
  - Repeat testing of new points





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  - Exponentially expand to the following layer
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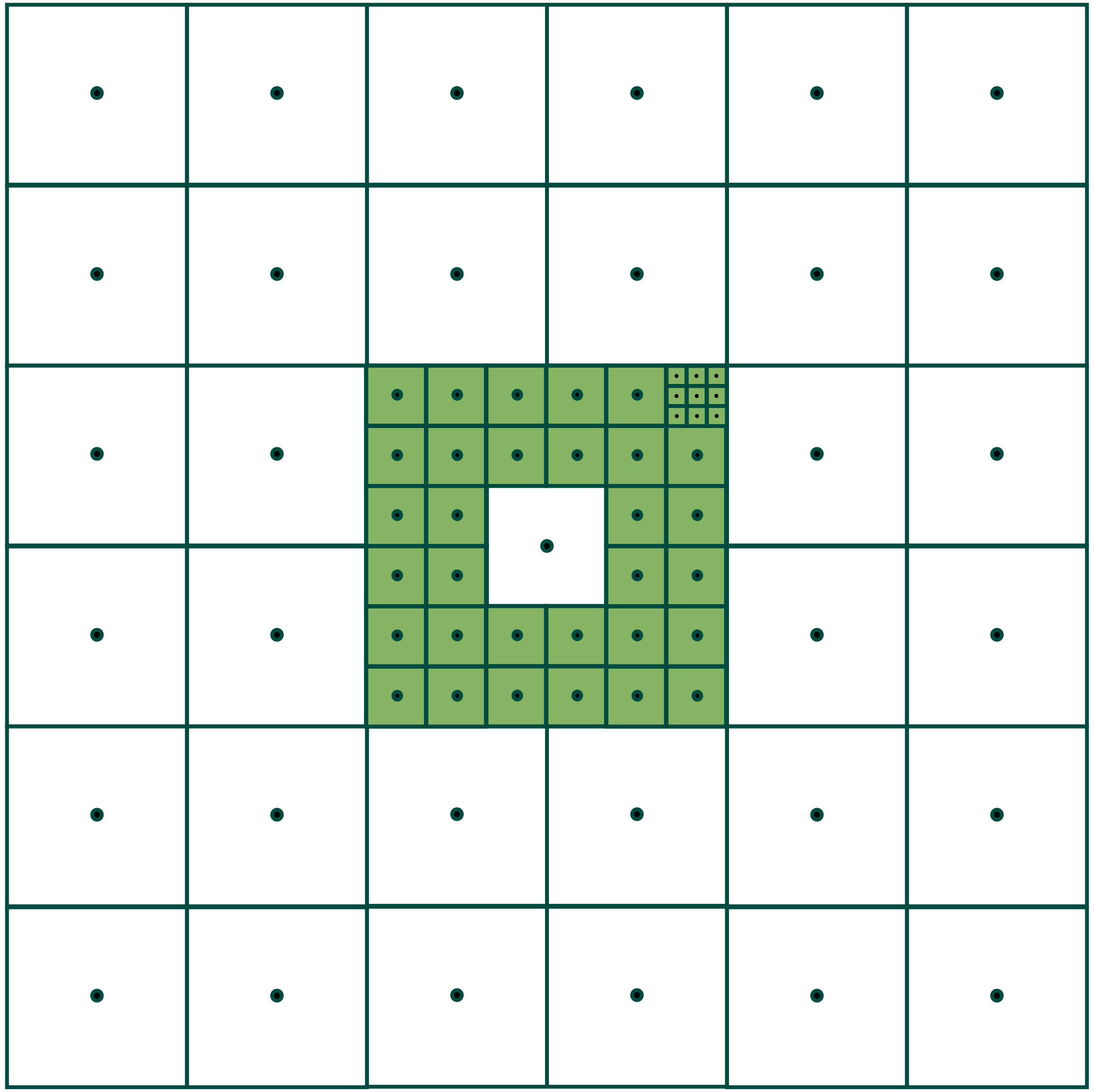

