Model-Free Analysis of Dynamical Systems Using Recurrent Sets

Enrique Mallada

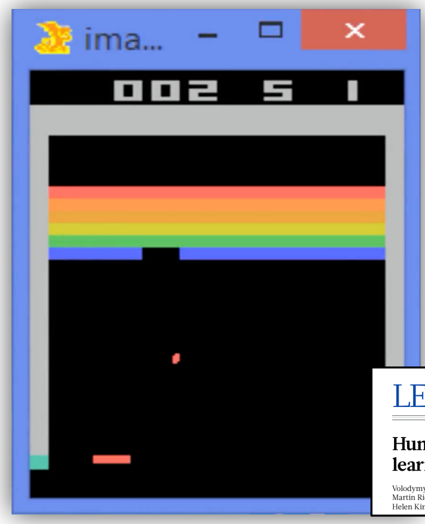
JOHNS HOPKINS UNIVERSITY

Data Science Seminar
Johns Hopkins University

November 2, 2022

A World of Success Stories

2017 Google DeepMind's DQN




LETTER

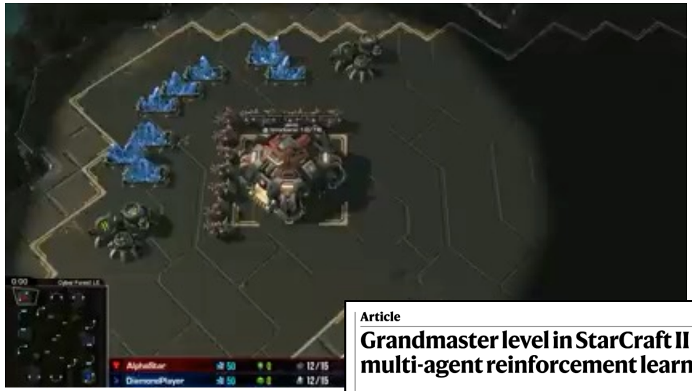
doi:10.1038/nature14236

Human-level control through deep reinforcement learning

Volodymyr Mnih[1]*, Koray Kavukcuoglu[1]*, David Silver[1]*, Andrei A. Rusu[1], Joel Veness[1], Marc G. Bellemare[1], Alex Graves[1], Martin Riedmiller[1], Andreas K. Fidjeland[1], Georg Ostrovski[1], Stig Petersen[1], Charles Beattie[1], Amir Sadik[1], Ioannis Antonoglou[1], Helen King[1], Dharshan Kumaran[1], Daan Wierstra[1], Shane Legg[1] & Demis Hassabis[1]

2017 AlphaZero – Chess, Shogi, Go

2019 AlphaStar – Starcraft II




Article

Grandmaster level in StarCraft II using multi-agent reinforcement learning

https://doi.org/10.1038/s41586-019-1724-z

Received: 30 August 2019

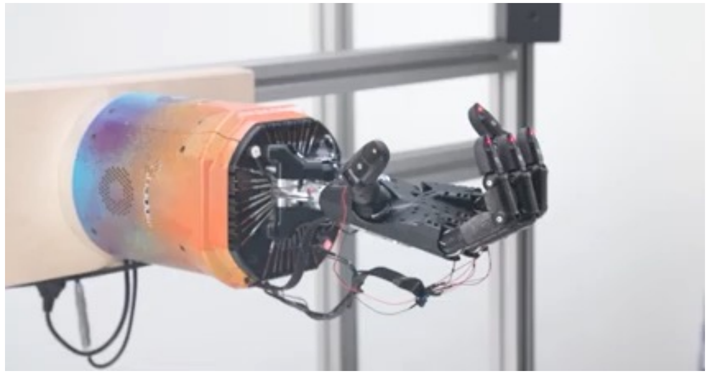
Accepted: 10 October 2019

Published online: 30 October 2019

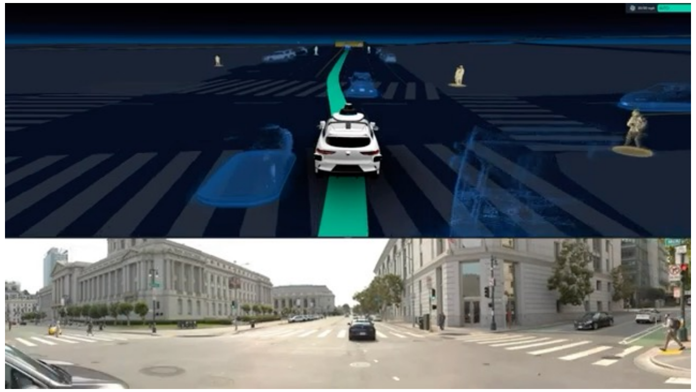
Oriol Vinyals[1,3]*, Igor Babuschkin[1,3], Wojciech M. Czarnecki[1,3], Michaël Mathieu[1,3], Andrew Dudzik[1,3], Junyoung Chung[1,3], David H. Choi[1,3], Richard Powell[1,3], Timo Ewalds[1,3], Petko Georgiev[1,3], Junhyuk Oh[1,3], Dan Horgan[1,3], Manuel Kroiss[1,3], Ivo Danihelka[1,3], Aja Huang[1,3], Laurent Sifre[1,3], Trevor Cai[1,3], John P. Agapiou[1,3], Max Jaderberg[1], Alexander S. Vezhnevets[1], Rémi Leblond[1], Tobias Pohlen[1], Valentin Dalibard[1], David Budden[1], Yury Sulsky[1], James Molloy[1], Tom L. Paine[1], Caglar Gulcehre[1], Ziyu Wang[1], Tobias Pfaff[1], Yuhuai Wu[1], Roman Ring[1], Dani Yogatama[1], Dario Wünsch[2], Katrina McKinney[1], Oliver Smith[1], Tom Schaul[1], Timothy Lillicrap[1], Koray Kavukcuoglu[1], Demis Hassabis[1], Chris Apps[1,3] & David Silver[1,3]*

Boston Dynamics

OpenAI – Rubik's Cube



Waymo



Reality Kicks In

Angry Residents, Abrupt Stops: Waymo Vehicles Are Still Causing Problems in Arizona

RAY STERN | MARCH 31, 2021 | 8:26AM

GARY MARCUS BUSINESS 08.14.2019 09:00 AM

DeepMind's Losses and the Future of Artificial Intelligence

Alphabet's DeepMind unit, conqueror of Go and other games, is losing lots of money. Continued deficits could imperil investments in AI.

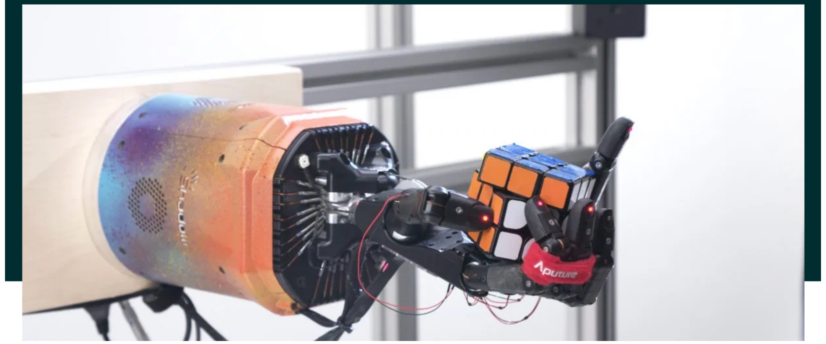
AARIAN MARSHALL BUSINESS 12.07.2020 04:06 PM

Uber Gives Up on the Self-Driving Dream

The ride-hail giant invested more than $1 billion in autonomous vehicles. Now it's selling the unit to Aurora, which makes self-driving tech.

OpenAI disbands its robotics research team

Kyle Wiggers @Kyle_L_Wiggers July 16, 2021 11:24 AM

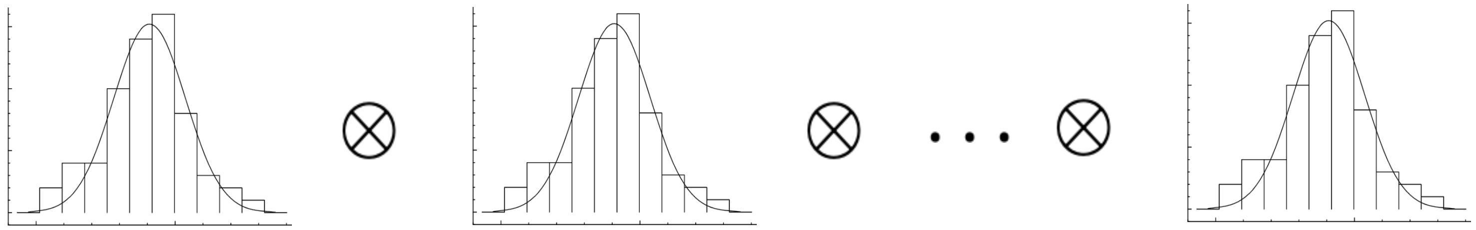


Self-driving Uber car that hit and killed woman did not recognize that pedestrians jaywalk

The automated car lacked "the capability to classify an object as a pedestrian unless that object was near a crosswalk," an NTSB report said.

