Model-Free Analysis of Dynamical Systems Using Recurrent Sets

Towards a GPU-based Approach to Control

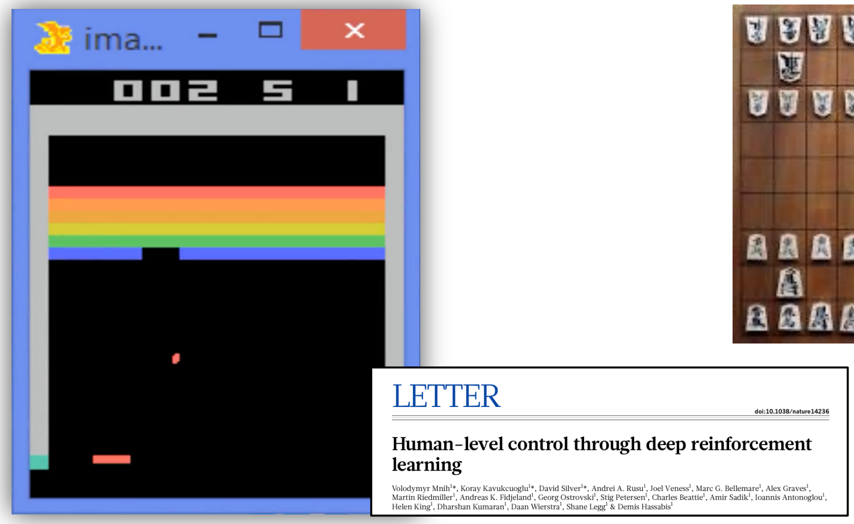
Enrique Mallada

Johns Hopkins University

Workshop on Uncertain Dynamical Systems
Kyoto, Japan
July 6, 2023

A World of Success Stories

2017 Google DeepMind's DQN

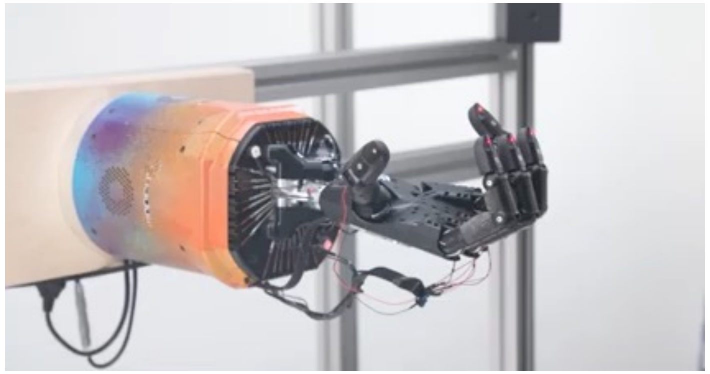


2017 AlphaZero – Chess, Shogi, Go

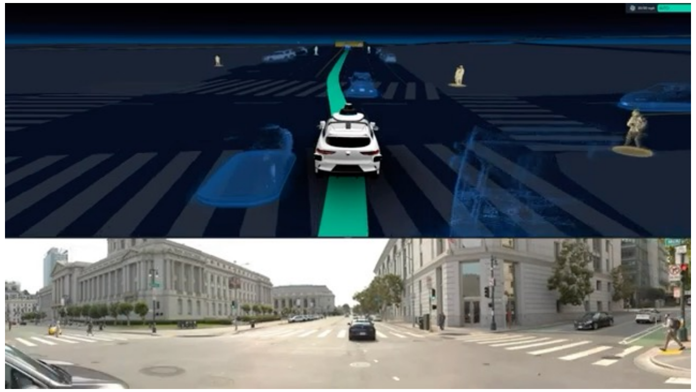
2019 AlphaStar – Starcraft II

Boston Dynamics

OpenAI – Rubik's Cube



Waymo



Reality Kicks In

Angry Residents, Abrupt Stops: Waymo Vehicles Are Still Causing Problems in Arizona

RAY STERN | MARCH 31, 2021 | 8:26AM

GARY MARCUS BUSINESS 08.14.2019 09:00 AM

DeepMind's Losses and the Future of Artificial Intelligence

Alphabet's DeepMind unit, conqueror of Go and other games, is losing lots of money. Continued deficits could imperil investments in AI.

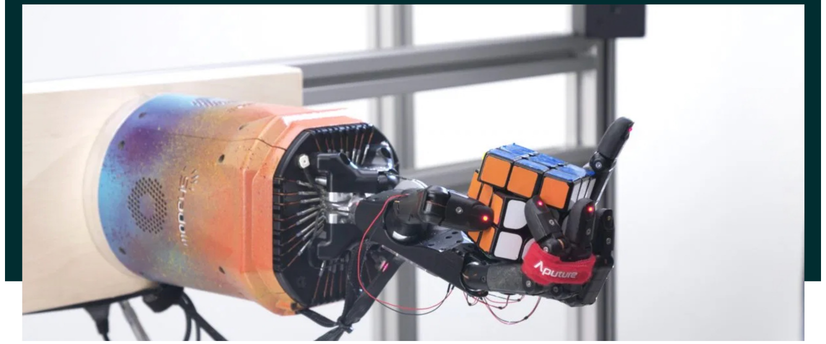
AARIAN MARSHALL BUSINESS 12.07.2020 04:06 PM

Uber Gives Up on the Self-Driving Dream

The ride-hail giant invested more than $1 billion in autonomous vehicles. Now it's selling the unit to Aurora, which makes self-driving tech.

OpenAI disbands its robotics research team

Kyle Wiggers @Kyle_L_Wiggers July 16, 2021 11:24 AM

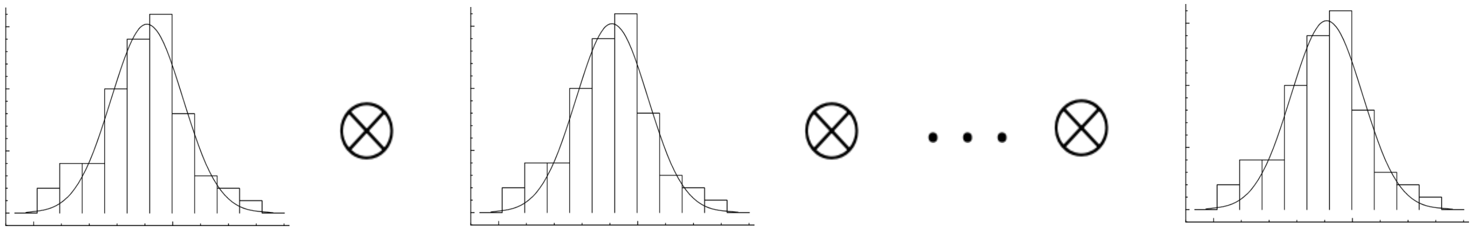


Self-driving Uber car that hit and killed woman did not recognize that pedestrians jaywalk

The automated car lacked "the capability to classify an object as a pedestrian unless that object was near a crosswalk," an NTSB report said.

Core challenge: The curse of dimensionality

- **Statistical: Sampling in d dimension with resolution ϵ**



Sample complexity:

$$O(\varepsilon^{-d})$$

For $\epsilon = 0.1$ and $d = 100$, we would need 10^{100} points.

- **Computational: Verifying non-negativity of polynomials**

Copositive matrices:

$$[x_1^2 \dots x_d^2] A [x_1^2 \dots x_d^2]^{\mathrm{T}} \geq 0$$

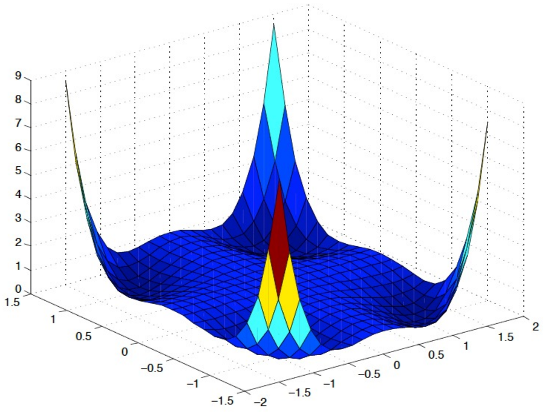
Murty&Kadabi [1987]: Testing co-positivity is NP-Hard

Sum of Squares (SoS):

$$z(x)^T Q z(x) \geq 0, \quad z_i(x) \in \mathbb{R}[x], \ x \in \mathbb{R}^d, Q \succcurlyeq 0$$

Artin [1927] (Hilbert's 17th problem):

Non-negative polynomials are sum of square of *rational* functions



Motzkin [1967]:

$p = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$

is nonnegative,

not a sum of squares,

but $(x^2 + y^2)^2 p$ is SoS

Question: Are we asking too much?

- Models are intrinsically valid across the ***entire domain***

  Q: Can we provide local guarantees, and progressively expand as needed?

  [arXiv ‘22] Shen, Bichuch, M - [CDC ’23] Siegelmann, Shen, Paganini, M

- Lyapunov functions and control barrier functions require strict and exhaustive notions of ***invariance***

  Q: Can we substitute invariance with less restrictive properties?

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- Control synthesis usually aims for the ***best*** (optimal) controller

  Q: Can we focus on feasibility, rather than optimality?

  [TAC ’23, L4DC 22] Castellano, Min, Bazerque, M

[arXiv 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* **CDC 2022,** journal preprint arXiv:2204.10372.
[CDC 23] Siegelmann, Shen, Paganini, M, *A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions*, **submitted CDC 2023**
[L4DC 22] Castellano, Min, Bazerque, M, *Reinforcement Learning with Almost Sure Constraints,* **Learning for Dynamics and Control (L4DC) Conference, 2022**
[TAC 23] Castellano, Min, Bazerque, M, *Learning to Act Safely with Limited Exposure and Almost Sure Certainty,* **IEEE TAC, 2023**

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University at Buffalo

Model-free Learning of Regions of Attractions via Recurrent Sets

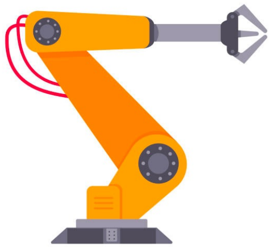
Y Shen, M. Bichuch, and E Mallada, “Model-free Learning of regions of attraction via recurrent sets.” CDC 2022.

Motivation: Estimation of regions of attraction

Having an approximation of the region of attraction allows us to

- **Test the limits of controller designs**
  especially for those based on (possibly linear) approximations of nonlinear systems

quadcopter



robot arm

. . .

- **Verify safety of certain operating condition**

HVAC system

power grids

. . .

Problem setup

Continuous time dynamical system: $\dot{x}(t) = f(x(t))$

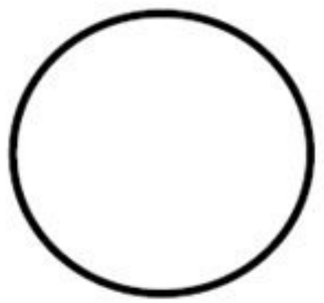
- Initial condition $x_0 = x(0)$, solution at time t: $\phi(t, x_0)$.

Ω-Limit Set $\Omega(f)$:

$$x \in \Omega(f) \iff \exists\, x_0, \{t_n\}_{n\geq 0}, \text{ s.t. } \lim_{n\to\infty} t_n = \infty \text{ and } \lim_{n\to\infty} \phi(t_n, x_0) = x$$

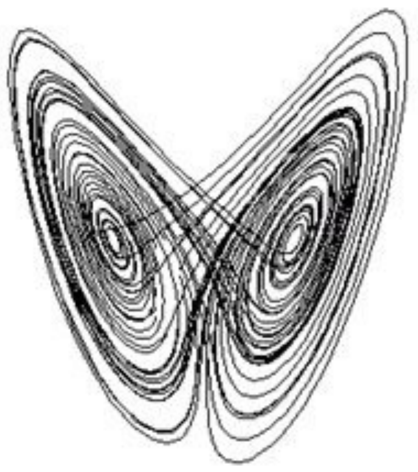
Types of Ω-limit set

equilibrium



limit cycle

limit torus



chaotic attractor

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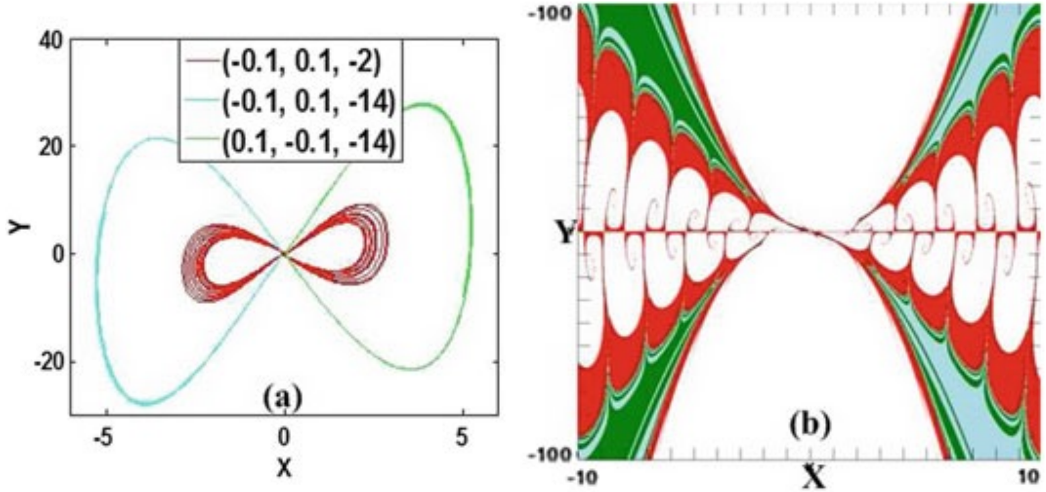
- The ω-limit set of the system: $\Omega(f)$

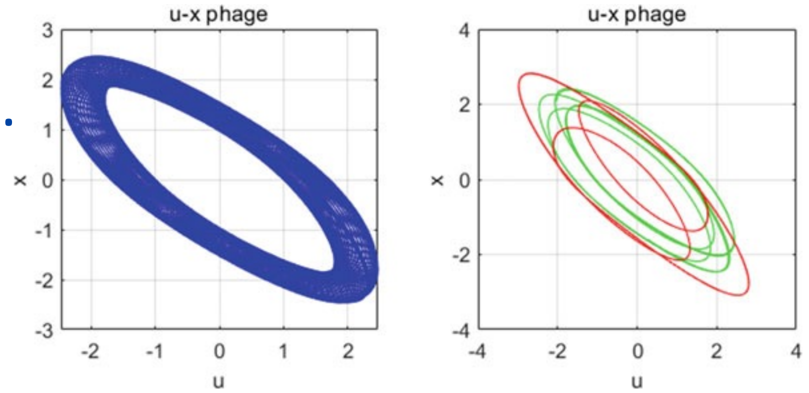
Region of attraction (ROA) of a set $S \subseteq \Omega(f)$:

$$\mathcal{A}(S) := \left\{ x \in \mathbb{R}^d \,|\, \liminf_{t\to\infty} d(\phi(t,x), S) = 0 \right\}$$

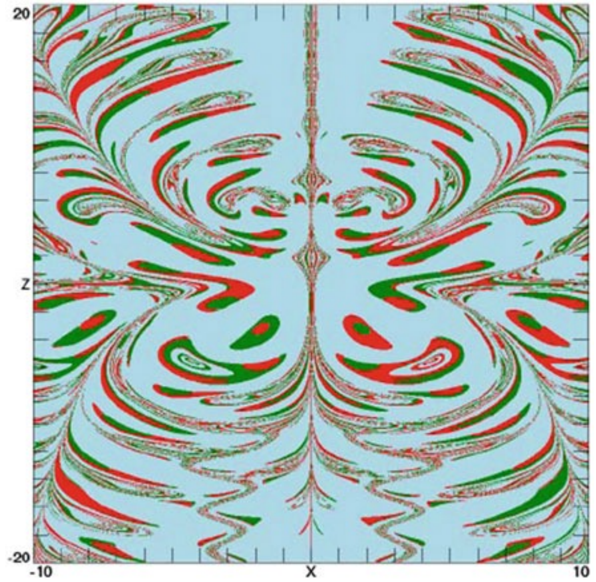
Example I: Limit set $\Omega(f)$ Basin of $\mathcal{A}(\Omega)$