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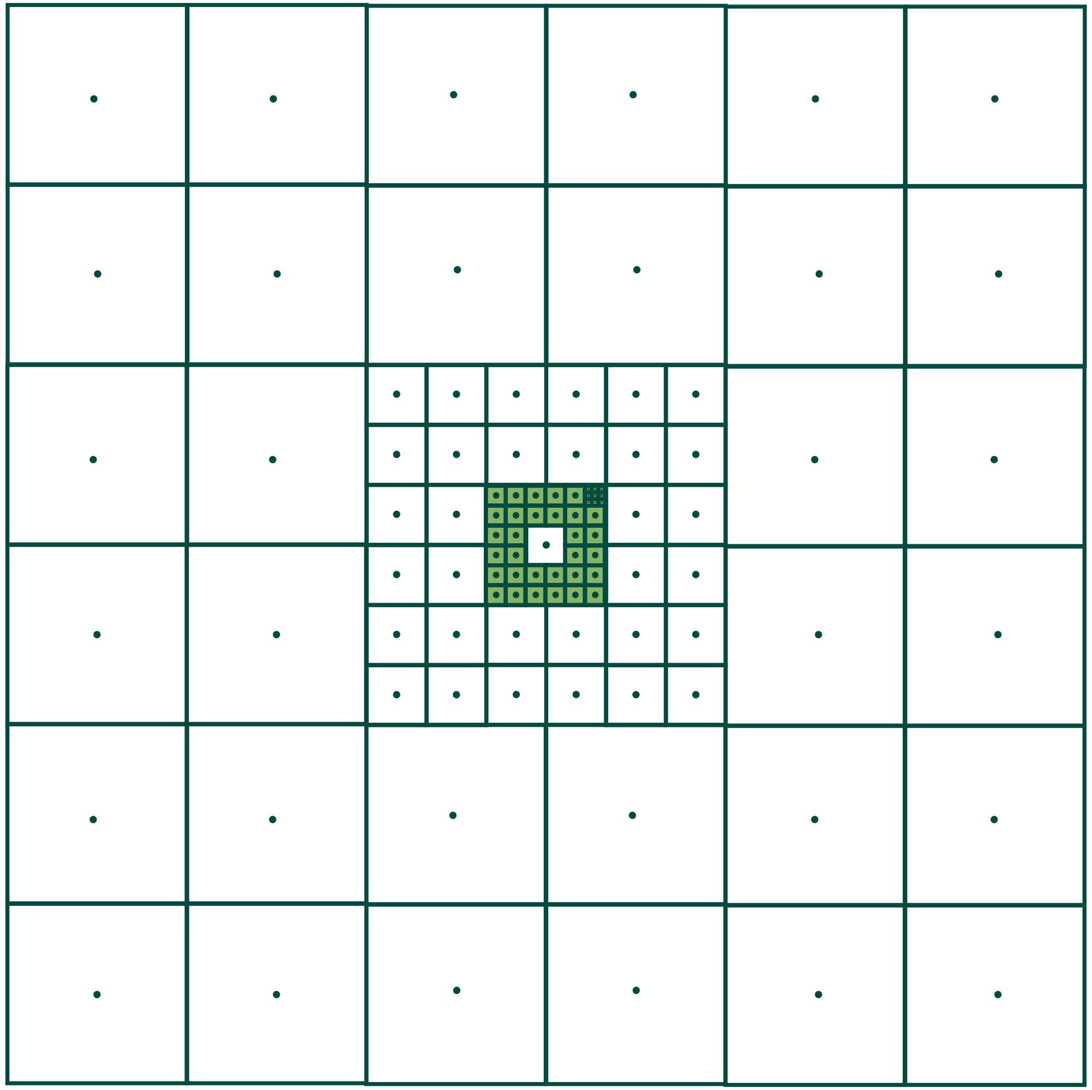