Core challenge: The curse of dimensionality

- **Statistical: Sampling in d dimension with resolution ϵ**



Sample complexity:

$$O(\varepsilon^{-d})$$

For $\epsilon = 0.1$ and $d = 100$, we would need 10^{100} points.

- **Computational: Verifying non-negativity of polynomials**

Copositive matrices:

$$[x_1^2 \dots x_d^2] A [x_1^2 \dots x_d^2]^{\mathrm{T}} \geq 0$$

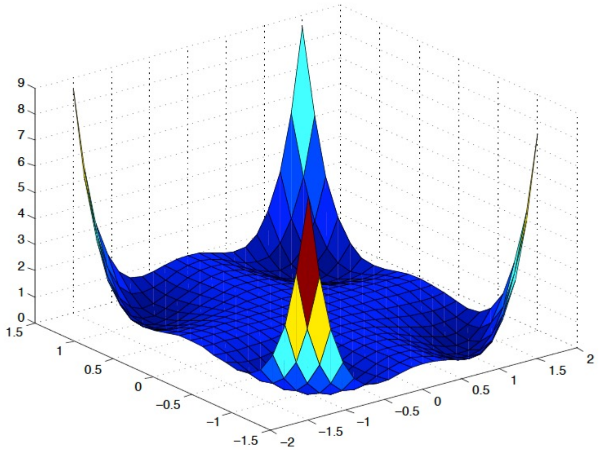
Murty&Kadabi [1987]: Testing co-positivity is NP-Hard

Sum of Squares (SoS):

$$z(x)^T Q z(x) \geq 0, \quad z_i(x) \in \mathbb{R}[x], \ x \in \mathbb{R}^d, Q \succcurlyeq 0$$

Artin [1927] (Hilbert's 17th problem):

Non-negative polynomials are sum of square of *rational* functions



Motzkin [1967]:

$p = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$

is nonnegative,

not a sum of squares,

but $(x^2 + y^2)^2 p$ is SoS

Question: Are we asking too much?

- Learnability requires uniform approximation errors across the ***entire domain***

  Q: Can we provide local guarantees, and progressively expand as needed?

  [arXiv '22] Shen, Bichuch, M

- Lyapunov functions and control barrier functions require strict and exhaustive notions of ***invariance***

  Q: Can we substitute invariance with less restrictive properties?

  [arXiv '22] Shen, Bichuch, M

- Control synthesis usually aims for the ***best*** (optimal) controller

  Q: Can we focus on feasibility, rather than optimality?

  [arXiv '21, L4DC 22] Castellano, Min, Bazerque, M

[arXiv 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* **submitted to CDC 2022,** preprint arXiv:2204.10372.
[L4DC 22] Castellano, Min, Bazerque, M, *Reinforcement Learning with Almost Sure Constraints,* **Learning for Dynamics and Control (L4DC) Conference, 2022**
[arXiv 21] Castellano, Min, Bazerque, M, *Learning to Act Safely with Limited Exposure and Almost Sure Certainty,* **submitted to IEEE TAC, 2021, under review**, preprint arXiv:2105.08748

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Yue Shen
JOHNS HOPKINS UNIVERSITY

Maxim Bichuch
University at Buffalo

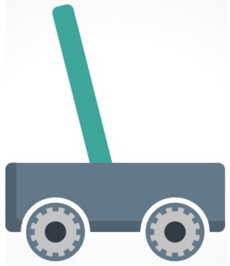
Model-free Learning of Regions of Attractions via Recurrent Sets

Y Shen, M. Bichuch, and E Mallada, “Model-free Learning of regions of attraction via recurrent sets.” CDC 2022.

Motivation: Estimation of regions of attraction

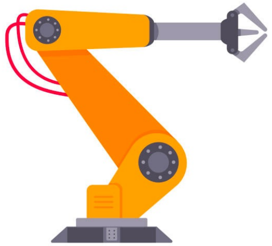
Having an approximation of the region of attraction allows us to

- **Test the limits of controller designs**
  especially for those based on (possibly linear) approximations of nonlinear systems



cart-pole

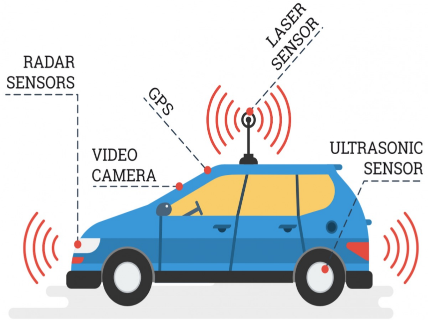
quadcopter



robot arm

. . .

- **Verify safety of certain operating condition**



self-driving

HVAC system

power grids

. . .

Problem setup

Continuous time dynamical system: $\dot{x}(t) = f(x(t))$

- Initial condition $x_0 = x(0)$, solution at time t: $\phi(t, x_0)$.

Ω-Limit Set $\Omega(f)$:

$$x \in \Omega(f) \iff \exists\, x_0, \{t_n\}_{n\geq 0}, \text{ s.t. } \lim_{n\to\infty} t_n = \infty \text{ and } \lim_{n\to\infty} \phi(t_n, x_0) = x$$

Types of Ω-limit set

equilibrium   limit cycle   limit torus   chaotic attractor

Problem setup

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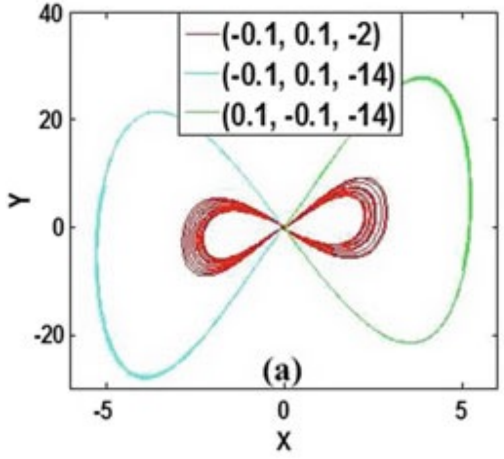
- Initial condition $x_0 = x(0)$, solution at time t: $\phi(t, x_0)$.
- The ω-limit set of the system: $\Omega(f)$

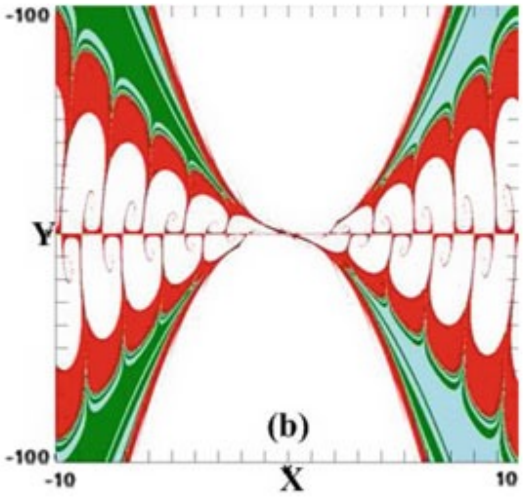
Region of attraction (ROA) of a set $S \subseteq \Omega(f)$:

$$\mathcal{A}(S) := \left\{ x_0 \in \mathbb{R}^d \middle| \lim_{t \to \infty} d(\phi(t, x_0), S) = 0 \right\}$$