Example II: Limit set $\Omega(f)$



Basin of $\mathcal{A}(\Omega)$



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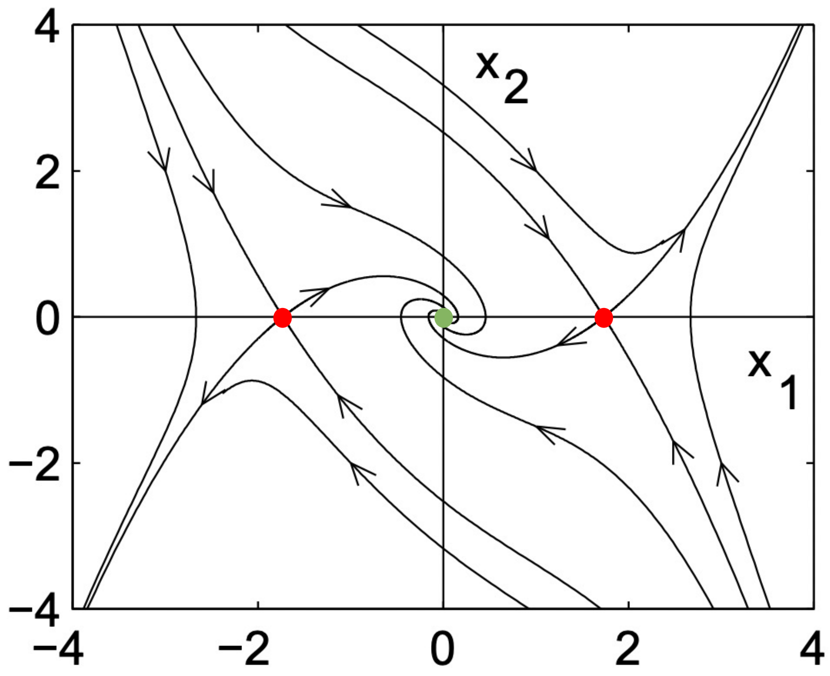
Example III

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 + \frac{1}{3}x_1^3 - x_2 \end{bmatrix}$$

$$\Omega(f) = \{(0,0), (-\sqrt{3}, 0), (\sqrt{3}, 0)\}$$



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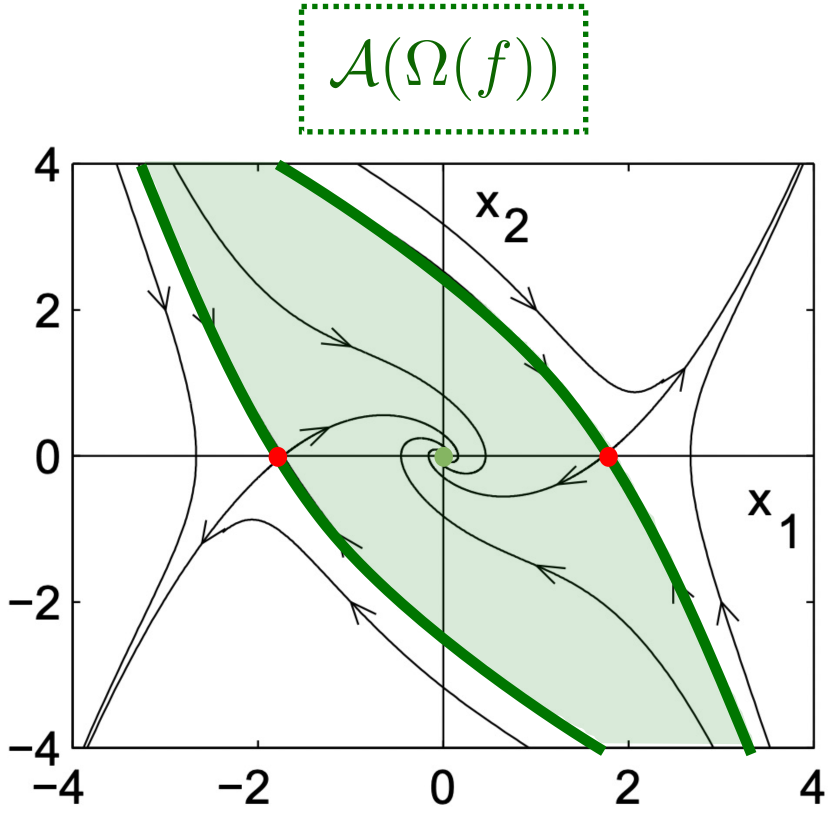

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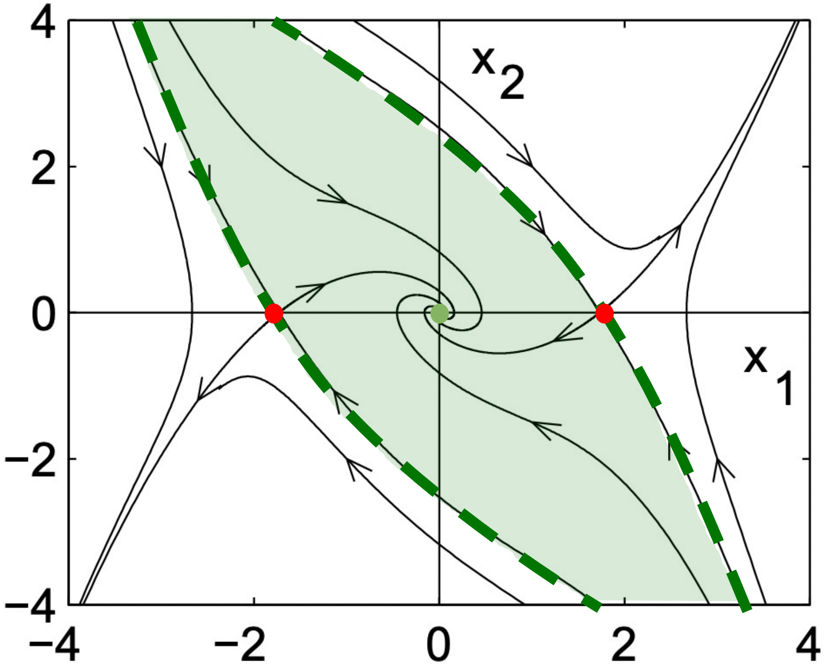
Asymptotically stable equilibrium at $x^* = (0,0)$

$\mathcal{A}(x^*)$



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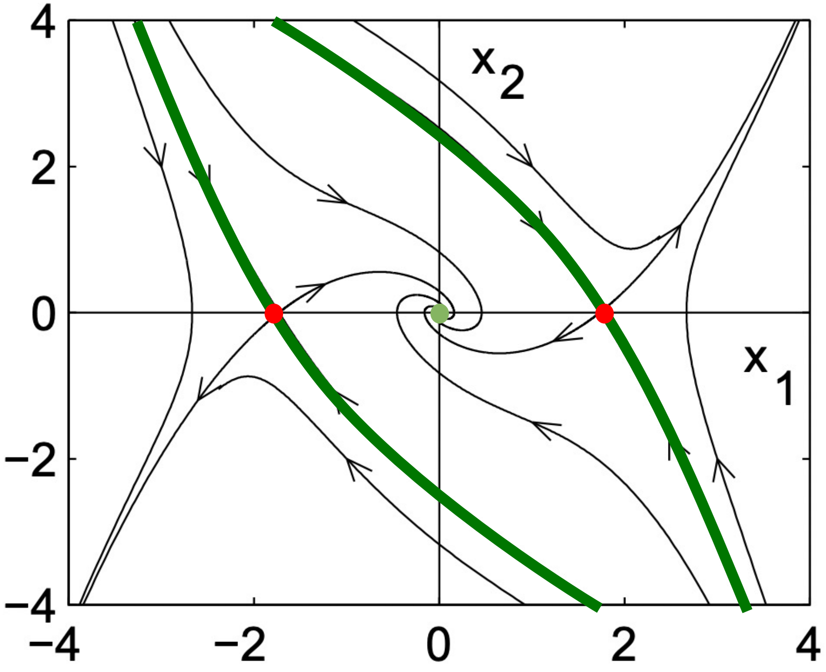
Unstable equilibria $\{(\sqrt{3},0), (-\sqrt{3},0)\}$

$\mathcal{A}(\{(\sqrt{3},0),(-\sqrt{3},0)\})$



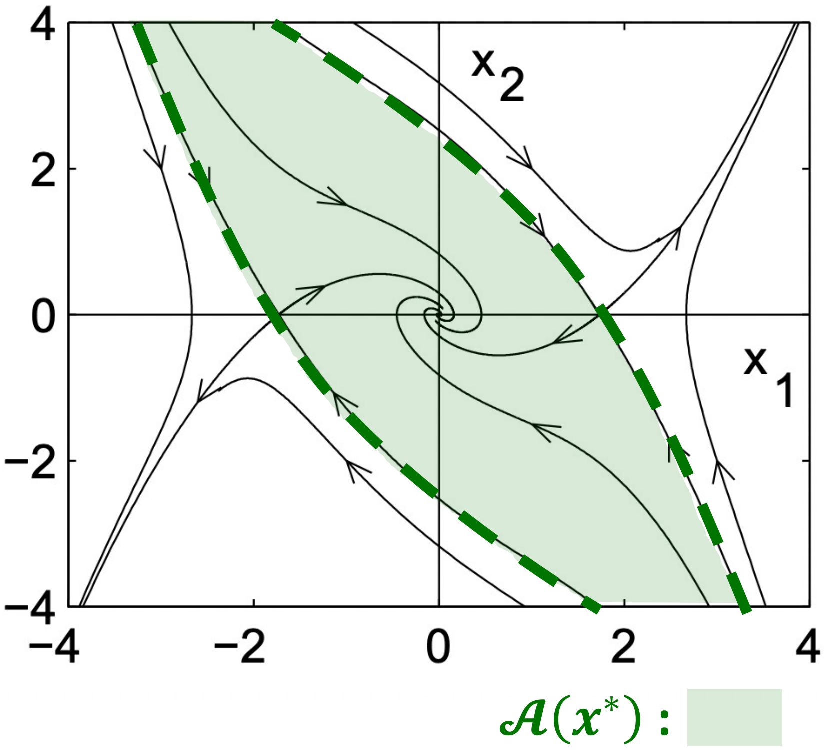
Region of attraction of stable equilibria

Region of attraction (ROA) of a set $S \subseteq \Omega(f)$:

$$\mathcal{A}(S) := \left\{ x_0 \in \mathbb{R}^d \middle| \lim_{t\to\infty} \phi(t, x_0) \in S \right\}$$

***Assumption** 1. The system $\dot{x}(t) = f(x(t))$ has an asymptotically stable equilibrium at x^*.*

***Remark** 1. It follows from Assumption 1 that the **positively invariant** ROA $\mathcal{A}(x^*)$ **is an open contractible set** [Sontag, 2013], i.e., the identity map of $\mathcal{A}(x^*)$ to itself is null-homotopic [Munkres, 2000].*



E. Sontag. "Mathematical Control Theory: Deterministic Finite Dimensional Systems." Springer 2013

J. R. Munkres. "Topology." Prentice Hall 2000

Invariant sets

A set $I \subseteq \mathbb{R}^d$ is **positively invariant** if and only if: $x_0 \in \mathcal{I} \implies \phi(t, x_0) \in \mathcal{I}, \quad \forall t \in \mathbb{R}^+$

Any trajectory starting in the set remains in inside it

- **Invariant sets guarantee stability**

  Lyapunov stability: solutions starting "close enough" to the equilibrium (within a distance δ) remain "close enough" forever (within a distance ε))

- **Invariant sets further certify asymptotic stability via Lyapunov's direct method**

  Asymptotic stability: solutions that start close enough not only remain close enough but also eventually converge to the equilibrium.)

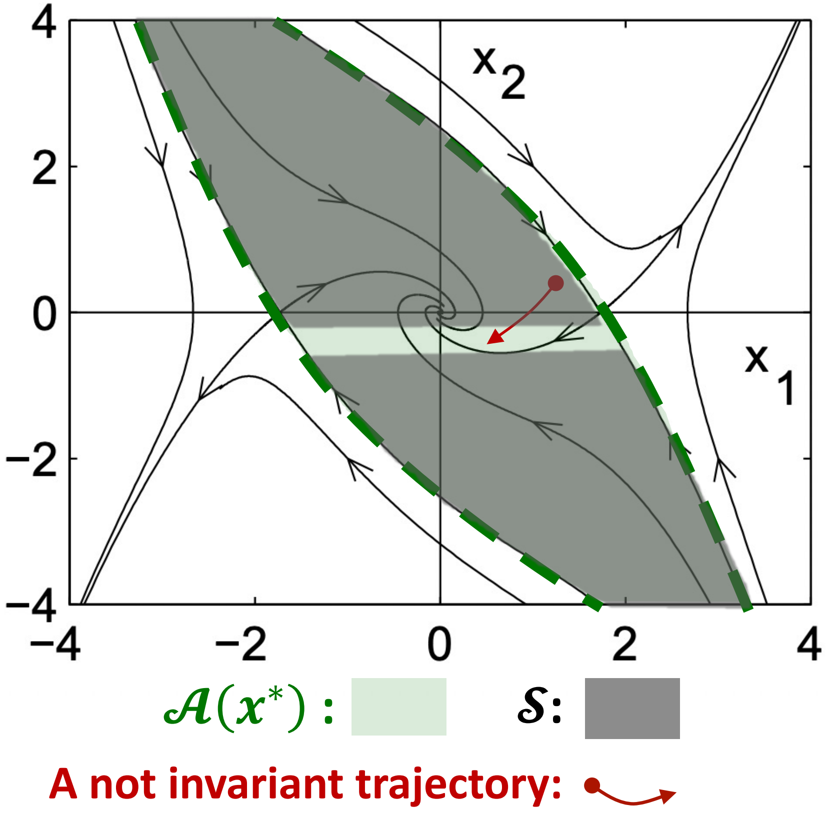
- **Regions of attraction are invariant sets, and so are the outcome of most approximation methods!**

Challenges of working with invariant set

Learning ROA $\mathcal{A}(x^*)$ by finding an invariant set $\mathcal{S} \subseteq \mathcal{A}(x^*)$

- $\mathcal{S}$ is topologically constrained
  - If $\mathcal{S} \cap \Omega(f) = \{x^*\}$, then $\mathcal{S}$ is connected

Example 1: $\mathcal{S} \subseteq \mathcal{A}(x^*)$ is not connected, not invariant!



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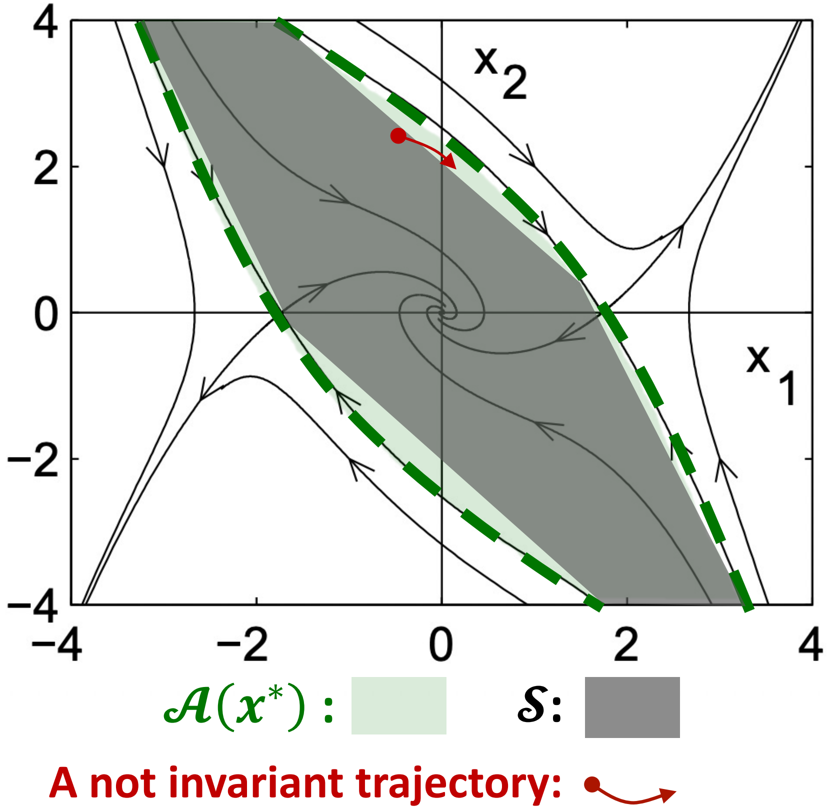
  - If $\mathcal{S} \cap \Omega(f) = \{x^*\}$, then $\mathcal{S}$ is connected
- $\mathcal{S}$ is geometrically constrained
  - f should point inwards for $x \in \partial\mathcal{S}$

Example 2: $\mathcal{S} \subseteq \mathcal{A}(x^*)$, f points outward on $\partial\mathcal{S}$, not invariant