Numerical Illustration

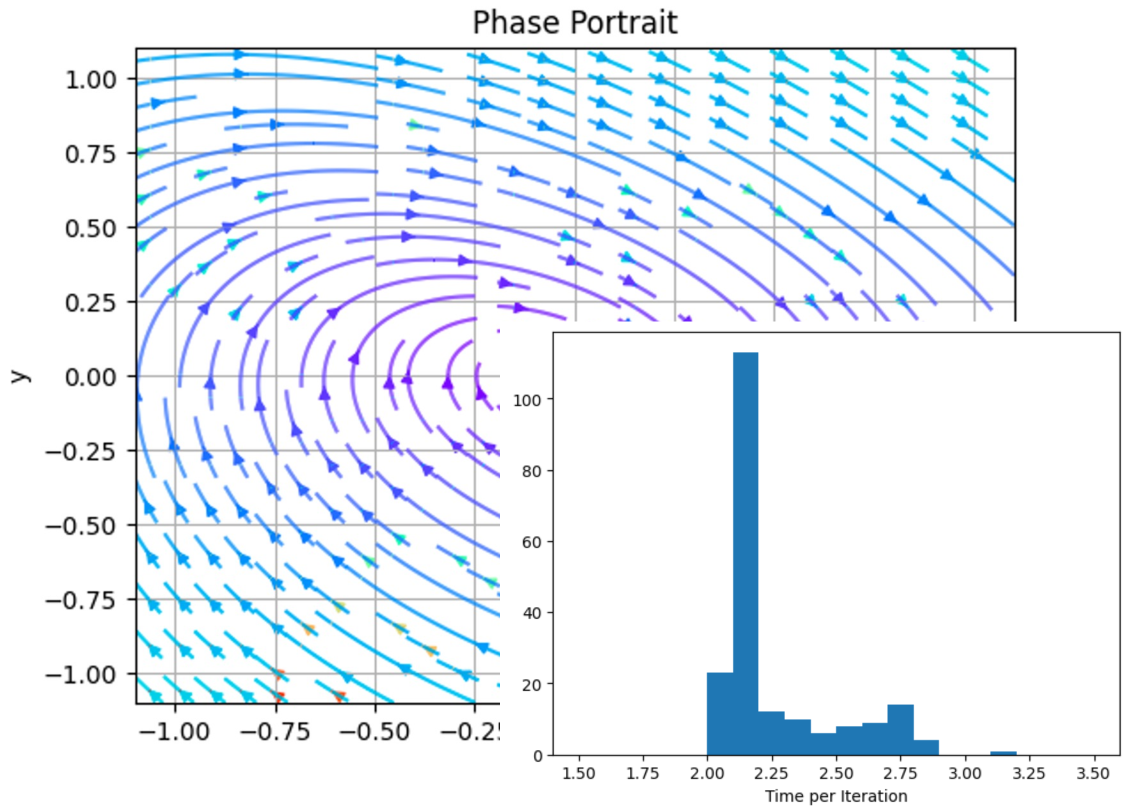
Consider the 2-d non-linear system:

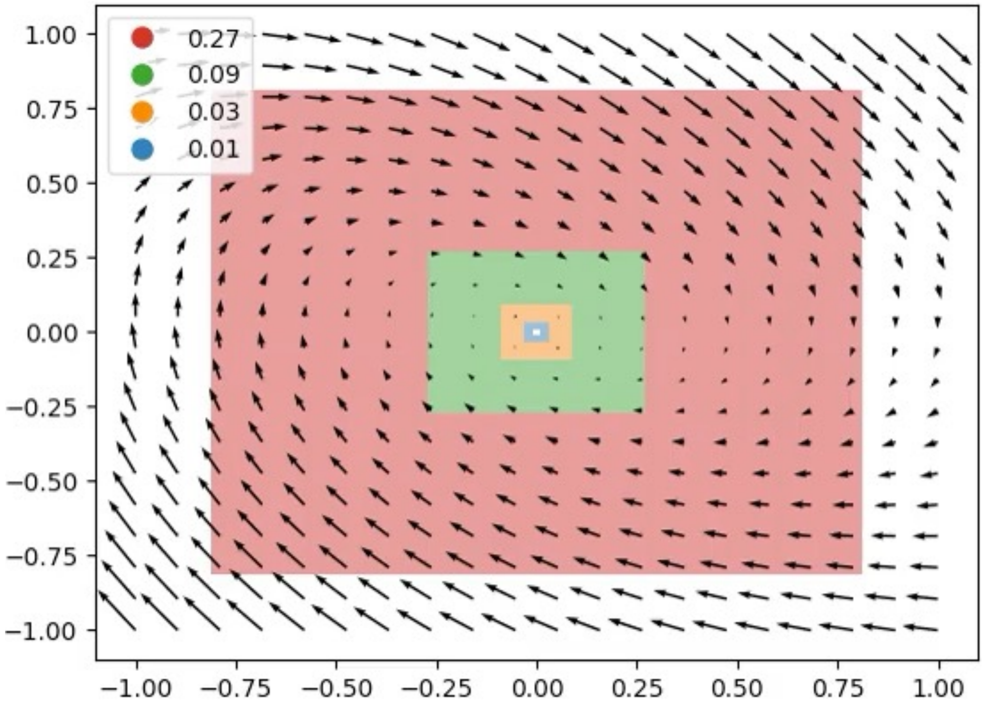
$$\dot{x} = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} x + B \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$$

with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

Parameter	Value
L	1.8
τ	1.5
ℓ	0.01

$\sigma = 0.2$





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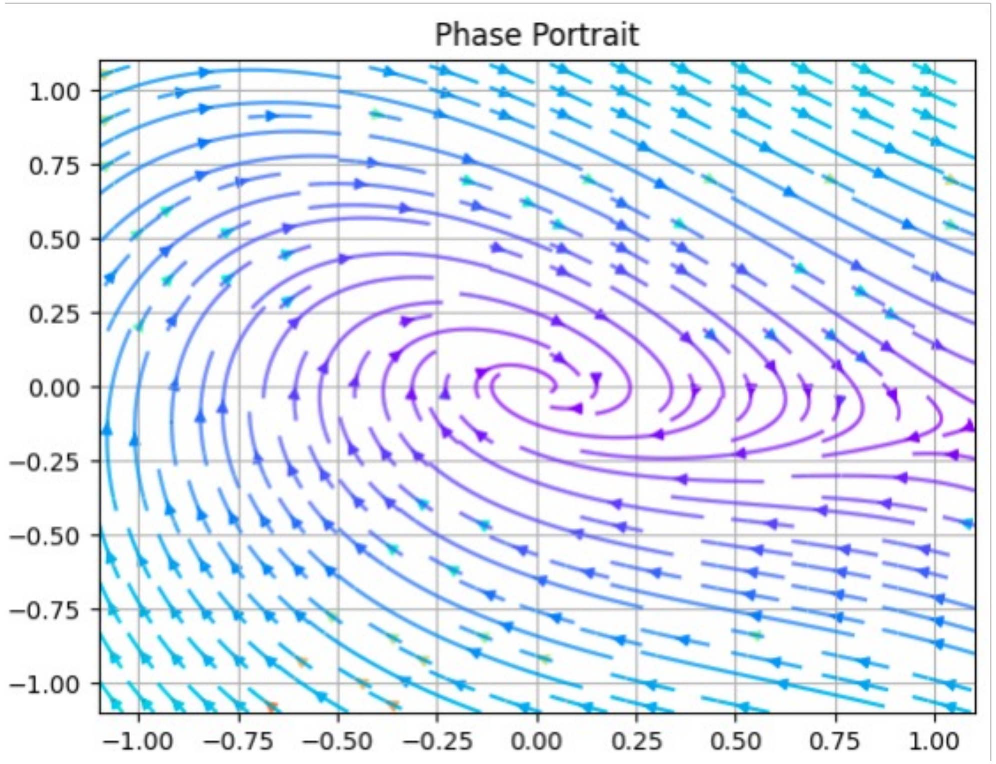