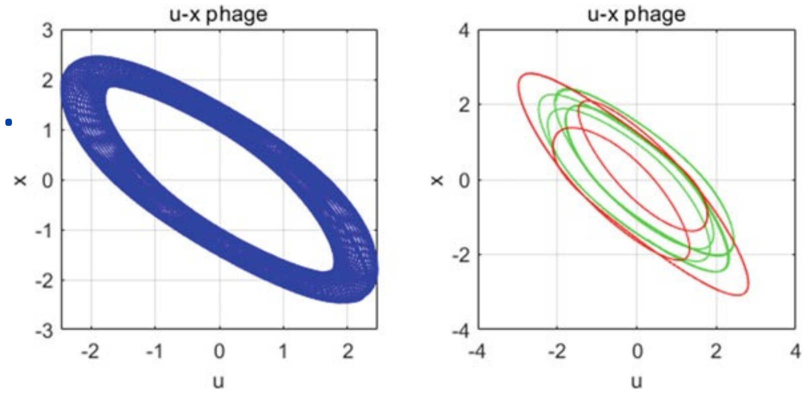
Example I: Limit set $\Omega(f)$



Basin of $\mathcal{A}(\Omega)$



Example II: Limit set $\Omega(f)$



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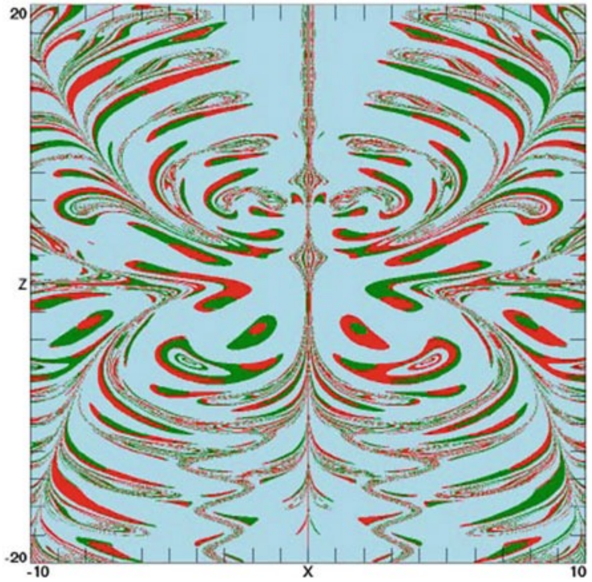

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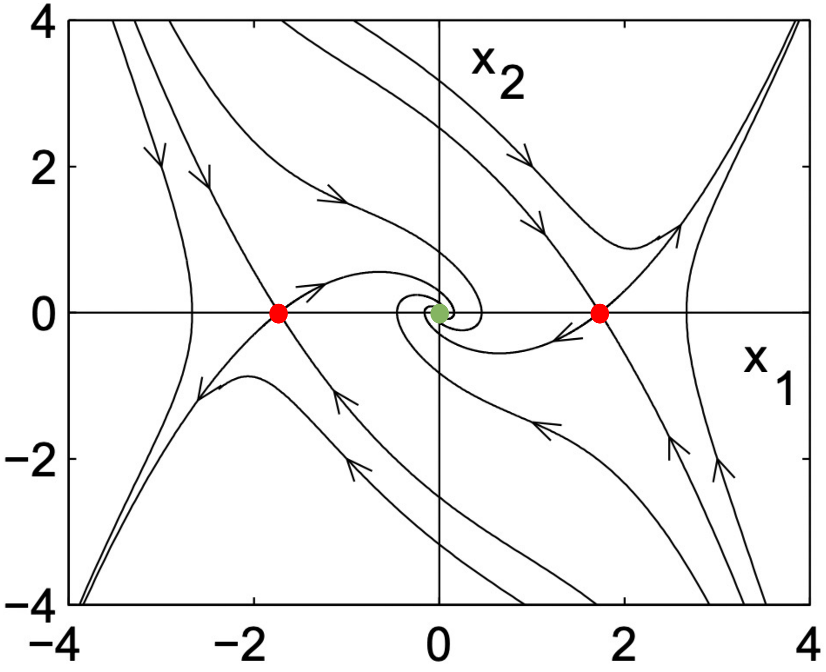
Example III

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 + \frac{1}{3}x_1^3 - x_2\end{bmatrix}$$

$$\Omega(f) = \{(0,0), (-\sqrt{3}, 0), (\sqrt{3}, 0)\}$$



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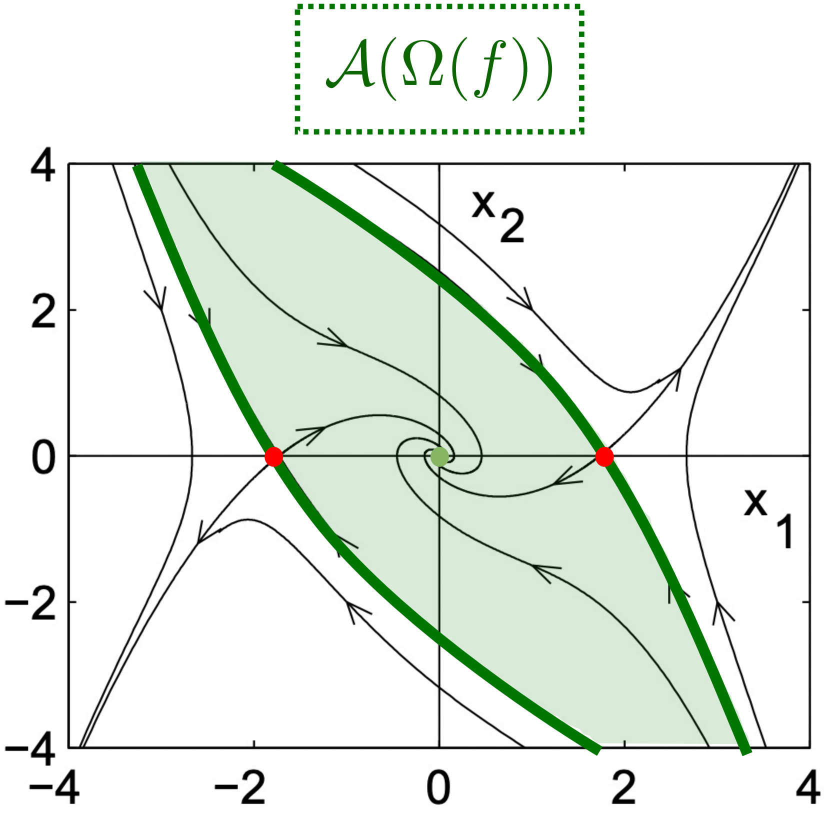

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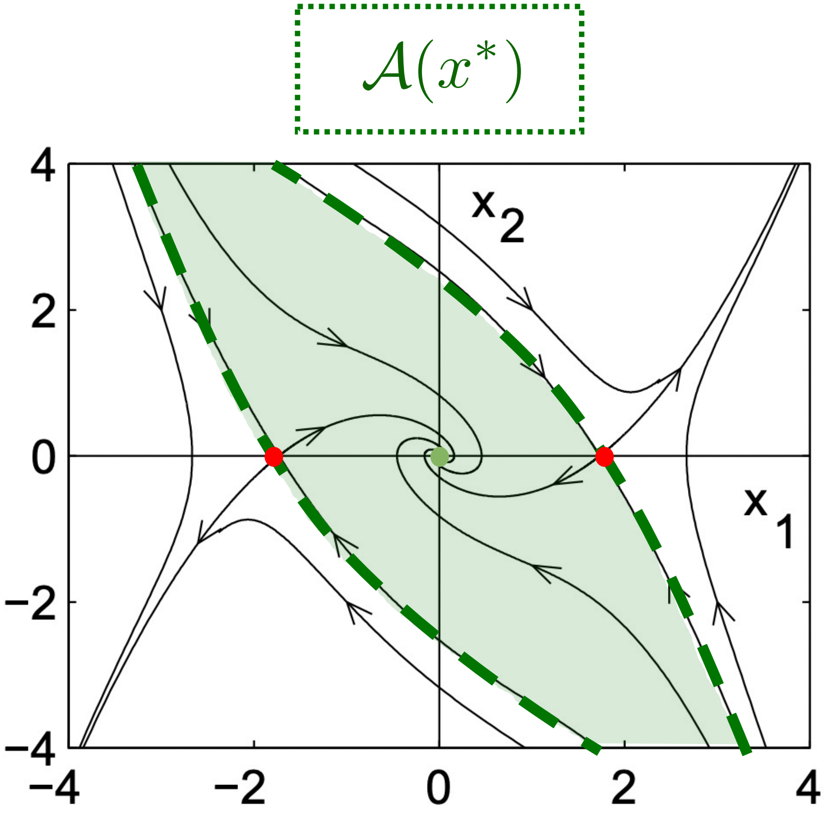
Asymptotically stable equilibrium at $x^* = (0,0)$