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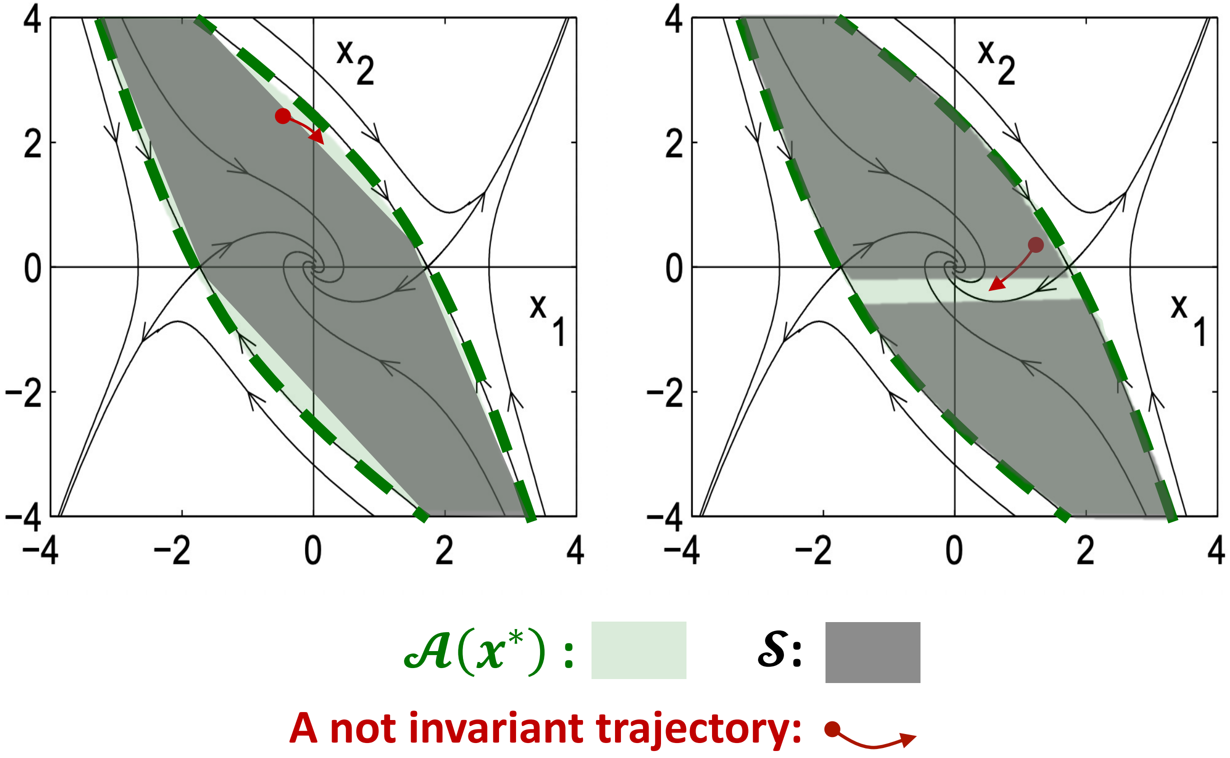
A subset of an invariant set is not necessary an invariant set

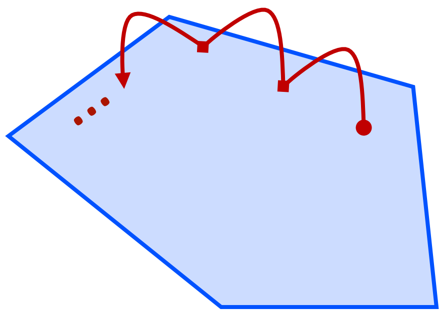


Recurrent sets: Letting things go, and come back

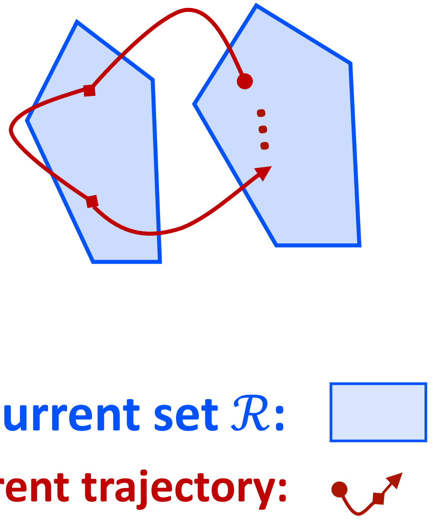
A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \geq t$ s.t. $\phi(t', x_0) \in \mathcal{R}$.

Property of Recurrent Sets

- $\mathcal{R}$ need **not** be **connected**
- $\mathcal{R}$ does **not** require f to **point inwards** on all $\partial\mathcal{R}$



Recurrent sets, while not invariant, guarantee that solutions that start in this set, will come back **infinitely often, forever!**



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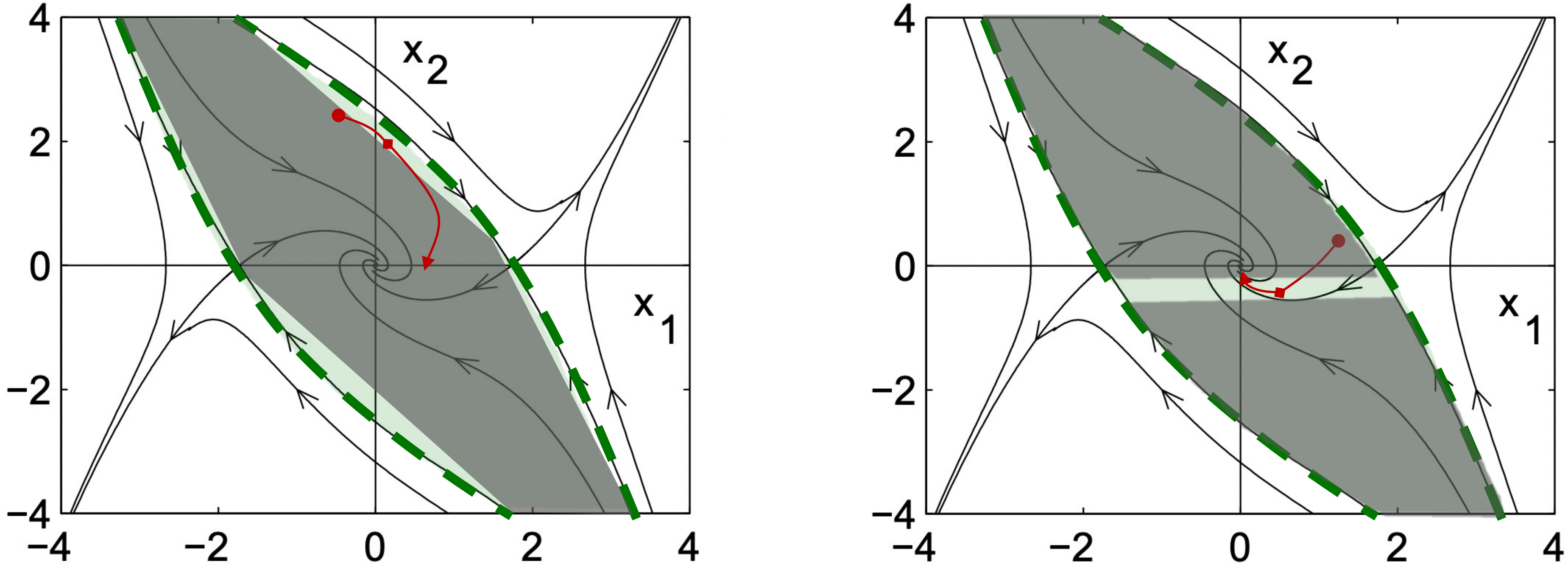
Previous two good inner approximations of $\mathcal{A}(x^*)$ are recurrent sets



Recurrent sets are subsets of the region of attraction

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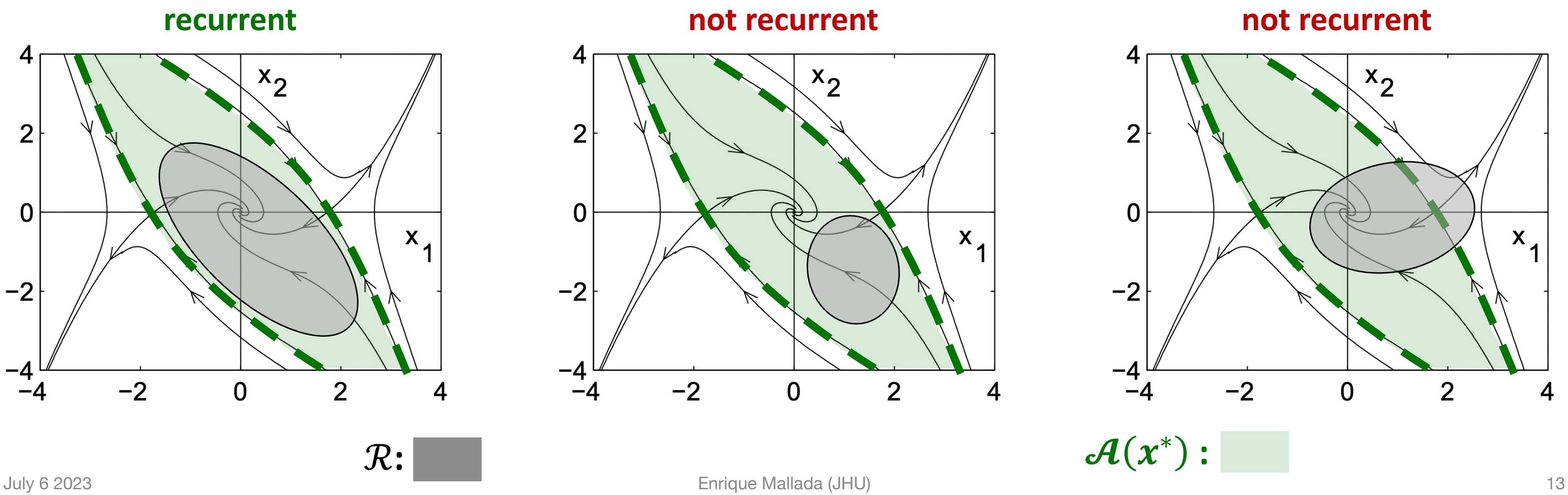
Theorem *1. Let $\mathcal{R} \subset \mathbb{R}^d$ be a* <u>*compact*</u> *set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$.*

Then:

$$\mathcal{R} \text{ is recurrent} \Longleftrightarrow \begin{array}{c} \mathcal{R} \cap \Omega(f) \neq \emptyset \\ \mathcal{R} \subset \mathcal{A}(\mathcal{R} \cap \Omega(f)) \end{array}$$



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***Proof**: [Sketch]*

(⟹)

- *If $x_0 \in \mathcal{R}$, the solution $\phi(t, x_0)$ visits $\mathcal{R}$ infinitely often, forever.*
- *We can build a sequence $\{x(t_n)\}_{n=0}^{\infty} \in \mathcal{R}$ with $\lim_{n \to +\infty} t_n = +\infty$*
- *Bolzano-Weierstrass ⟹ convergent subsequence $x(t_{n_i}) \to \bar{\boldsymbol{x}} \in \Omega(f) \cap \mathcal{R} \neq \emptyset$*
- *$\partial\mathcal{R} \cap \Omega(f) = \emptyset$ + $\mathcal{R}$ recurrent ⟹ $\phi(t, x_0)$ leaves $\mathcal{R}$ finitely many times*

(⟸) Trivial.

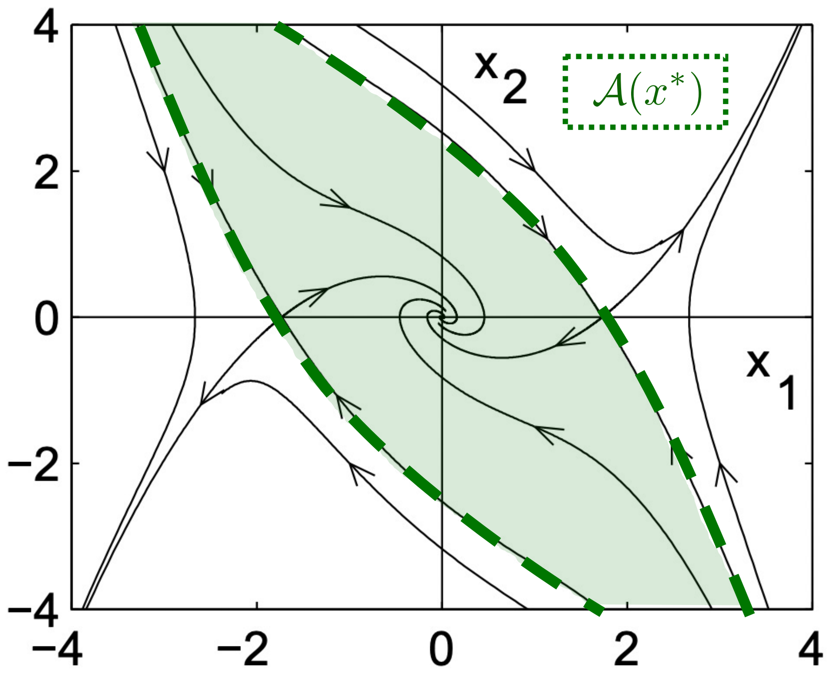
Recurrent sets are subsets of the region of attraction

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \geq t$ s.t. $\phi(t', x_0) \in \mathcal{R}$.

***Assumption** 2. The ω-limit set $\Omega(f)$ is composed by **hyperbolic equilibrium points**, with only one of them, say x^*, being asymptotically stable.*

***Corollary** 2. Let Assumptions 1 and 2 hold, and $\mathcal{R} \subset \mathbb{R}^d$ be a compact set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$ and $\mathcal{R} \cap \Omega(f) = \{x^*\}$ Then:*

$\mathcal{R}$ is recurrent ⬌ $\mathcal{R} \subset \mathcal{A}(x^*)$



Recurrent sets are subsets of the region of attraction

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \geq t$ s.t. $\phi(t', x_0) \in \mathcal{R}$.

?

Corollary *2. Let Assumptions 1 and 2 hold, and $\mathcal{R} \subset \mathbb{R}^d$ be a compact set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$ and $\mathcal{R} \cap \Omega(f) = \{x^*\}$*

Then:

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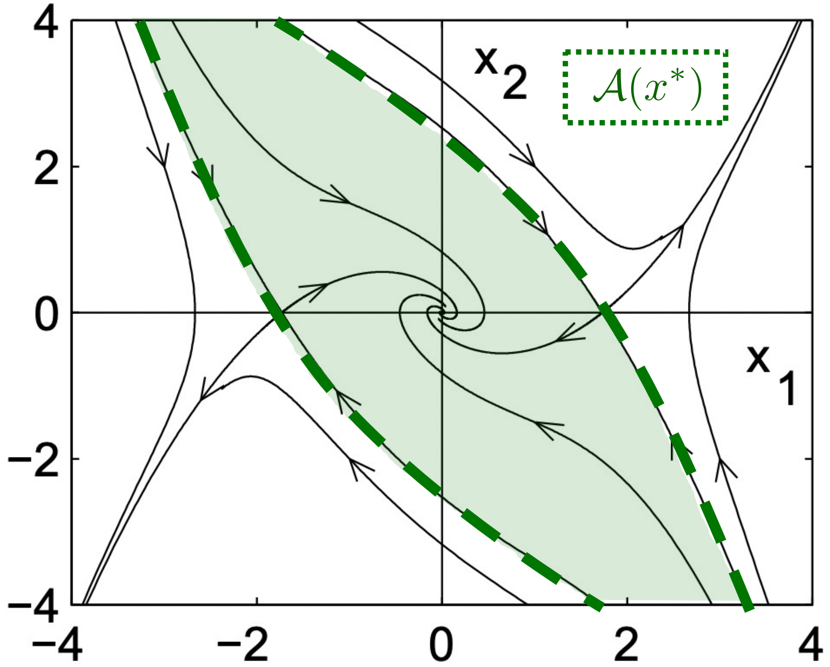


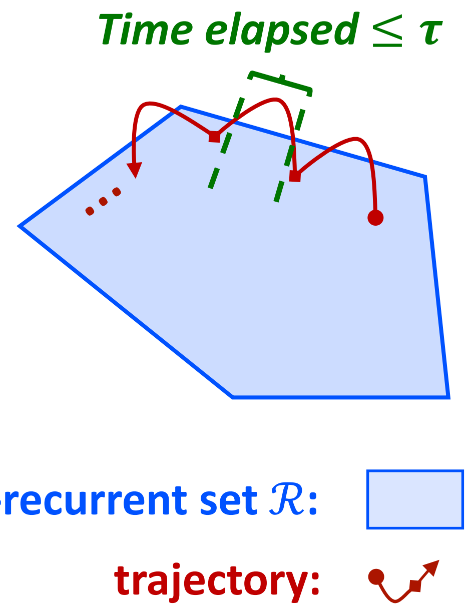
Idea: Use recurrence as a mechanism for finding inner approximations of $\mathcal{A}(x^*)$

Potential Issues:

- We do not know how long it takes to come back!
- We need to adapt results to trajectory samples

τ-recurrent sets

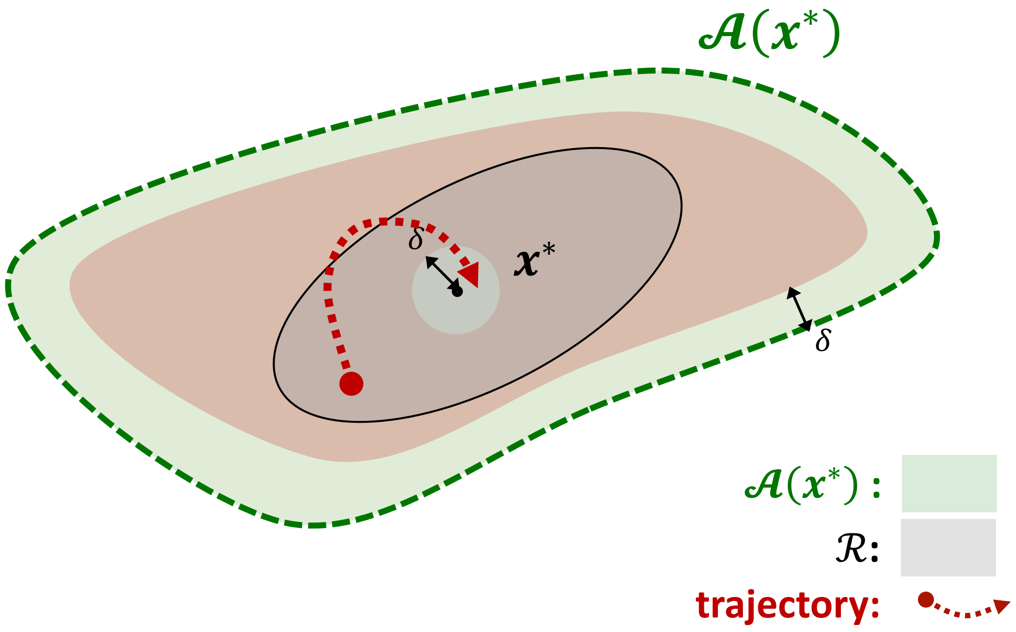
A set $\mathcal{R}$ is **τ-recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists\, t' \in [t, t+\tau]$ such that $\phi(t', x_0) \in \mathcal{R}$



***Theorem** 2. Under Assumption 1, any compact set $\mathcal{R}$* satisfying:

$$x^* + \mathcal{B}_\delta \subseteq \mathcal{R} \subseteq \mathcal{A}(x^*) \backslash \{\partial\mathcal{A}(x^*) + \text{int}\, \mathcal{B}_\delta\}$$

is τ-recurrent for $\tau \geq \bar{\tau}(\delta) := \frac{\underline{c}(\delta) - \bar{c}(\delta)}{a(\delta)}$.