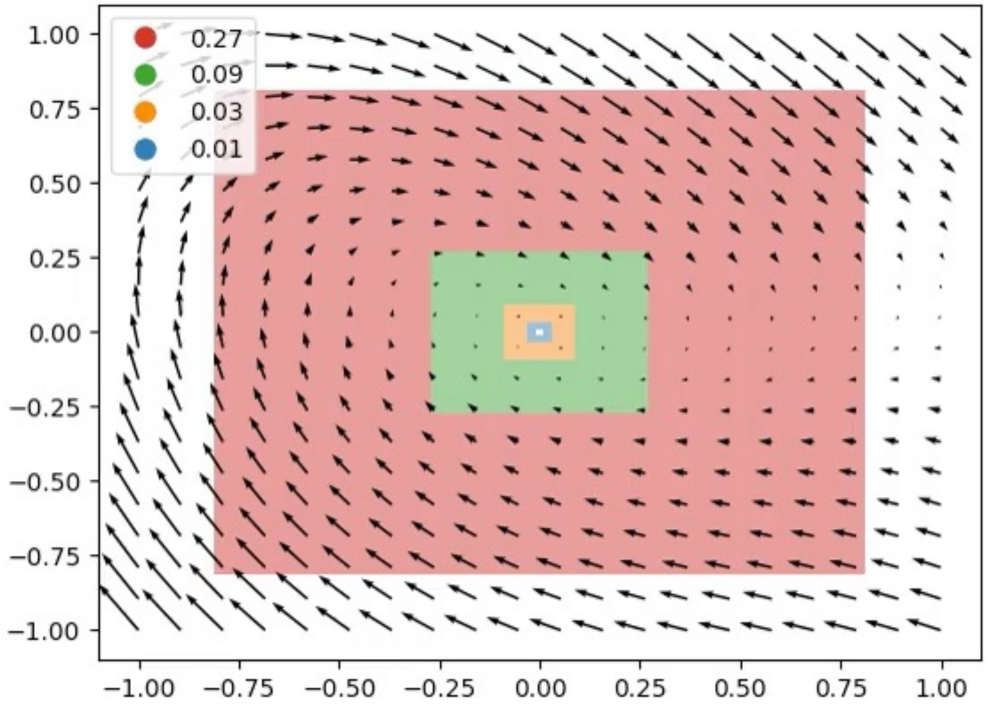

Conclusions and Future work

- **Takeaways**
  - Proposed a **relaxed notion of invariance** known as **recurrence.**
  - Provide **necessary and sufficient conditions** for a recurrent set to be an **inner approximation** of the ROA.
  - Generalized Lyapunov Theory **for recurrently decreasing functions** using recurrent sets
  - Our algorithms are **parallelizable via GPUs and progressive/sequential.**

- **Ongoing work**
  - **Recurrent Sets:** Smart choice of multi-points, control recurrent sets, GPU implementation
  - **Lyapunov Functions:** Generalize other Lyapunov notions, Control Lyapunov Functions, Barrier Functions, Control Barrier Functions, Contraction, etc.
  - **Recurrence Entropy:** Understanding the complexity of making a set recurrent when compared with invariance.

Thanks!

Related Publications:

[arXiv 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* **CDC 2022, journal preprint arXiv:2204.10372**.

[CDC 23] Siegelmann, Shen, Paganini, M, *A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions*, **CDC 2023**

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