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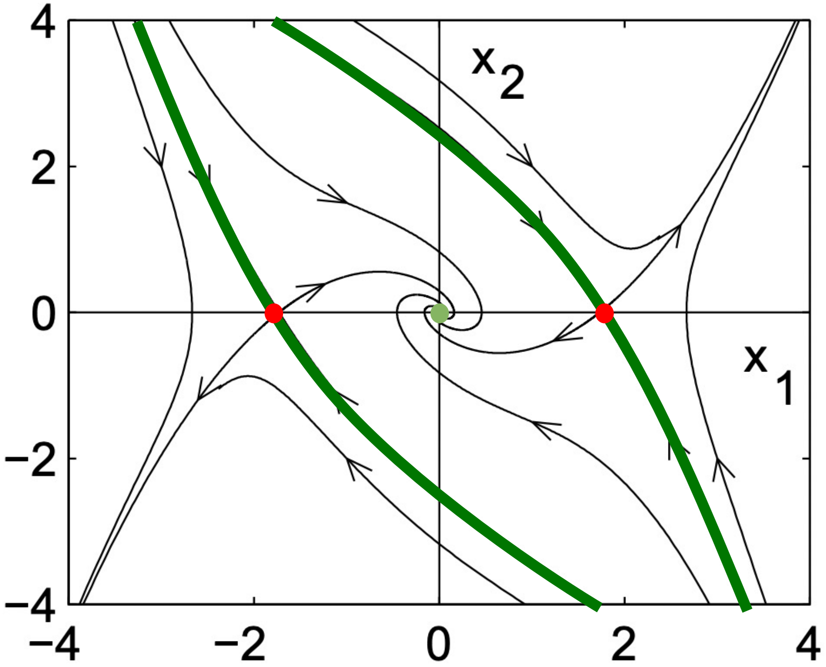
Unstable equilibria $\{(\sqrt{3}, 0), (-\sqrt{3}, 0)\}$

$\mathcal{A}(\{(\sqrt{3}, 0), (-\sqrt{3}, 0)\})$



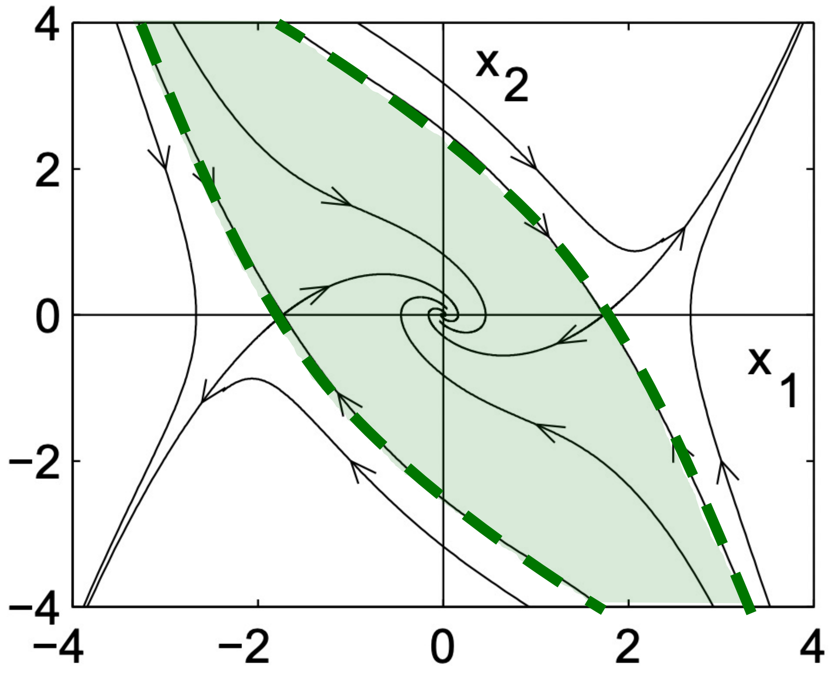
Region of attraction of stable equilibria

Region of attraction (ROA) of a set $S \subseteq \Omega(f)$:

$$\mathcal{A}(S) := \left\{ x_0 \in \mathbb{R}^d \middle| \lim_{t\to\infty} \phi(t, x_0) \in S \right\}$$

***Assumption** 1. The system $\dot{x}(t) = f(x(t))$ has an asymptotically stable equilibrium at x^*.*

***Remark** 1. It follows from Assumption 1 that the **positively invariant** ROA $\mathcal{A}(x^*)$ **is an open contractible set** [Sontag, 2013], i.e., the identity map of $\mathcal{A}(x^*)$ to itself is null-homotopic [Munkres, 2000].*



$\mathcal{A}(x^*)$:

E. Sontag. "Mathematical Control Theory: Deterministic Finite Dimensional Systems." Springer 2013
J. R. Munkres. "Topology." Prentice Hall 2000

Invariant sets

A set $I \subseteq \mathbb{R}^d$ is **positively invariant** if and only if: $x_0 \in \mathcal{I} \implies \phi(t, x_0) \in \mathcal{I}, \quad \forall t \in \mathbb{R}^+$

Any trajectory starting in the set remains in inside it

- **Invariant sets guarantee stability**
  Lyapunov stability: solutions starting "close enough" to the equilibrium (within a distance δ) remain "close enough" forever (within a distance ε))

- **Invariant sets further certify asymptotic stability via Lyapunov's direct method**
  Asymptotic stability: solutions that start close enough not only remain close enough but also eventually converge to the equilibrium.)

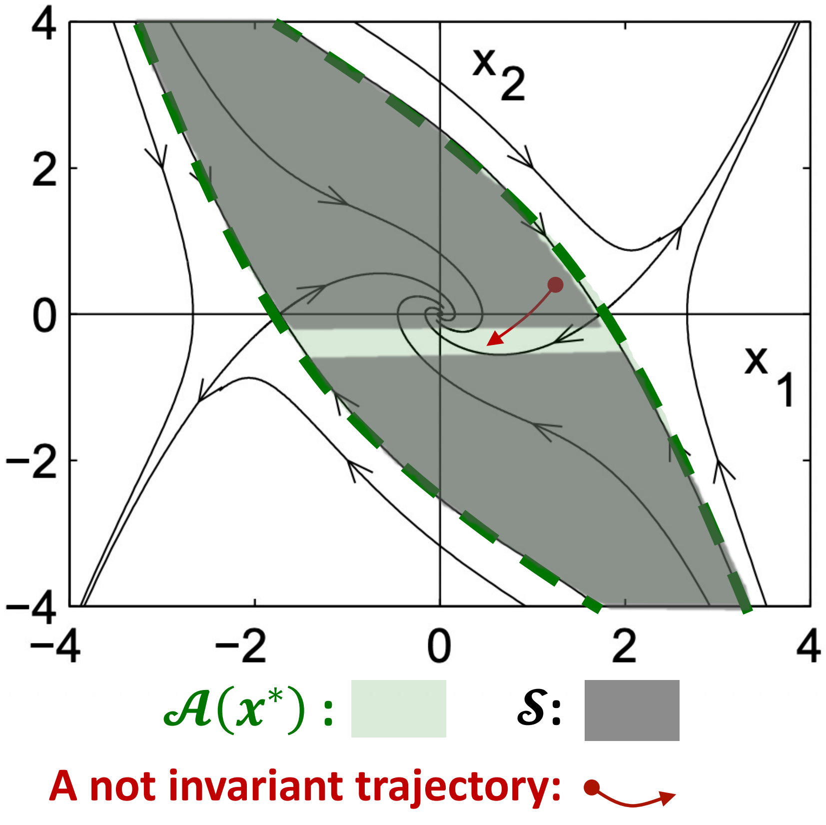
- **Regions of attraction are invariant sets, and so are the outcome of most approximation methods!**

Challenges of working with invariant set

Learning ROA $\mathcal{A}(x^*)$ by finding an invariant set $\mathcal{S} \subseteq \mathcal{A}(x^*)$

- $\mathcal{S}$ is topologically constrained
  - If $\mathcal{S} \cap \Omega(f) = \{x^*\}$, then $\mathcal{S}$ is connected

Example 1: $\boldsymbol{\mathcal{S}} \subseteq \boldsymbol{\mathcal{A}}(\boldsymbol{x}^*)$ is not connected, not invariant!