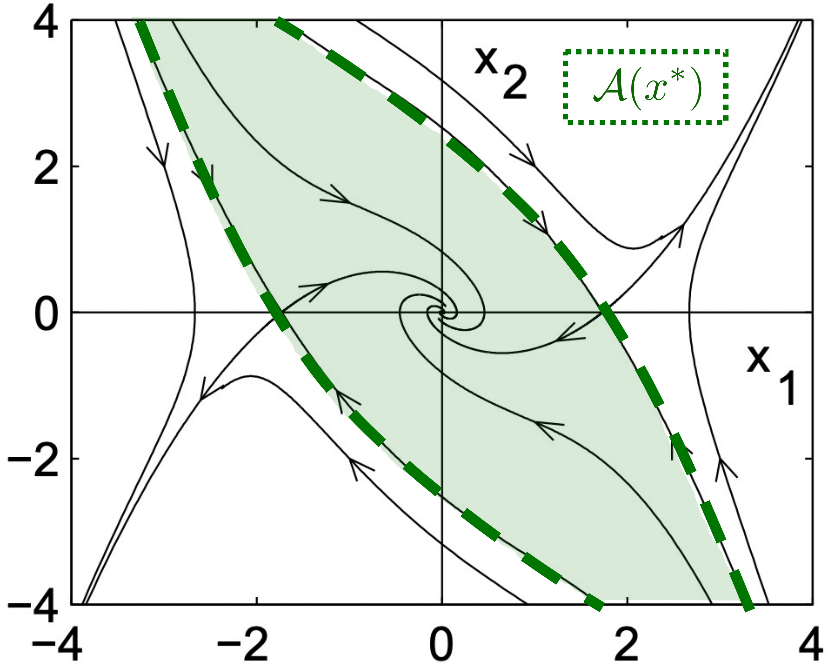
Recurrent sets are subsets of the region of attraction

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for $x_0 \in \mathcal{R}$, for any t $\geq 0 \Rightarrow \exists t' > t,$ s.t. $\phi(t', x_0) \in \mathcal{R}$ **?**

Corollary *2. Let Assumptions 1 and 2 hold, and $\mathcal{R} \subset \mathbb{R}^d$ be a* <u>*compact*</u> *set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$ and $\mathcal{R} \cap \Omega(f) = \{x^*\}$*

Then:

$\mathcal{R}$ is recurrent ⟺ $\mathcal{R} \subset \mathcal{A}(x^*)$



Idea: Use recurrence as a mechanism for finding inner approximations of $\mathcal{A}(x^*)$

Potential Issues:

- We do not know how long it takes to come back! ✔
- We need to adapt results to trajectory samples

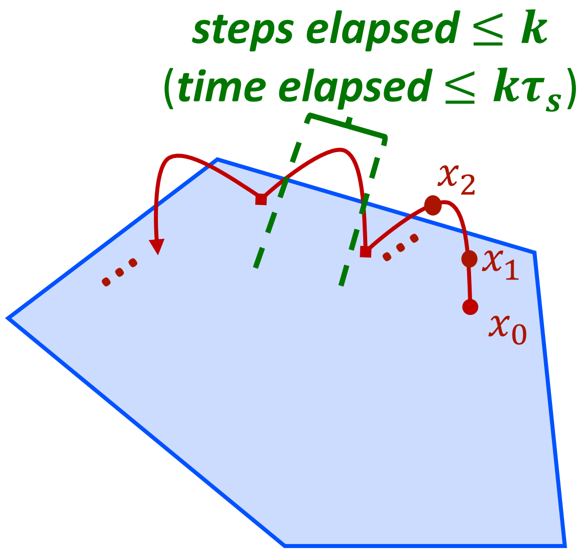
Learning recurrent sets from k-length trajectory samples

- Consider **finite length** trajectories:

$$x_n = \phi(n\tau_s, x_0), \qquad x_0 \in \mathbb{R}^d, n \in \mathbb{N},$$

  where $\tau_s > 0$ is the sampling period.

- A set $\mathcal{R} \subseteq \mathbb{R}^d$ is k**-recurrent** if whenever $x_0 \in \mathcal{R}$, then $\exists\, n \in \{1, \ldots, k\}$ s.t. $x_n \in \mathcal{R}$



***k*-recurrent set $\mathcal{R}$:**

trajectory:

Sufficiency:

$\mathcal{R}$ is k-recurrent ➡ $\mathcal{R}$ is τ-recurrent with $\tau = k\tau_s$ ➕ $\mathcal{R}$ is compact $\partial\mathcal{R} \cap \Omega(f) = \emptyset$ ➡ $\mathcal{R} \subset \mathcal{A}(x^*)$

(Corollary 2, under Assumption 2)

Necessity:

Theorem *3. Under Assumption 1, any compact set* $\mathcal{R}$ satisfying:

$$\mathcal{B}_\delta + x^* \subseteq \mathcal{R} \subseteq \mathcal{A}(x^*) \backslash \{\partial\mathcal{A}(x^*) + \text{int}\, \mathcal{B}_\delta\}$$

is k*-recurrent for any* $k > \bar{k} := \bar{\tau}(\delta)/\tau_s$.

Recurrent sets are subsets of the region of attraction

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for $x_0 \in \mathcal{R}$, $\phi(t, x_0) \notin \mathcal{R} \Rightarrow \exists t' > t,$ s.t. $\phi(t', x_0) \in \mathcal{R}$ **?**

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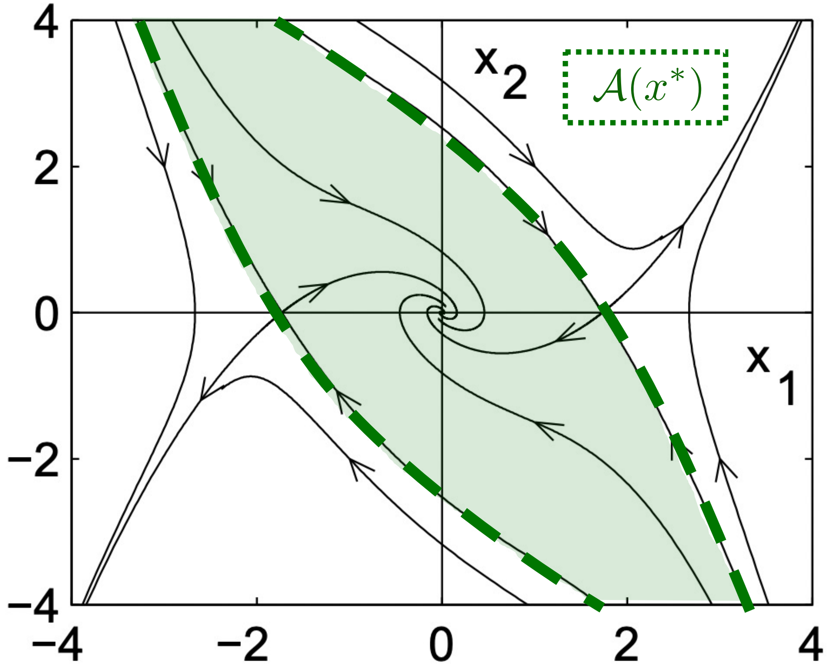

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Potential Issues:

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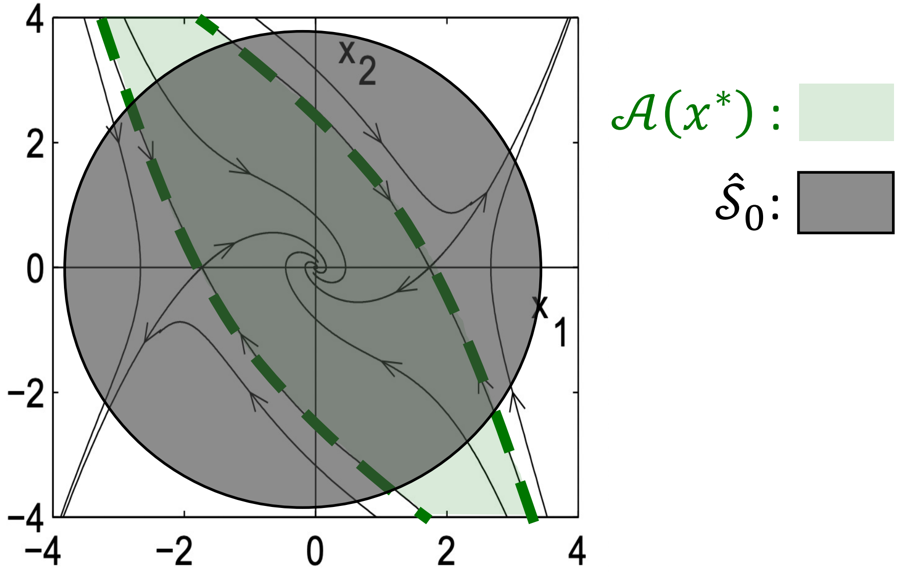
Sphere approximations of RoA

Algorithm: Given k and $\varepsilon > 0$:

At each iteration t

- *Sample trajectories of length k from the sphere $\hat{\mathcal{S}}_t$ until recurrence is violated (counter-example)*



$$\hat{\mathcal{S}}_t \coloneqq \{x \,|\, \|x - x^*\|_2 \leq b_t\}$$

Sphere approximations of RoA

Algorithm: Given k and $\varepsilon > 0$:

At each iteration t

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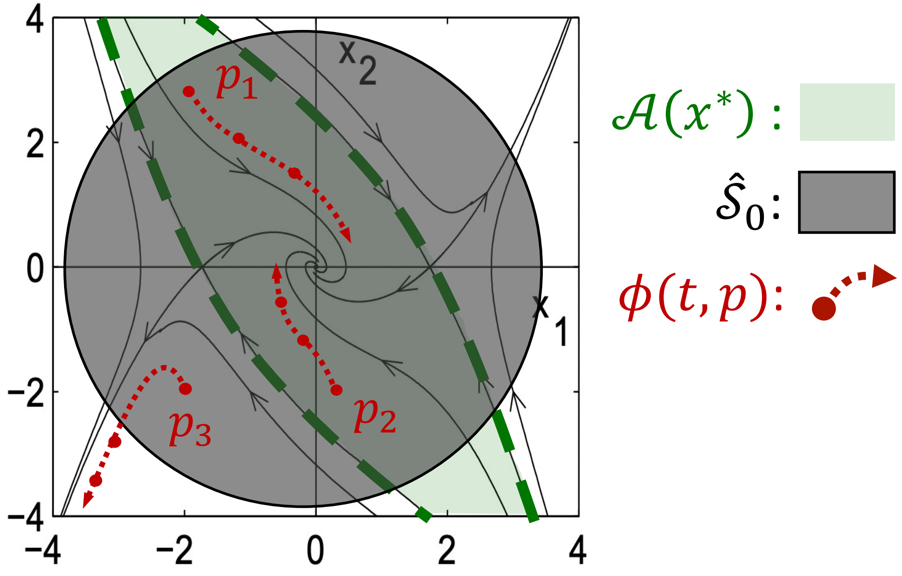
$t = 0$



p_3: counter example

$\hat{\mathcal{S}}_t \coloneqq \{x \,|\, \|x - x^*\|_2 \le b_t\}$

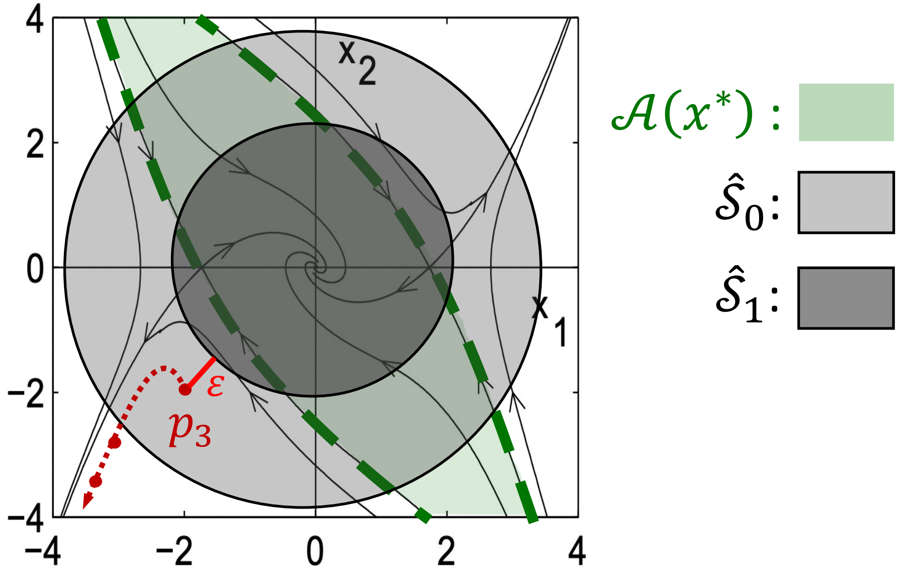
Sphere approximations of RoA

Algorithm: Given k and $\varepsilon > 0$:

At each iteration t

- *Sample trajectories of length k from the sphere $\hat{\mathcal{S}}_t$ until recurrence is violated (counter-example)*
- *Update sphere $\hat{\mathcal{S}}_{t+1}$ to exclude counter example point p_j*

$t = 0$



p_3: counter example

$$\hat{\mathcal{S}}_t \coloneqq \{x \,|\, \|x - x^*\|_2 \leq b_t\}$$

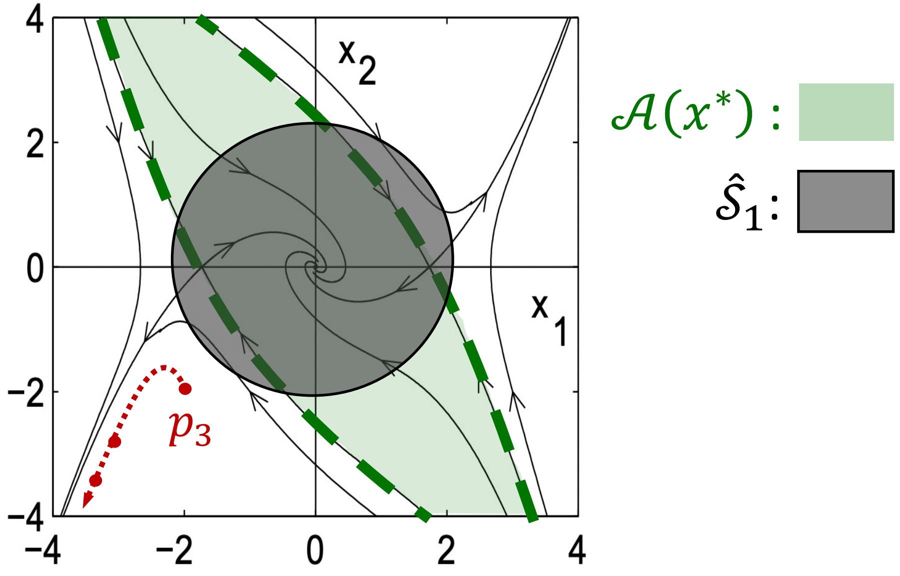
Sphere approximations of RoA

Algorithm: Given k and $\varepsilon > 0$:

At each iteration t

- *Sample trajectories of length k from the sphere $\hat{\mathcal{S}}_t$ until recurrence is violated (counter-example)*
- *Update sphere $\hat{\mathcal{S}}_{t+1}$ to exclude counter example point p_j, and start again*