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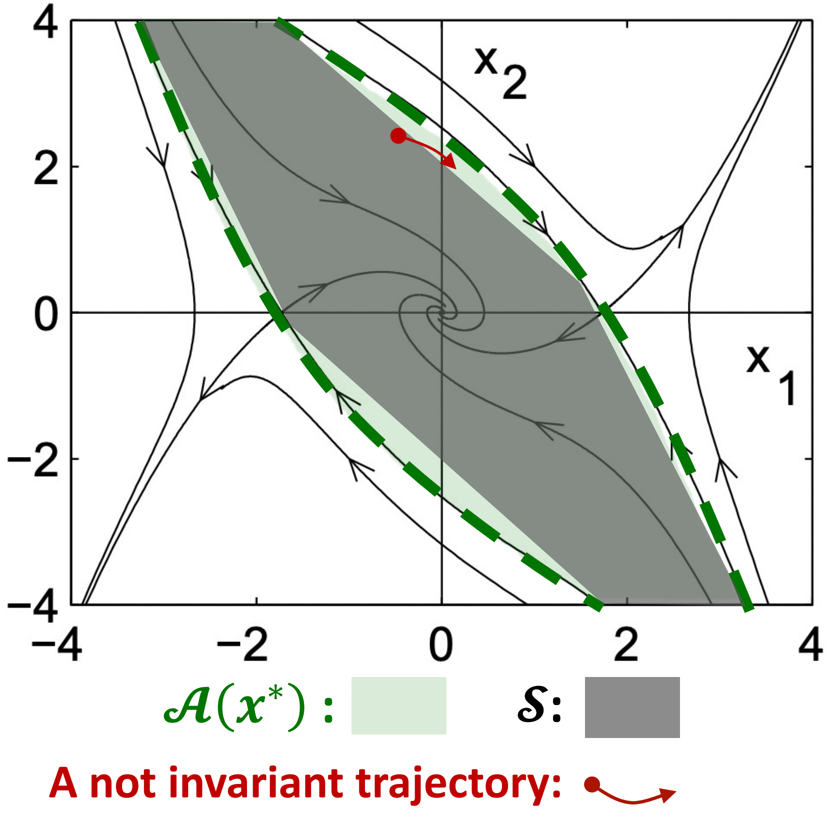
- $\mathcal{S}$ is topologically constrained
  - If $\mathcal{S} \cap \Omega(f) = \{x^*\}$, then $\mathcal{S}$ is connected
- $\mathcal{S}$ is geometrically constrained
  - f should point inwards for $x \in \partial\mathcal{S}$

Example 2: $\boldsymbol{\mathcal{S}} \subseteq \boldsymbol{\mathcal{A}}(\boldsymbol{x}^*)$, f points outward on $\boldsymbol{\partial\mathcal{S}}$, not invariant



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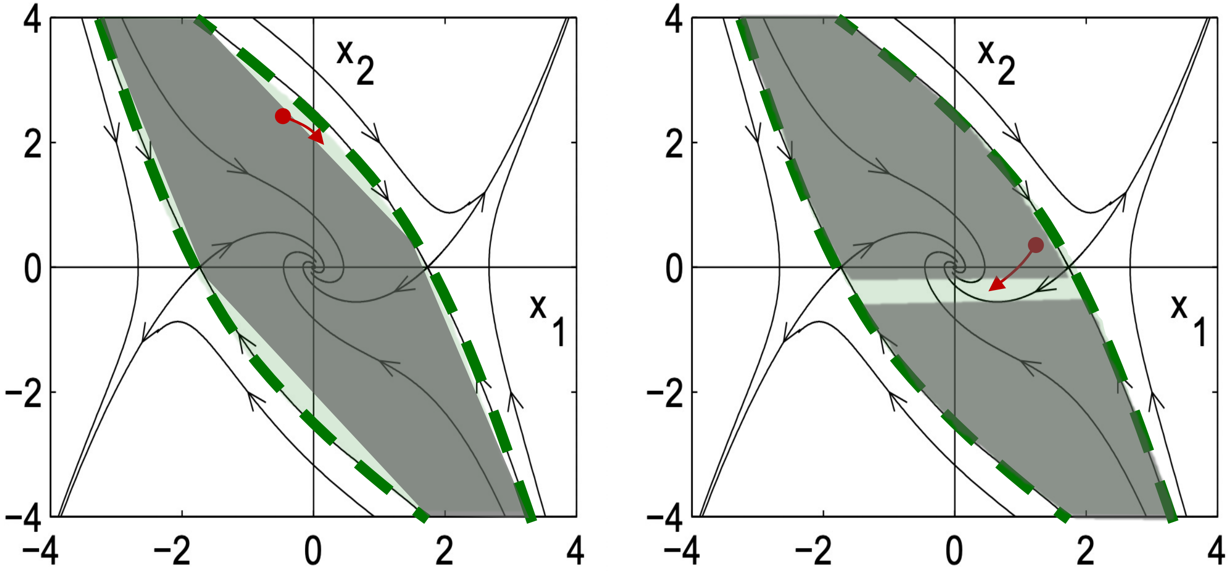
A subset of an invariant set is not necessary an invariant set

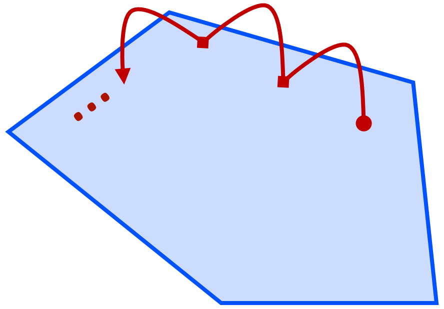


$\mathcal{A}(x^*)$: $\mathcal{S}$:

A not invariant trajectory:

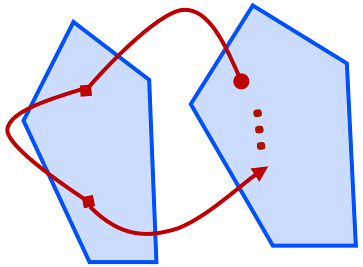
Recurrent sets: Letting things go, and come back

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \geq t$ s.t. $\phi(t', x_0) \in \mathcal{R}$.

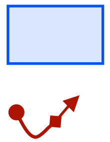


Property of Recurrent Sets

- $\mathcal{R}$ need **not** be **connected**
- $\mathcal{R}$ does **not** require f to **point inwards** on all $\partial\mathcal{R}$



> Recurrent sets, while not invariant, guarantee that solutions that start in this set, will come back **infinitely often, forever!**



Recurrent sets: Letting things go, and come back

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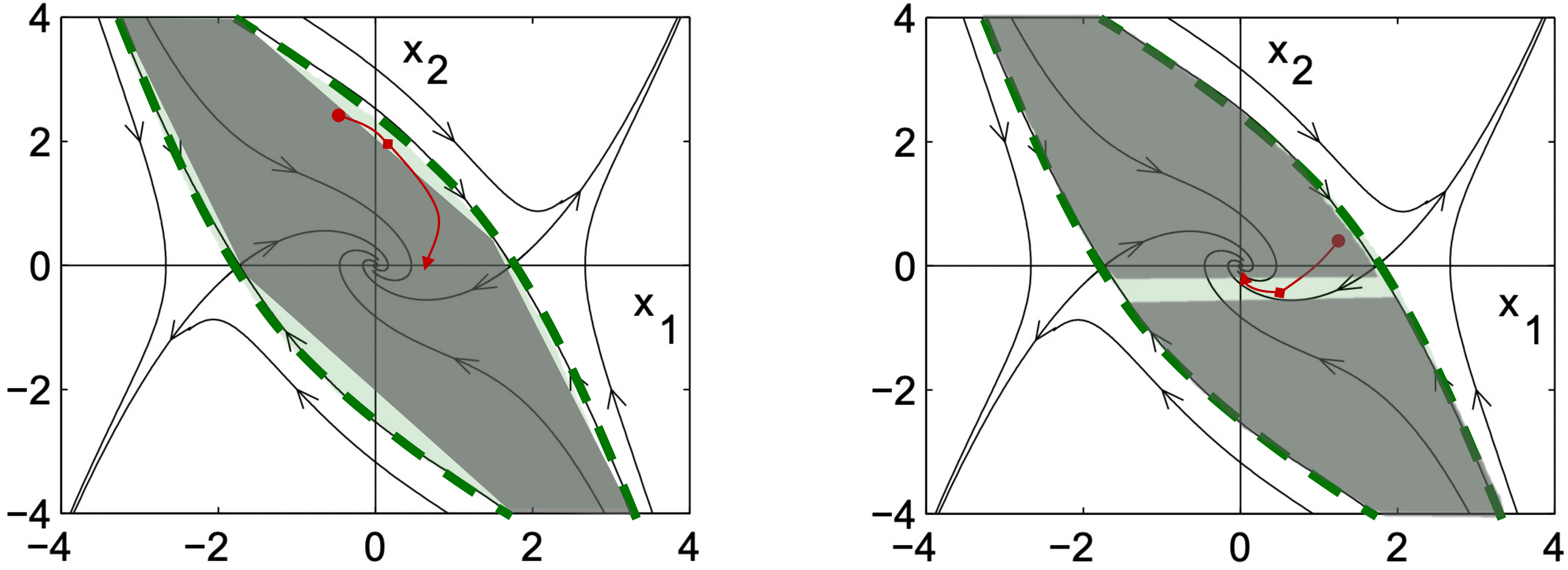
Previous two good inner approximations of $\mathcal{A}(x^*)$ are recurrent sets