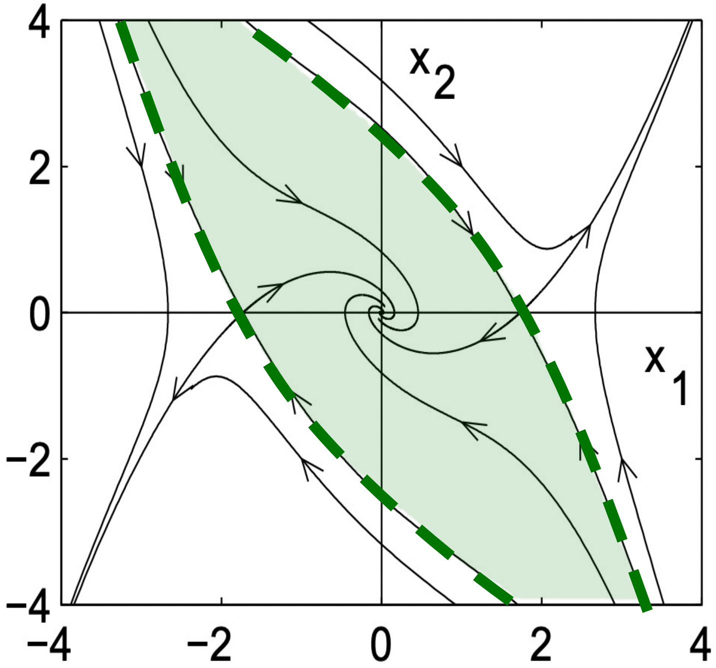
$t = 1$



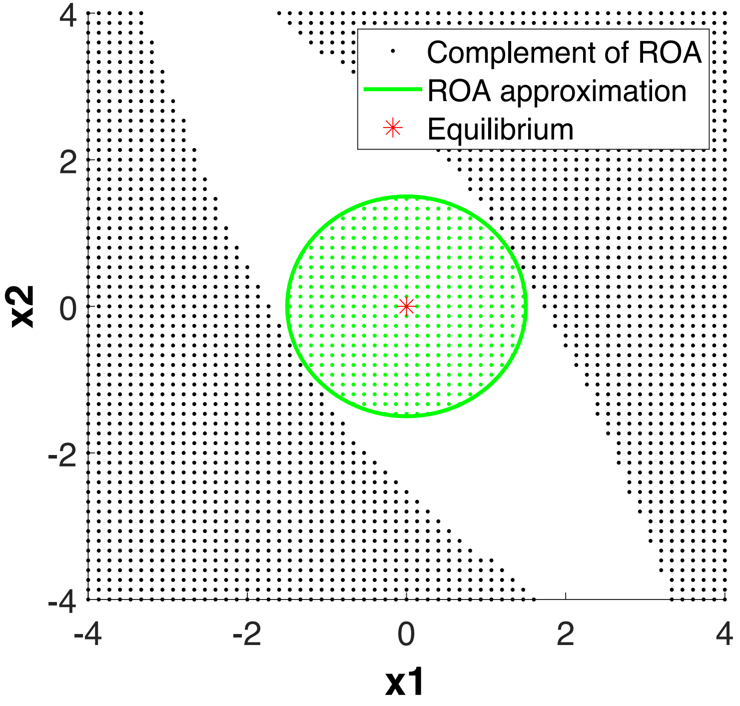
p_3: counter example

$$\hat{\mathcal{S}}_t \coloneqq \{x \,|\, \|x - x^*\|_2 \le b_t\}$$

Algorithm Result - Sphere Approximations

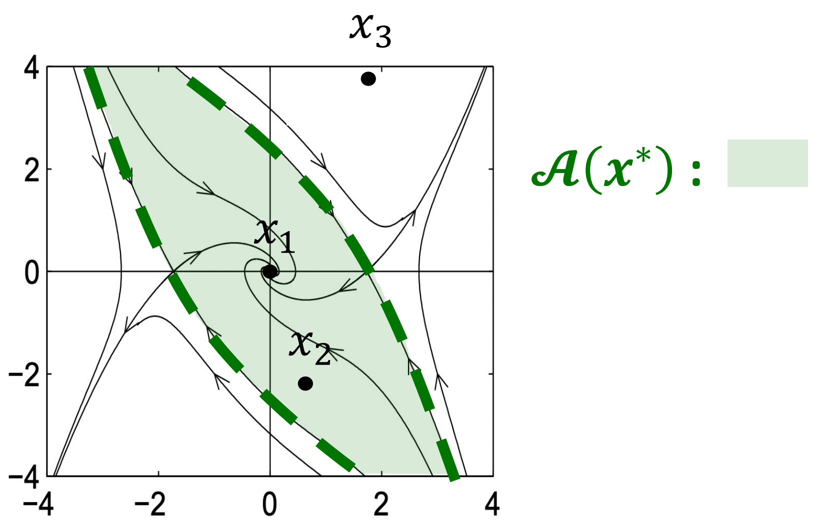


$\mathcal{A}(0)$:



Multi-center approximation

- **Consider $\boldsymbol{h} \in \mathbb{N}^+$ center points $\boldsymbol{x_q}$ indexed by $\boldsymbol{q} \in \{\mathbf{1}, \dots, \boldsymbol{h}\}$.**
  - Let the first center point $x_1 = x^* = 0$
  - Additional center point $x_2, \dots, x_h$ can be designed chosen uniformly.



Multi-center approximation

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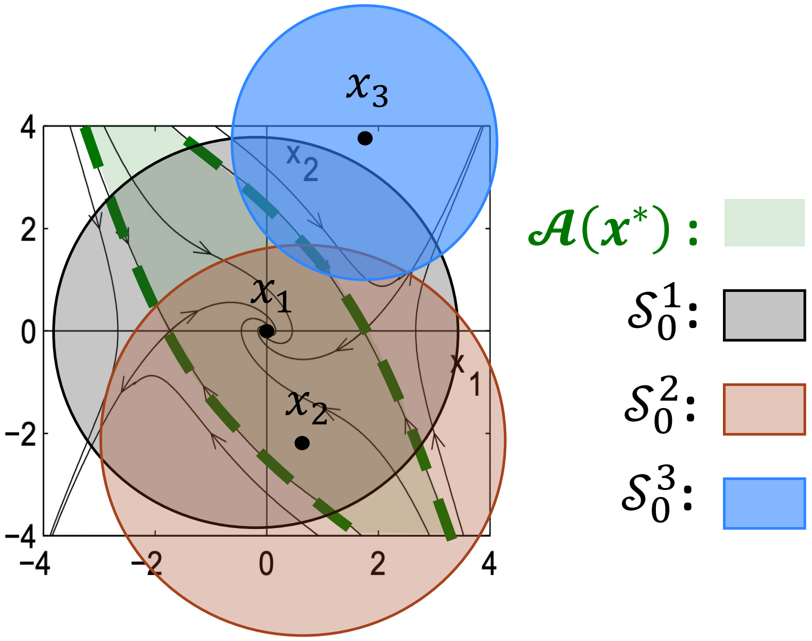
- **Respectively define approximations centered at each x_q**
  - $\mathcal{S}_t^q := \{x | \|x - x_q\|_2 \leq b_q^{(i)}\}$



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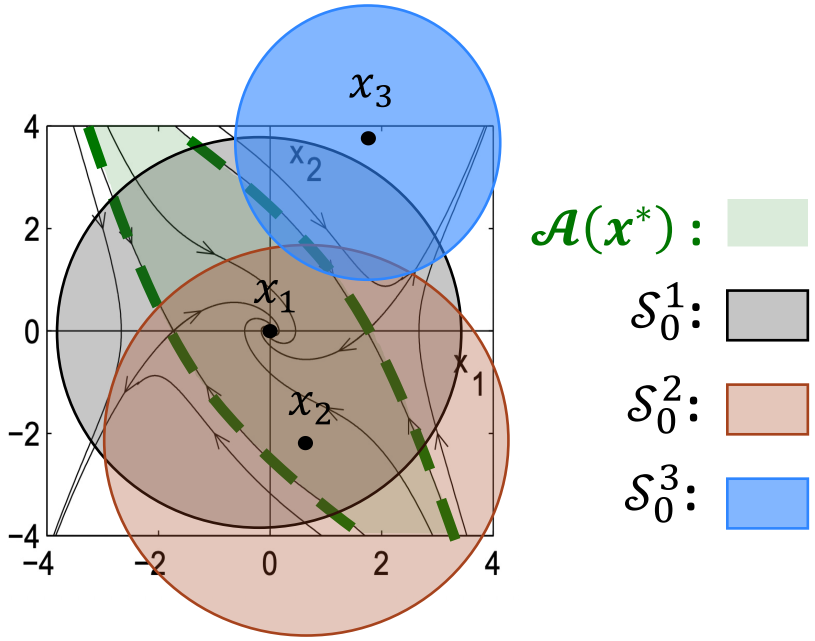
- **Multi-center approximation given by** $\hat{\mathcal{S}}_t = \cup_{q=1}^h \mathcal{S}_t^q$



Multi-center approximation

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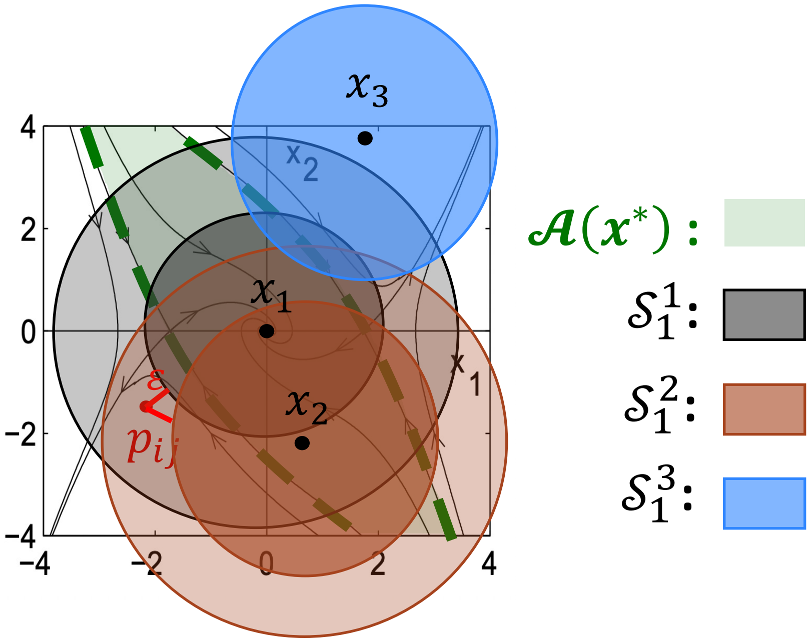
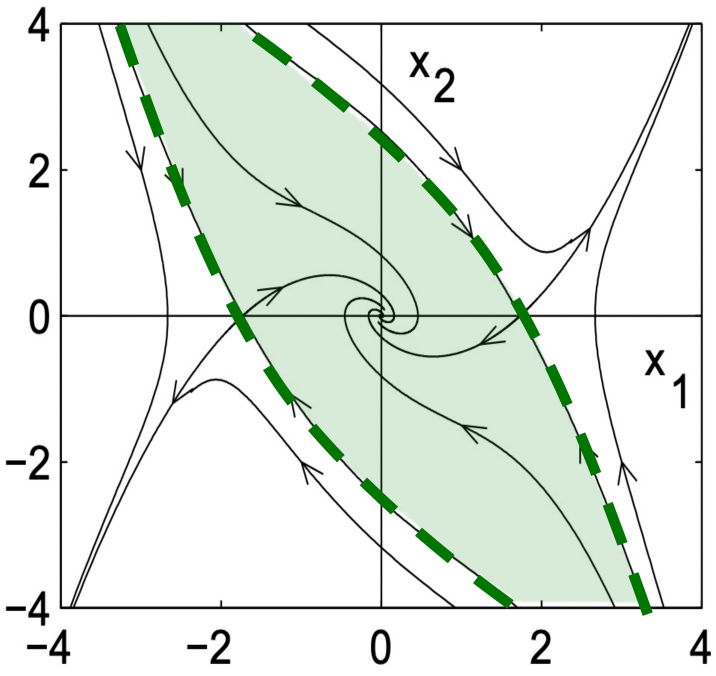
- **Multi-center approximation given by** $\hat{\mathcal{S}}_t = \cup_{q=1}^{h} \mathcal{S}_t^q$

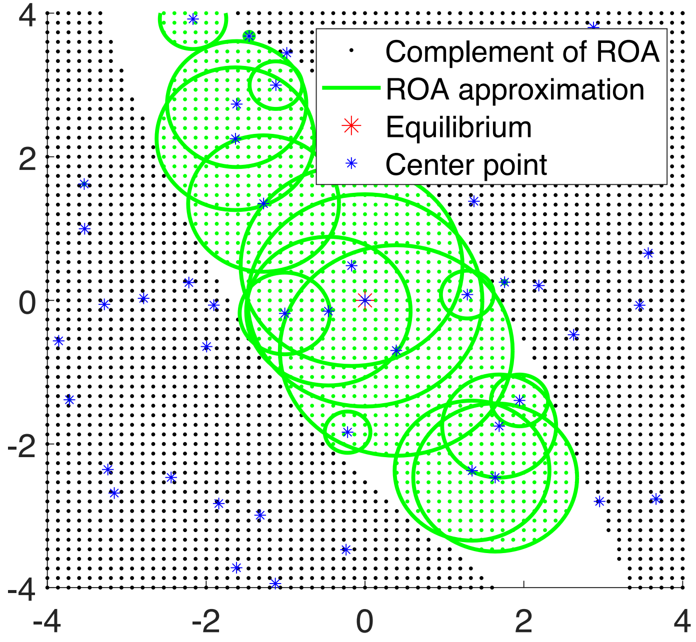
- **If $\mathbf{p_{ij}}$ is a counter-example w.r.t** $\hat{\mathcal{S}}_{\text{multi}}^{(i)}$
  - We shrink every $\hat{\mathcal{S}}_q^{(i)}$ satisfying $p_{ij} \in \hat{\mathcal{S}}_q^{(i)}$
  - For the rest approximations, we simply let $\hat{\mathcal{S}}_q^{(i+1)} = \hat{\mathcal{S}}_q^{(i)}$



Numerical illustrations – Multi-center approximation



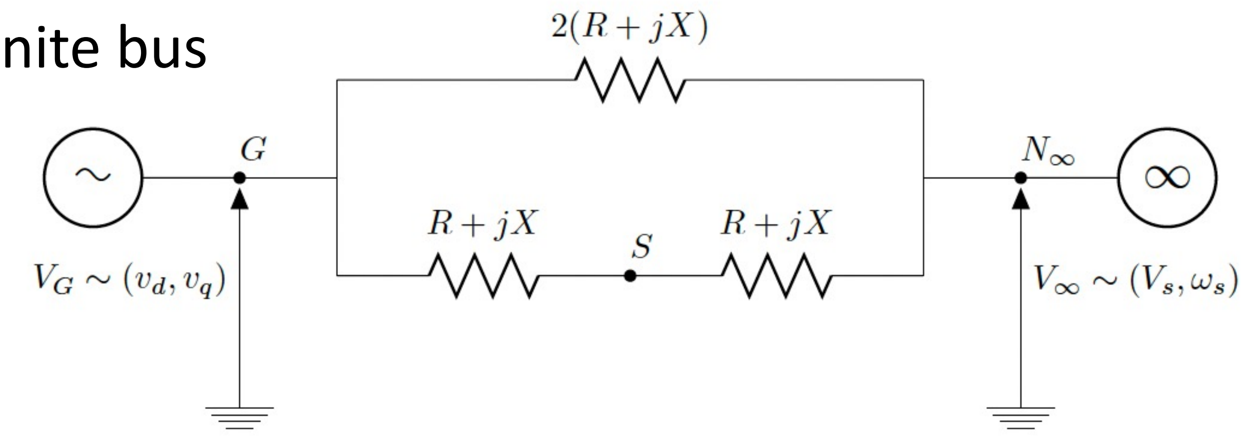
$\mathcal{A}(x^*)$:



50 sphere approximation

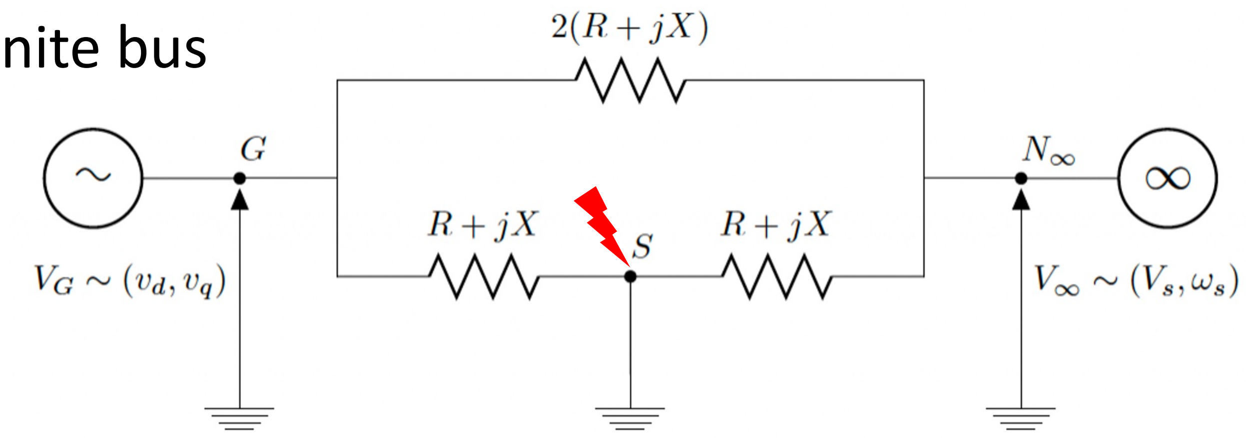
Transient Stability Analysis

- Synchronous machine connected to infinite bus



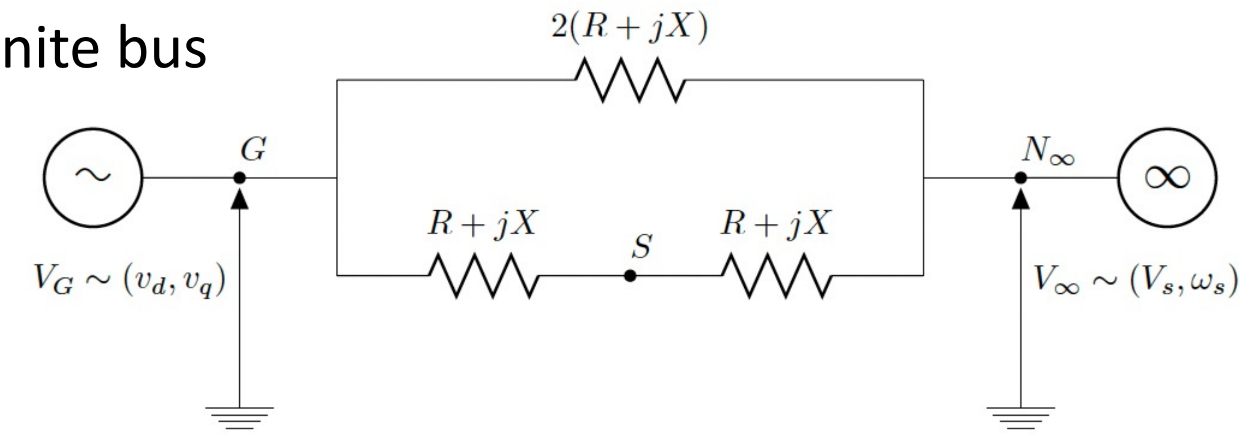
Transient Stability Analysis

- Synchronous machine connected to infinite bus
- t_1 lower line is short-circuited



Transient Stability Analysis

- Synchronous machine connected to infinite bus
- t_1 lower line is short-circuited
- t_2 fault is cleared



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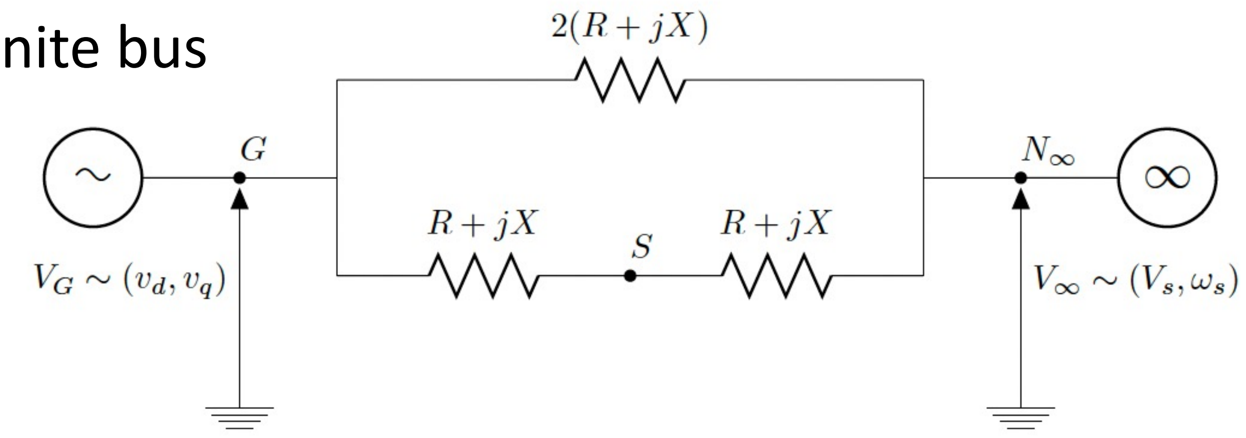

$$\frac{d\delta}{dt} = \omega - \omega_s$$

$$2H\frac{d\omega}{dt} = P_m - (v_d i_d + v_q i_q + e i_d^2 + r i_q^2)$$

$$T'_{d_0}\frac{de'_q}{dt} = -e'_q - (x_d - x'_d)i_d + E_{fd}$$

$$T_a\frac{dE_{fd}}{dt} = -E_{fd} + K_a(V_{ref} - V_t)$$

$$T_g\frac{dP_m}{dt} = -P_m + P_{ref} + K_g(\omega_{ref} - \omega)$$

$$i_q = \frac{(X - x'_d)V_s\sin(\delta) - (R+r)(V_s\cos(\delta) - e'_q)}{(R+r)^2 + (X + x'_d)(X + x_q)}$$

$$i_d = \frac{X - x_q}{R+r}i_q - \frac{1}{R+r}V_s\sin(\delta)$$

$$v_d = x_q i_q - r - i_d$$

$$v_q = R i_q + X i_d + V_s\cos(\delta)$$

$$V_t = \sqrt{v_d^2 + v_q^2}$$

$T'_{d_0} = 9.67$	$x_d = 2.38$	$x'_d = 0.336$	$x_q = 1.21$
$H = 3$	$r = 0.002$	$\omega_s = \omega_{ref} = 1$	$R = 0.01$
$X = 1.185$	$V_s = 1$	$T_a = 1$	$K_a = 70$
$V_{ref} = 1$	$T_g = 0.4$	$K_g = 0.5$	$P_{ref} = 0.7$

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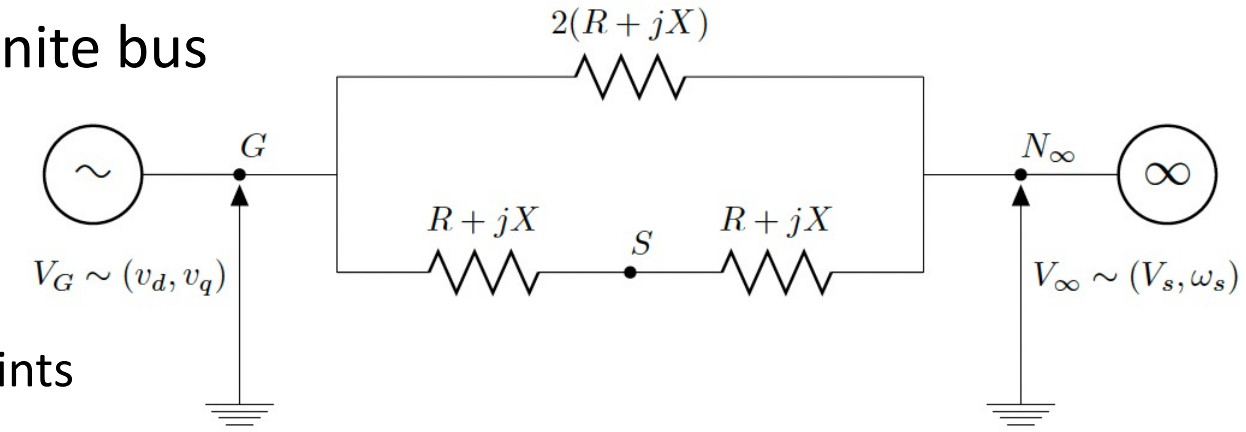
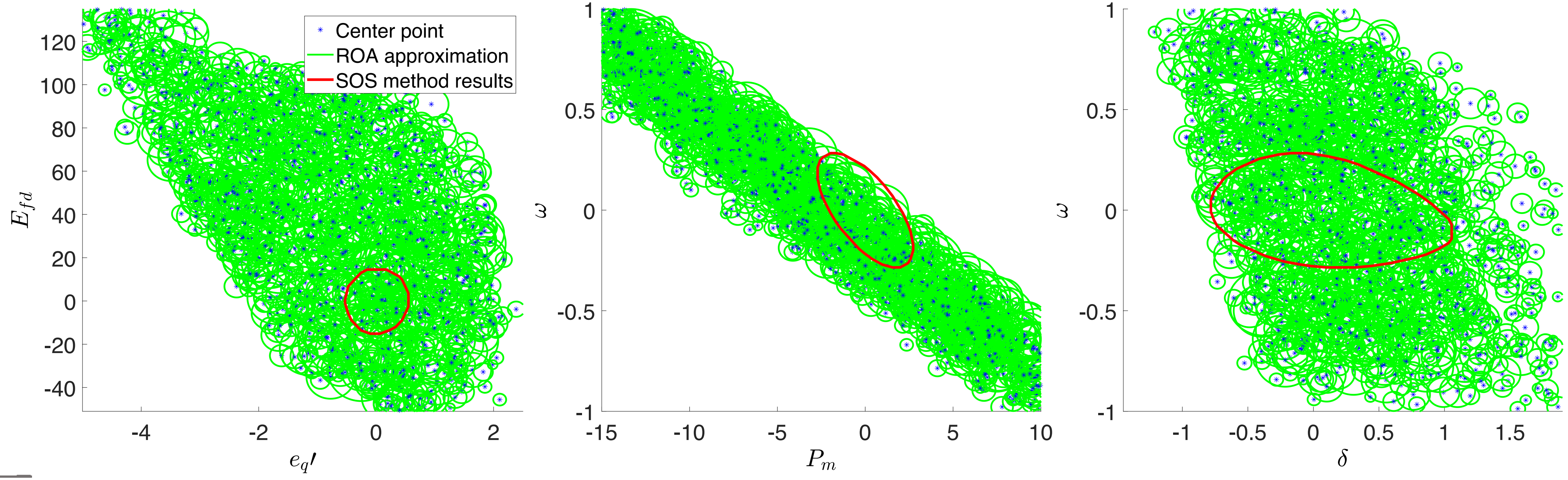


Multi-center in **green**: $\tau_s = 1$, $k = 40$, 1.5K points



M. Tacchi et al *Power system transient stability analysis using SoS programming,* Power System Computation Conference (PSCC) 2018
Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* Control and Decision Conference (**CDC) 2022**

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Recurrently Non-Increasing Lyapunov Functions

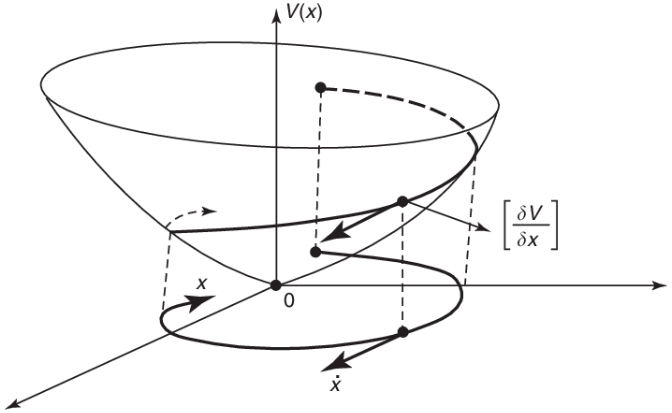
R. Siegelmann, Y. Shen, F. Paganini, and E. Mallada, “A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions”, submitted CDC 2023

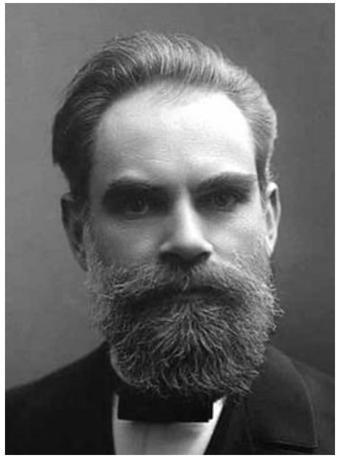
Lyapunov's Direct Method

Key idea: Make sub-level sets invariant to trap trajectories

Theorem [Lyapunov '1892]*. Given $V: \mathbb{R}^d \to \mathbb{R}_{\geq 0}$, with $V(x) > 0, \forall x \in \mathbb{R}^d \backslash \{x^*\}$, then:*

- $\dot{V} \leq 0 \to x^*$ *stable*
- $\dot{V} < 0 \to x^*$ *as. stable*





Challenge: Couples shape of V and vector field f

- Towards decoupling the $V - f$ geometry
  - Controlling regions where $\dot{V} \geq 0$ **[Karalfyllis '09, Liu et al '20]**
  - Higher order conditions: $g(V^{(q)}, \dots, \dot{V}, V) \leq 0$ **[Butz '69, Gunderson '71, Ahmadi '06, Meigoli '12]**
  - Discretization approach: $V\big(x(T)\big) \leq V(x(0))$ **[Coron et al '94, Aeyels et. al '98, Karafyllis '12]**

Karafyllis, Kravaris, Kalogerakis. Relaxed Lyapunov criteria for robust global stabilisation of non-linear systems. International Journal of Control, 2009
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Ahmadi. Non-monotonic Lyapunov functions for stability of nonlinear and switched systems: theory and computation, 2008
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Coron, Lionel Rosier. A relation between continuous time-varying and discontinuous feedback stabilization. J. Math. Syst., Estimation, Control, 1994
Aeyels, Peuteman. A new asymptotic stability criterion for nonlinear time-variant differential equations. IEEE Transactions on automatic control, 1998
Karafyllis. Can we prove stability by using a positive definite function with non sign-definite derivative? IMA Journal of Mathematical Control and Information, 2012

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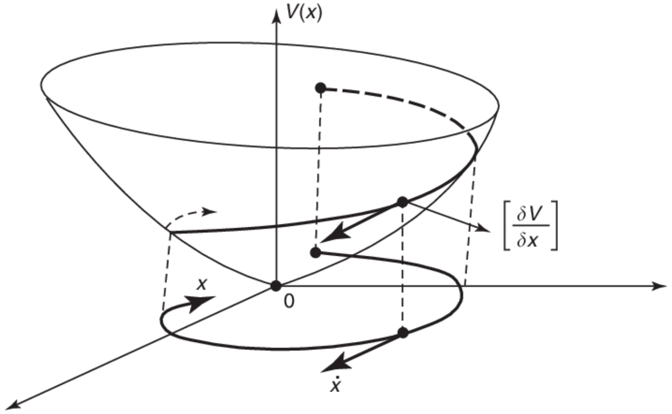

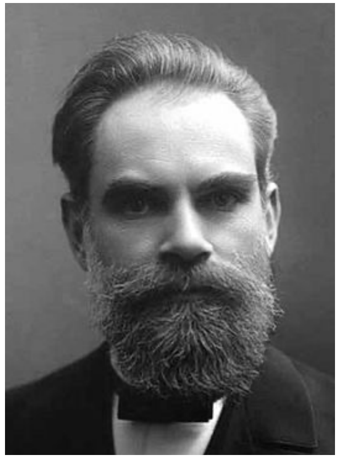

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Question: Can we provide stability conditions based on recurrence?

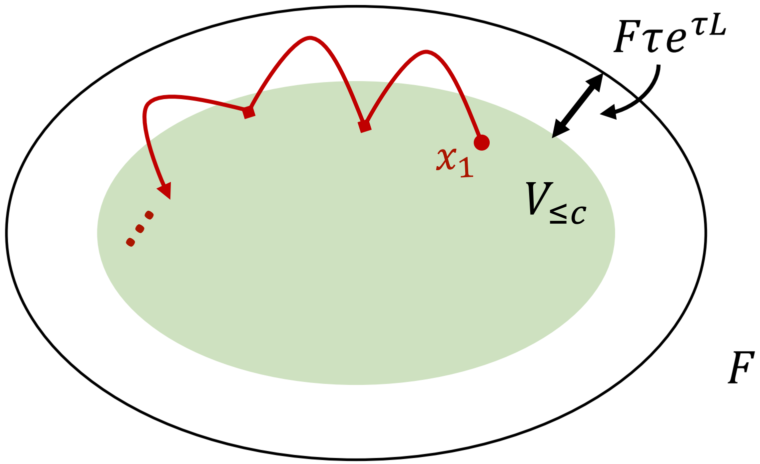
Recurrently Decreasing Lyapunov Functions

A continuously differentiable function $V: \mathbb{R}^d \to \mathbb{R}_+$ is a **recurrently non-increasing Lyapunov function** over intervals of length $\boldsymbol{\tau}$ if

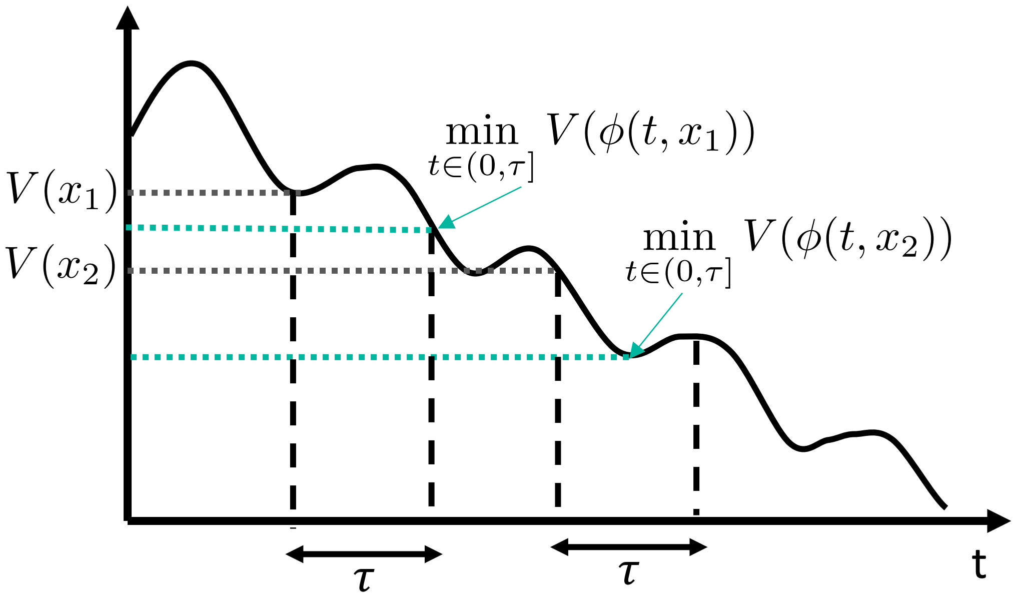
$$\mathcal{L}_f^{(0,\tau]} V(x) := \min_{t \in (0,\tau]} V(\phi(t,x)) - V(x) \leq 0 \quad \forall x \in \mathbb{R}^d$$

Preliminaries:

- Sub-level sets $\{V(x) \leq c\}$ are $\boldsymbol{\tau}$-recurrent sets.
- When f is globally L-Lipschitz, one can trap trajectories.