Recurrent sets are subsets of the region of attraction

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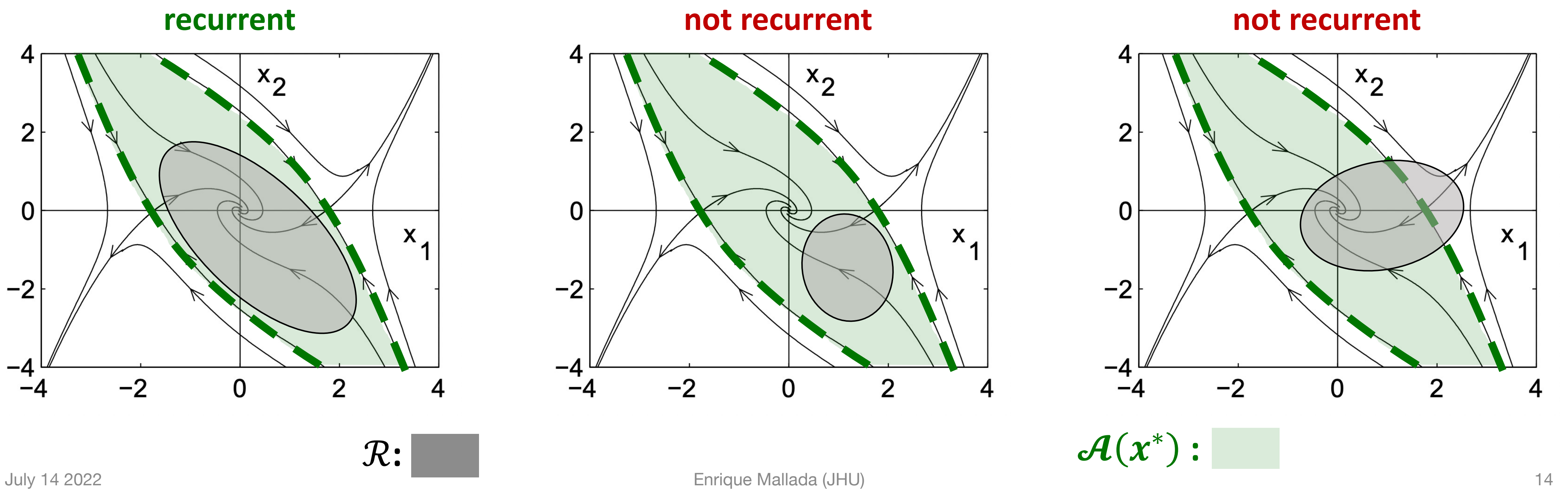
Theorem *1. Let $\mathcal{R} \subset \mathbb{R}^d$ be a* *compact* *set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$.*

Then:

$\mathcal{R}$ *is recurrent* $\Longleftrightarrow$ $\mathcal{R} \cap \Omega(f) \neq \emptyset$, $\mathcal{R} \subset \mathcal{A}(\mathcal{R} \cap \Omega(f))$



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***Theorem** 1. Let $\mathcal{R} \subset \mathbb{R}^d$ be a <u>compact</u> set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$.*

Then:

$$\mathcal{R} \text{ is recurrent} \iff \begin{array}{c} \mathcal{R} \cap \Omega(f) \neq \emptyset \\ \mathcal{R} \subset \mathcal{A}(\mathcal{R} \cap \Omega(f)) \end{array}$$

***Proof**: [Sketch]*

($\Longrightarrow$)

- *If $x_0 \in \mathcal{R}$, the solution $\phi(t, x_0)$ visits $\mathcal{R}$ infinitely often, forever.*
- *We can build a sequence $\{x(t_n)\}_{n=0}^{\infty} \in \mathcal{R}$ with $\lim_{n \to +\infty} t_n = +\infty$*
- *Bolzano-Weierstrass $\Longrightarrow$ convergent subsequence $x(t_{n_i}) \to \bar{\boldsymbol{x}} \in \Omega(f) \cap \mathcal{R} \neq \emptyset$*
- *$\partial\mathcal{R} \cap \Omega(f) = \emptyset$ + $\mathcal{R}$ recurrent $\Longrightarrow$ $\phi(t, x_0)$ leaves $\mathcal{R}$ finitely many times*

($\Longleftarrow$) Trivial.

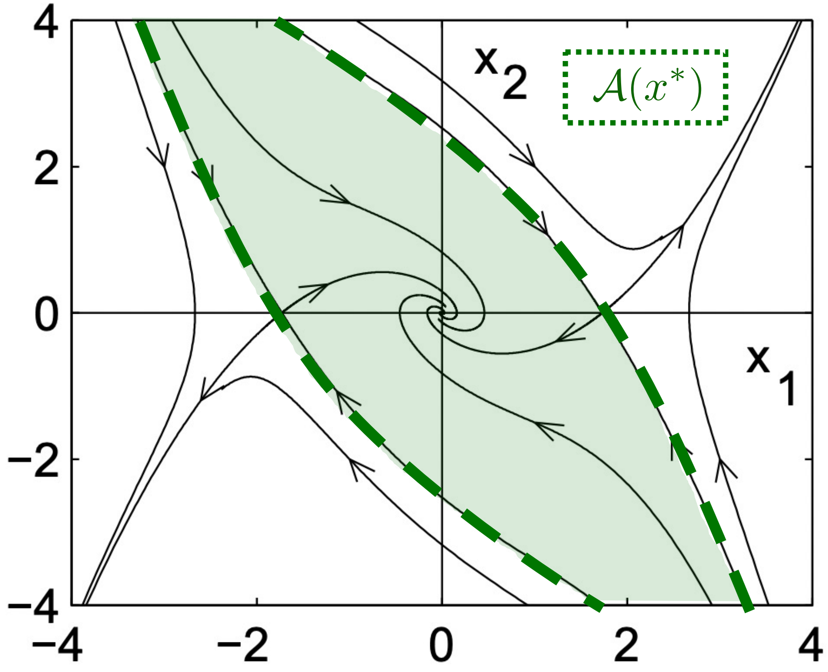
Recurrent sets are subsets of the region of attraction

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists t' \geq t$ s.t. $\phi(t', x_0) \in \mathcal{R}$.

***Assumption** 2. The ω-limit set $\Omega(f)$ is composed by **hyperbolic equilibrium points**, with only one of them, say x^*, being asymptotically stable.*

***Corollary** 2. Let Assumptions 1 and 2 hold, and $\mathcal{R} \subset \mathbb{R}^d$ be a <u>compact</u> set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$. Then:*

$\mathcal{R}$ is recurrent $\Longleftrightarrow$ $\mathcal{R} \cap \Omega(f) = \{x^*\}$, $\mathcal{R} \subset \mathcal{A}(x^*)$



Recurrent sets are subsets of the region of attraction

?