$F = \max_{x \in S} ||f(x)||$



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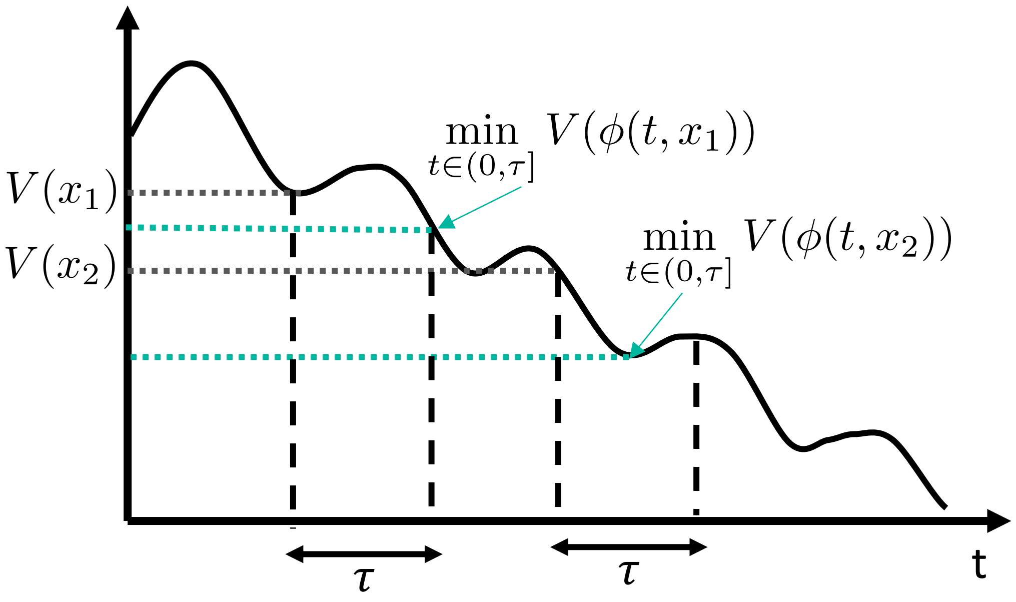
$$\mathcal{L}_f^{(0,\tau]} V(x) := \min_{t\in(0,\tau]} V(\phi(t,x)) - V(x) \leq 0 \quad \forall x \in \mathbb{R}^d \qquad \textbf{(*)}$$

Theorem [CDC 23*]: *Let $V: \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ be a recurrently non-increasing Lyapunov function over intervals of length τ.*

- *Then when f is L-Lipschitz, the equilibrium x^* is stable.*
- *Further, if the **inequality is strict**, then x^* is asymptotically stable!*



Siegelmann, Shen, Paganini, M, *A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions*, **submitted CDC 2023**

Exponential Stability Analysis

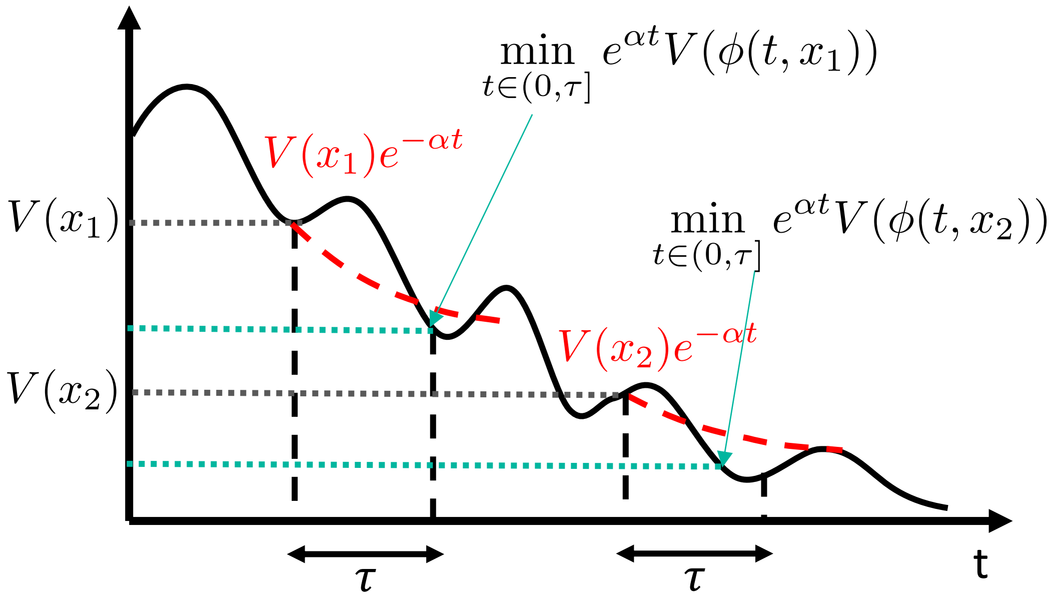
The function $V: \mathbb{R}^d \to \mathbb{R}_+$ is ***α*-exponentially recurrently non-increasing Lyapunov function** over intervals of length $\boldsymbol{\tau}$ if

$$\mathcal{L}_{f,\alpha}^{(0,\tau]} V(x) := \min_{t \in (0,\tau]} e^{\alpha t} V(\phi(t,x)) - V(x) \leq 0 \quad \forall x \in \mathbb{R}^d$$

Theorem [CDC 23*]: *Let $V: \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ satisfy*

$$\alpha_1 \|x - x^*\| \leq V(x) \leq \alpha_2 \|x - x^*\|.$$

Then, if V is α-exponentially recurrently non-increasing Lyapunov function over intervals of length τ, then x^* is exponentially stable.



Siegelmann, Shen, Paganini, M, *A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions,*
***submitted CDC 2023**

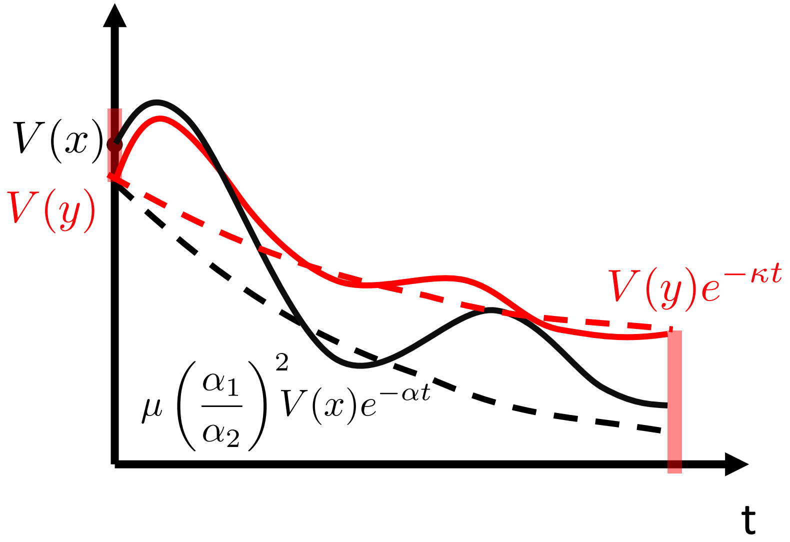
Verification of Exponential Stability

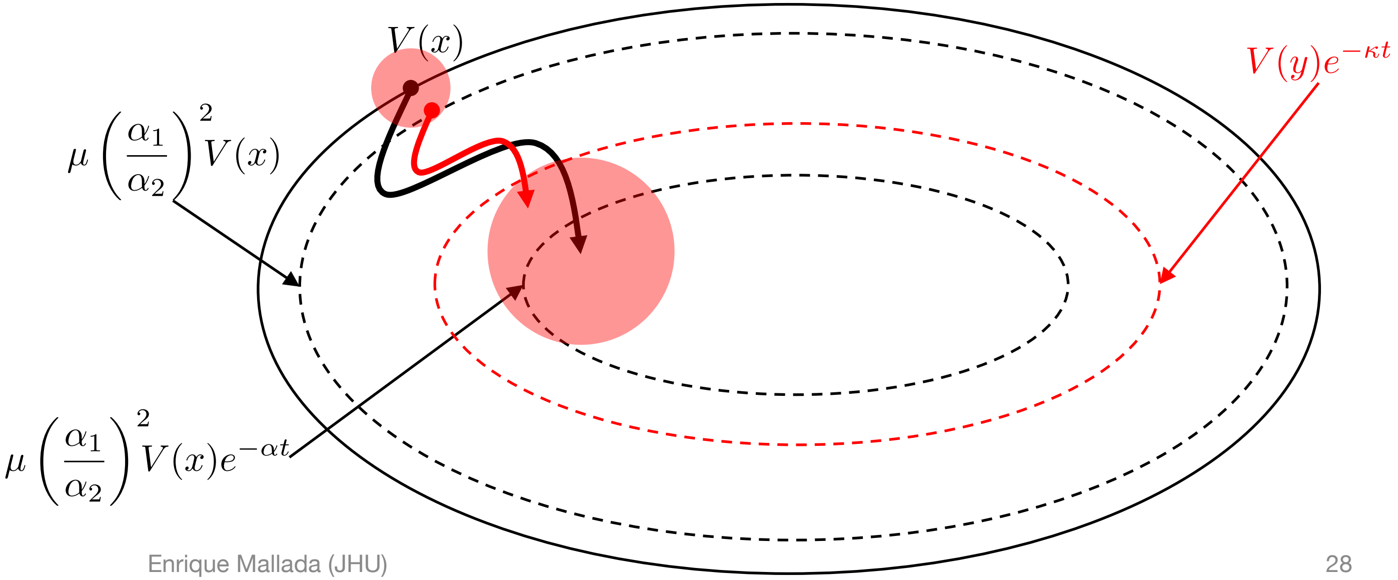
Proposition [CDC 23*]: *Let* $V: \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ *satisfy* $\alpha_1 \|x - x^*\| \leq V(x) \leq \alpha_2 \|x - x^*\|$, *and* $0 < \mu < 1$. *Then, whenever*

$$\min_{t \in (0,\tau]} e^{\alpha t} V(\phi(x,t)) \leq \mu \left(\frac{\alpha_1}{\alpha_2}\right)^2 V(x)$$

if $\exists \kappa, \rho > 0$ *s.t.* $\rho < g(\kappa, \mu, \alpha_1, \alpha_2)$, *for all* y *with* $\|y - x\| \leq r := \frac{\rho}{\alpha_2} V(x)$

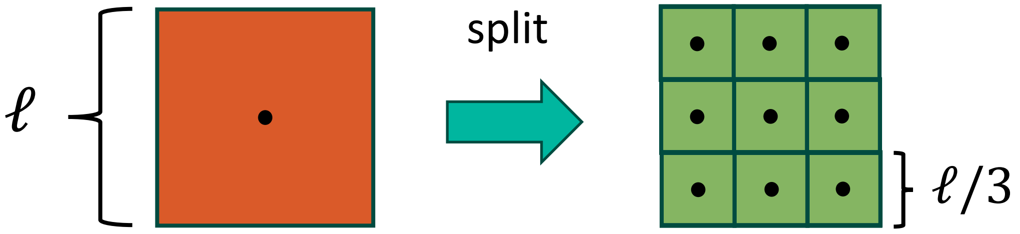
$$\min_{t \in (0,\tau]} e^{\kappa t} V(\phi(y,t)) \leq V(y)$$

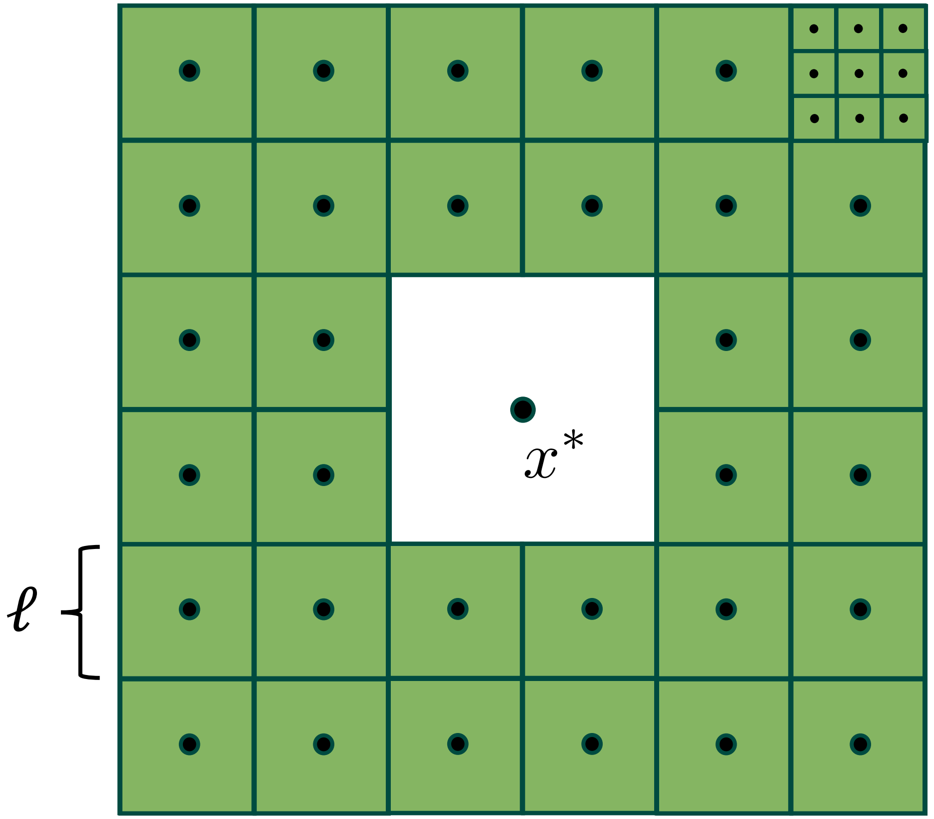




GPU-based Algorithm

- **Basic Algorithm:**
  - Consider $V(x) = ||x - x^*||_\infty$
  - Build a grid of hypercubes surrounding x^*
  - Test the center point and find κ s.t. the verified radius is $r \geq \ell/2$
  - If one hypercube is **not verified**, **split in $\mathbf{3^d}$** parts
  - Repeat testing of new points