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Corollary *2. Let Assumptions 1 and 2 hold, and $\mathcal{R} \subset \mathbb{R}^d$ be a* <u>*compact*</u> *set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$. Then:*

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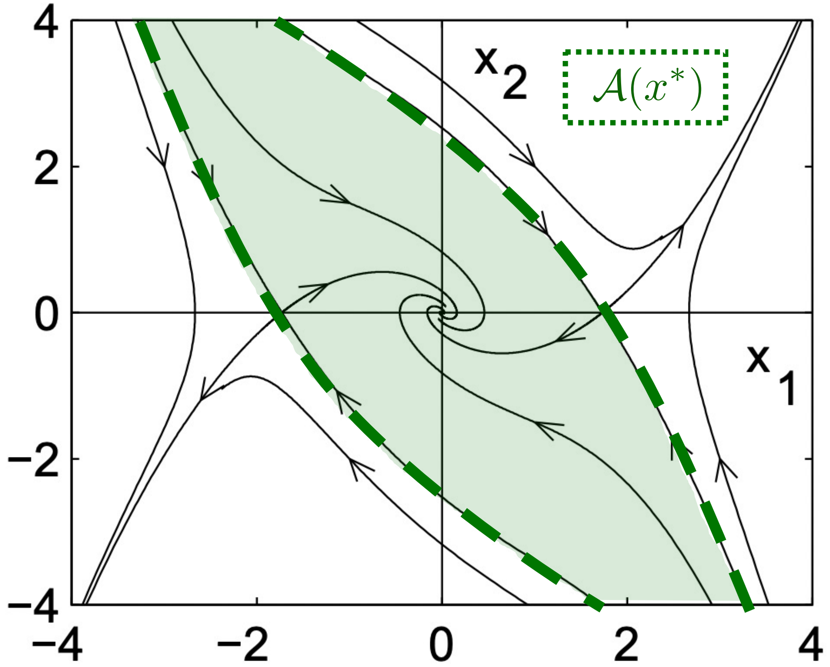
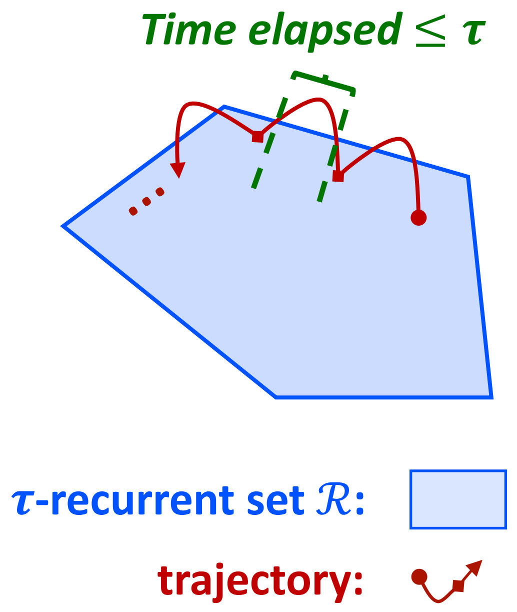


Idea: Use recurrence as a mechanism for finding inner approximations of $\mathcal{A}(x^*)$

Potential Issues:

- We do not know how long it takes to come back!
- We need to adapt results to trajectory samples

τ-recurrent sets

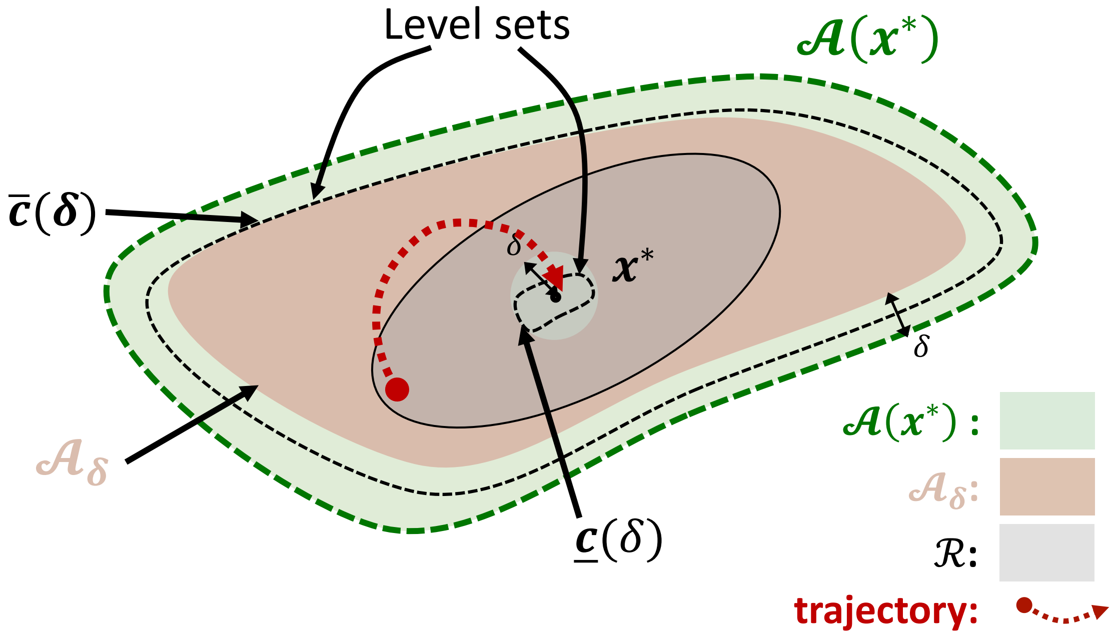


A set $\mathcal{R}$ is **τ-recurrent** if for any $x_0 \in \mathcal{R}$ and $t \geq 0$, $\exists\, t' \in [t, t+\tau]$ such that $\phi(t', x_0) \in \mathcal{R}$

Theorem *2. Under Assumption 1, any compact set* $\mathcal{R}$ satisfying:

$$x^* + \mathcal{B}_\delta \subseteq \mathcal{R} \subseteq \mathcal{A}(x^*) \backslash \{\partial \mathcal{A}(x^*) + \text{int}\, \mathcal{B}_\delta\}$$

is τ*-recurrent for* $\tau \geq \bar{\tau}(\delta) := \frac{\underline{c}(\delta) - \bar{c}(\delta)}{a(\delta)}$.



Proof: *[Sketch]*

- *Assumption 1* $\Longrightarrow \exists$ *Lyapunov function (Zubov '64)*
  - $V(x^*) = 0$, $0 < V(x) < 1$ for all $x \in \mathcal{A}(x^*) \backslash x^*$
  - $\nabla V(x^*)^T f(x^*) = 0$
  - $\nabla V(x)^T f(x) < 0$ for all $x \in \mathcal{A}(x^*) \backslash x^*$
- *Define* $\bar{c}(\delta) := \max_{x \in \mathcal{A}_\delta} V(x)$, $\quad \underline{c}(\delta) := \min_{x \in \mathcal{A}_\delta} V(x)$,

  and $\quad a(\delta) := \max_{x \in C_\delta} \nabla V(x)^T f(x)$,

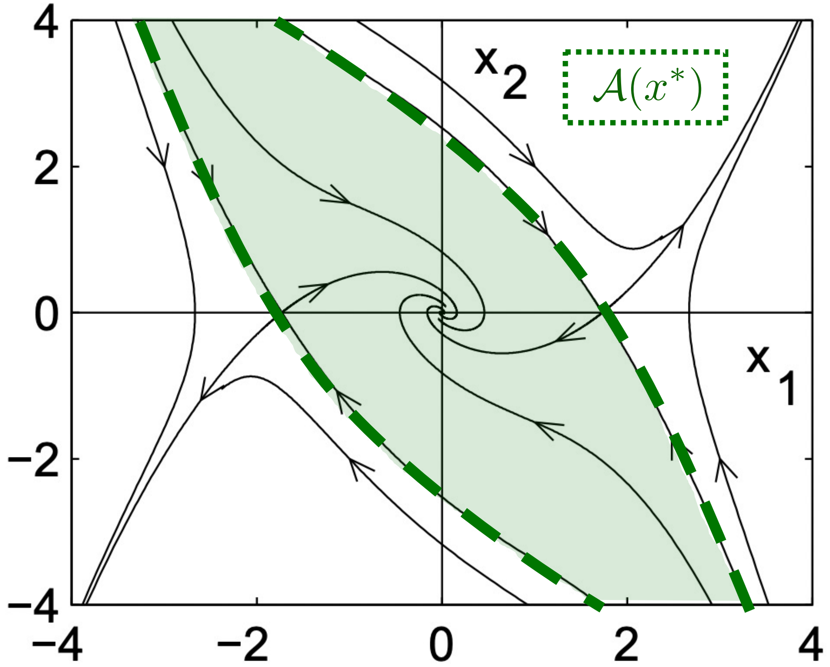
  where $C_\delta = \{x \in \mathbb{R}^d : \underline{c}(\delta) \leq V(x) \leq \bar{c}(\delta)\}$.

Recurrent sets are subsets of the region of attraction

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for $x_0 \in \mathcal{R}$, for any t $\geq 0 \Rightarrow \exists t' > t$, s.t. $\phi(t', x_0) \in \mathcal{R}$?