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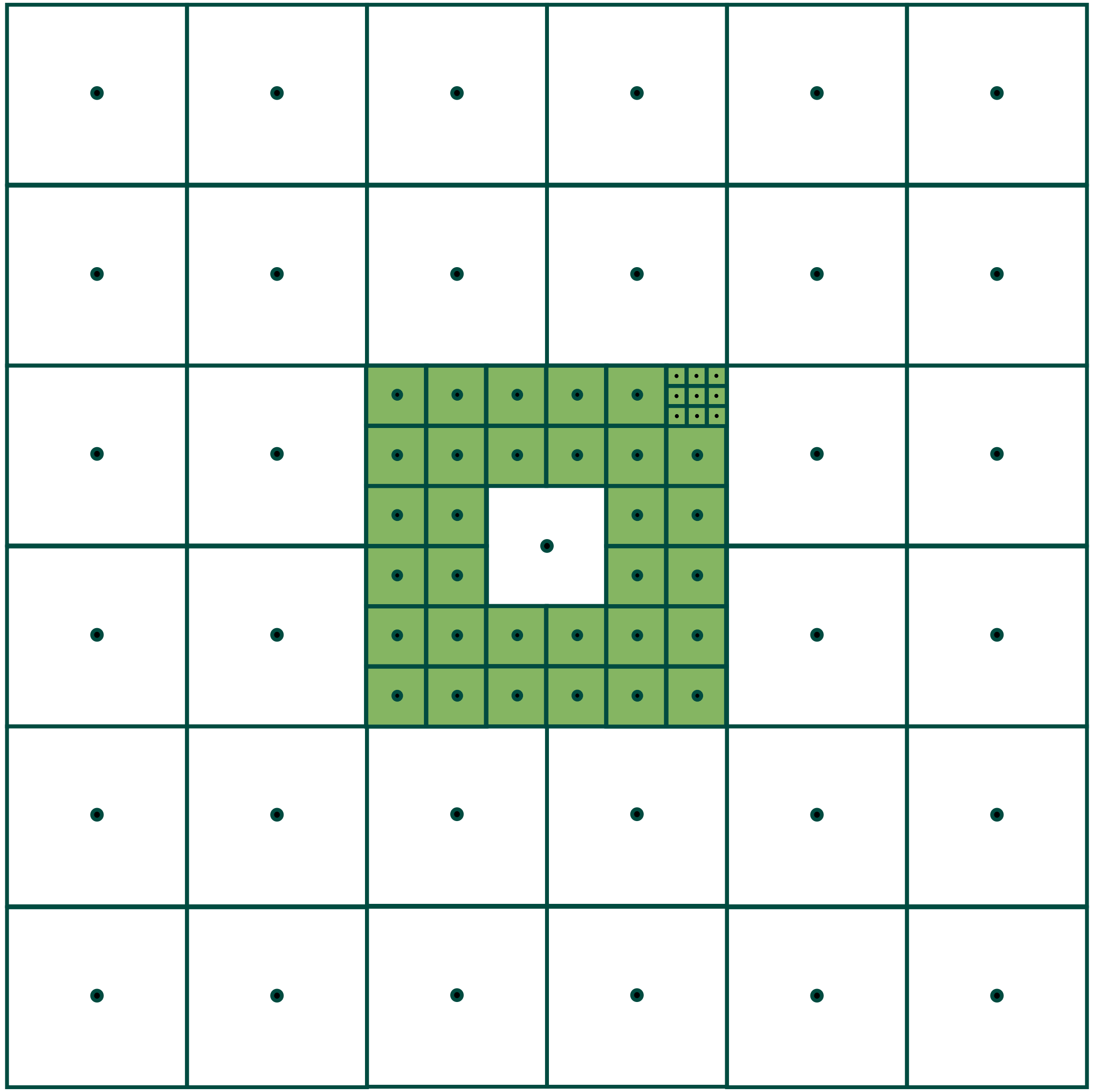

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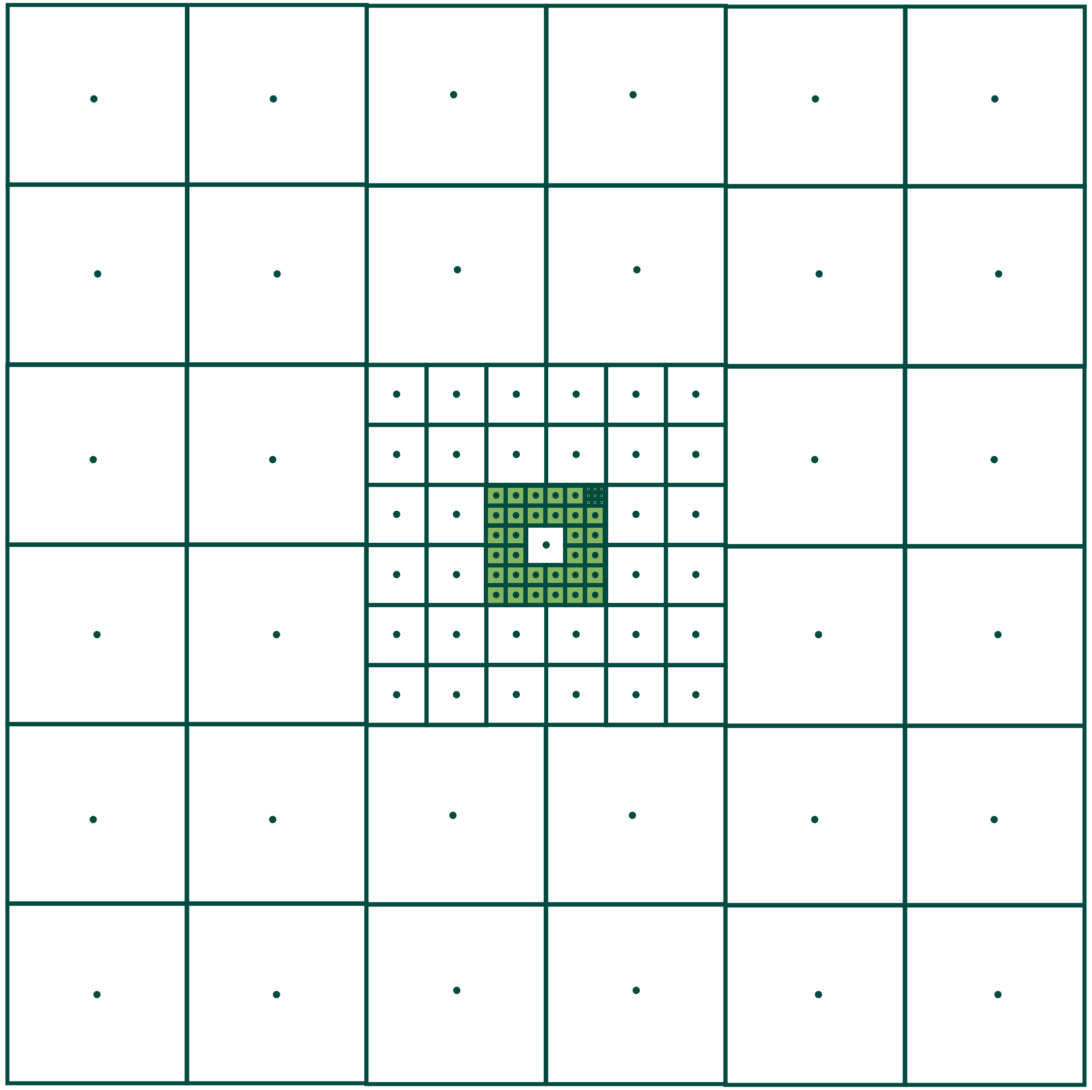


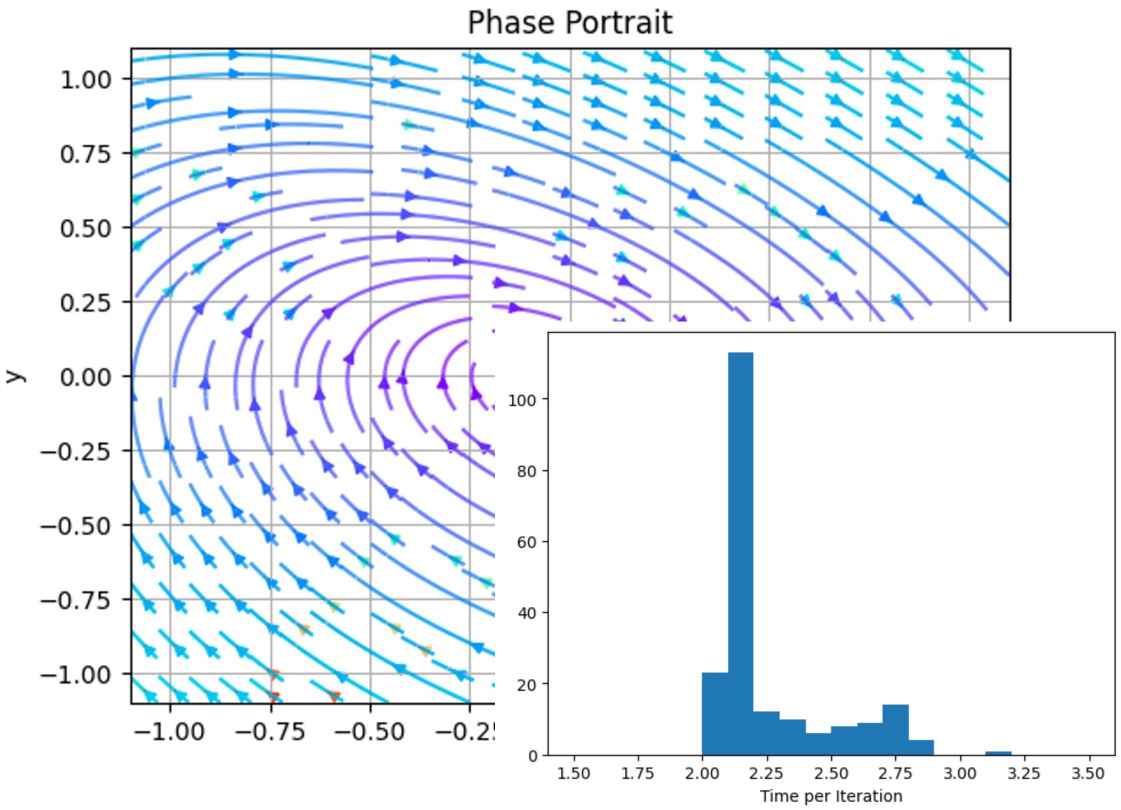
Numerical Illustration

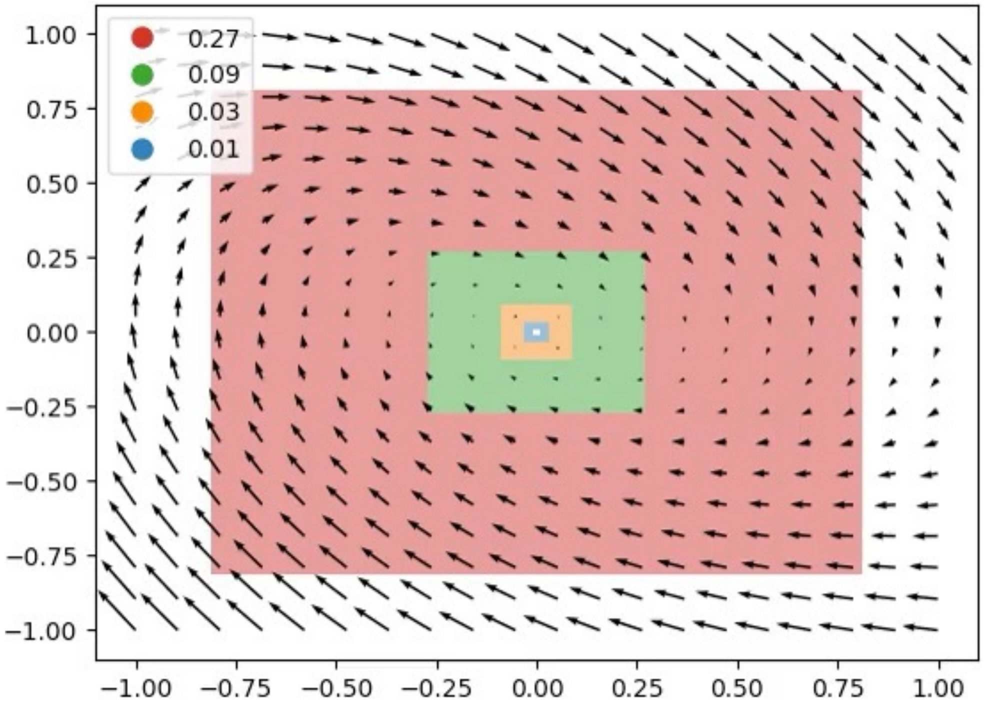
Consider the 2-d non-linear system: $\dot{x} = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} x + B \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$

with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

Parameter	Value
L	1.8
τ	1.5
ℓ	0.01

$\sigma = 0.2$





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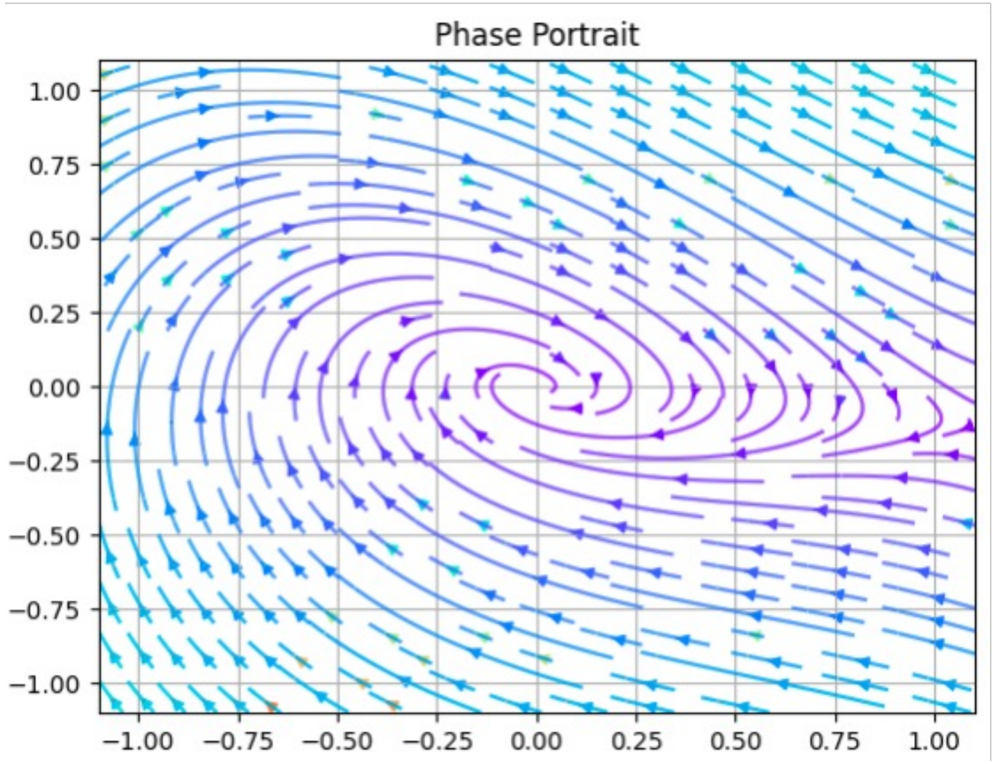
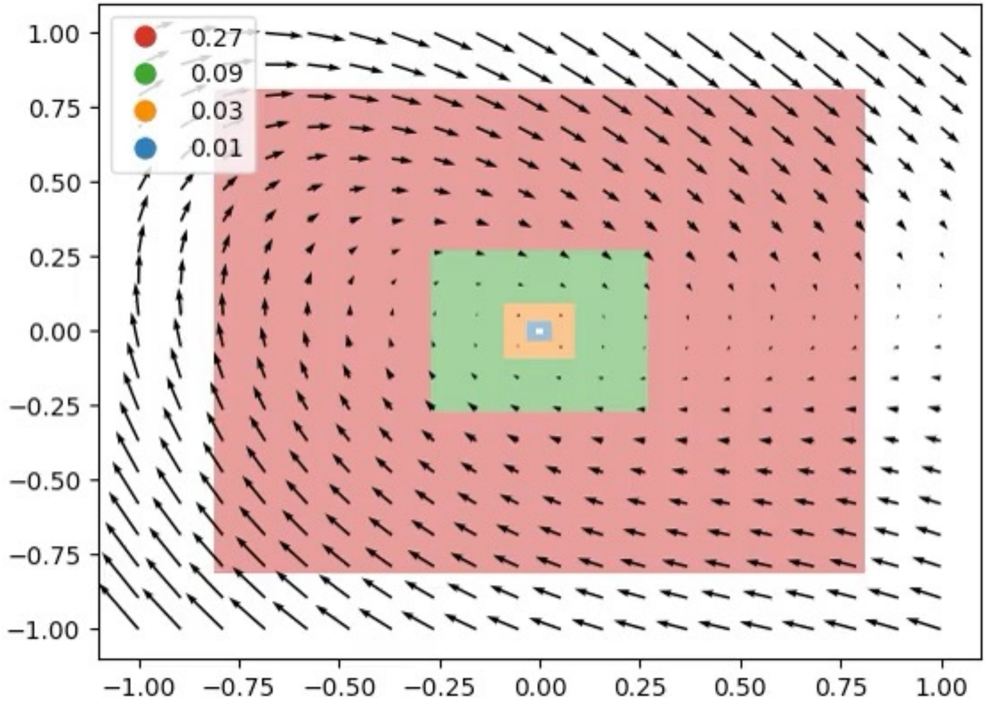
with $B_{ij} \sim \mathcal{N}(0, \sigma^2)$

$\sigma = 0.5$





Conclusions and Future work

- **Takeaways**
  - Proposed a **relaxed notion of invariance** known as **recurrence.**
  - Provide **necessary and sufficient conditions** for a recurrent set to be an **inner approximation** of the ROA.
  - Generalized Lyapunov Theory **for recurrently decreasing functions** using recurrent sets
  - Our algorithms are **parallelizable via GPUs and progressive/sequential.**

- **Ongoing work**
  - **Recurrent sets:** Sample complexity bounds, smart choice of multi-points, control recurrent sets, GPU implementation
  - **Lyapunov functions:** Generalize other Lyapunov notions, Control Lyapunov Functions, Barrier Functions, Control Barrier Functions, etc.

Thanks!

Related Publications:

[arXiv 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* **CDC 2022, journal preprint arXiv:2204.10372**.

[CDC 23] Siegelmann, Shen, Paganini, M, *A recurrence-based direct method for stability analysis and GPU-based verification of non-monotonic Lyapunov functions*, **submitted CDC 2023**

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