Corollary *2. Let Assumptions 1 and 2 hold, and $\mathcal{R} \subset \mathbb{R}^d$ be a compact set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$. Then:*

$\mathcal{R}$ is recurrent ⬌ $\mathcal{R} \cap \Omega(f) = \{x^*\}$, $\mathcal{R} \subset \mathcal{A}(x^*)$



Idea: Use recurrence as a mechanism for finding inner approximations of $\mathcal{A}(x^*)$

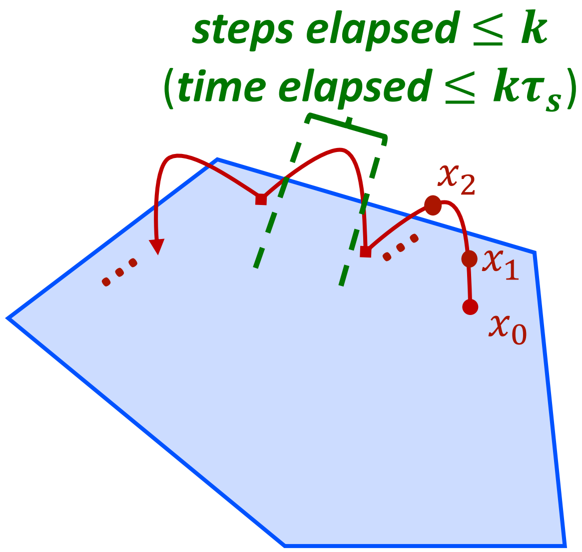
Potential Issues:

- We do not know how long it takes to come back! ✔
- We need to adapt results to trajectory samples

Learning recurrent sets from k-length trajectory samples

- Consider **finite length** trajectories:
  $$x_n = \phi(n\tau_s, x_0), \qquad x_0 \in \mathbb{R}^d, n \in \mathbb{N},$$
  where $\tau_s > 0$ is the sampling period.

- A set $\mathcal{R} \subseteq \mathbb{R}^d$ is k-**recurrent** if whenever $x_0 \in \mathcal{R}$, then $\exists\ n \in \{1, \dots, k\}$ s.t. $x_n \in \mathcal{R}$



***k*-recurrent set $\mathcal{R}$:**

trajectory:

Sufficiency:

$\mathcal{R}$ is k-recurrent ➡ $\mathcal{R}$ is τ-recurrent with $\tau = k\tau_s$ ✚ $\mathcal{R}$ is compact $\partial\mathcal{R} \cap \Omega(f) = \emptyset$ ➡ $\mathcal{R} \subset \mathcal{A}(x^*)$

(Corollary 2, under Assumption 2)

Necessity:

Theorem *3. Under Assumption 1, any compact set* $\mathcal{R}$ satisfying:
$$\mathcal{B}_\delta + x^* \subseteq \mathcal{R} \subseteq \mathcal{A}(x^*) \backslash \{\partial\mathcal{A}(x^*) + \text{int } \mathcal{B}_\delta\}$$
is k*-recurrent for any* $k > \bar{k} := \bar{\tau}(\delta)/\tau_s$.

Recurrent sets are subsets of the region of attraction

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if for $x_0 \in \mathcal{R}$, $\phi(t, x_0) \notin \mathcal{R} \Rightarrow \exists t' > t,$ s.t. $\phi(t', x_0) \in \mathcal{R}$?

***Corollary** 2. Let Assumptions 1 and 2 hold, and $\mathcal{R} \subset \mathbb{R}^d$ be a <u>compact</u> set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$. Then:*

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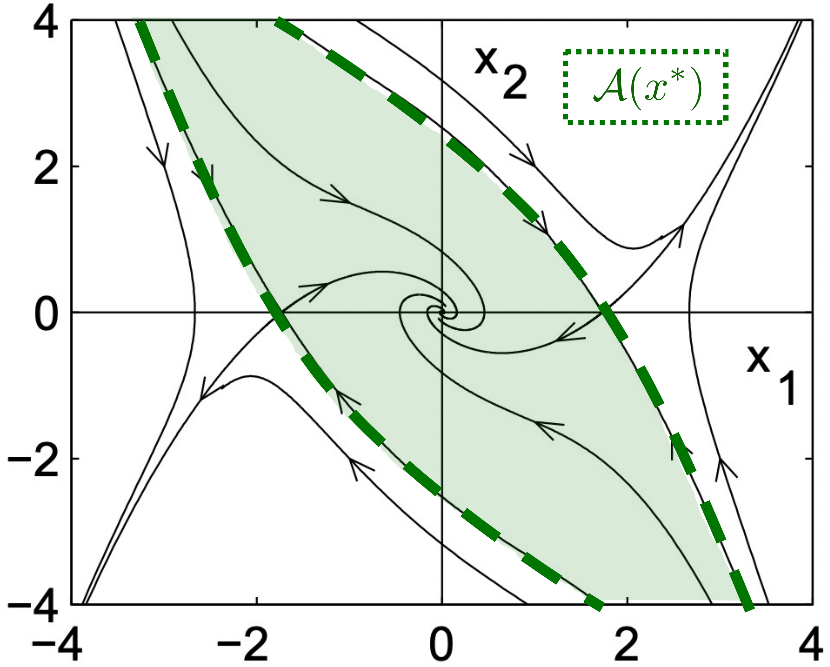


Idea: Use recurrence as a mechanism for finding inner approximations of $\mathcal{A}(x^*)$

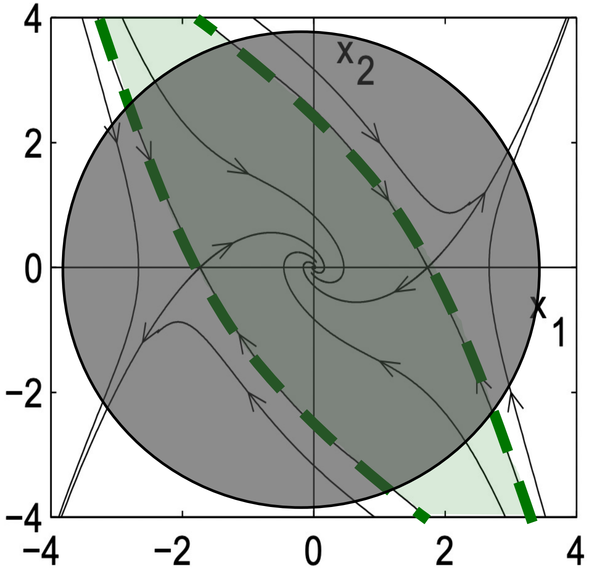
Potential Issues:

- We do not know how long it takes to come back! ✔
- We need to adapt results to trajectory samples ✔

Sphere approximations of RoA

Algorithm:

- **Initialize $\hat{\mathcal{S}}^{(0)}$ *as*** $\hat{\mathcal{S}}^{(0)} := \{x \,|\, \|x\|_2 \leq b^{(0)} := c\} \supseteq \mathcal{B}_\delta$



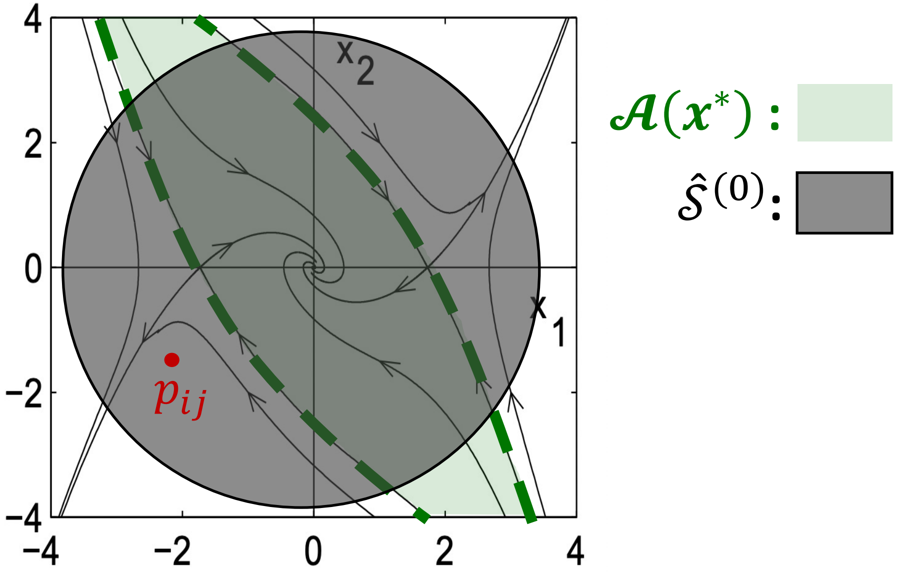
$\mathcal{A}(x^*)$:

$\hat{\mathcal{S}}^{(0)}$:

Sphere approximations of RoA

Algorithm:

- **Initialize** $\hat{\mathcal{S}}^{(0)}$ ***as*** $\hat{\mathcal{S}}^{(0)} := \{x \mid \|x\|_2 \leq b^{(0)} := c\} \supseteq \mathcal{B}_\delta$
- **For** *iteration* $i = 0,1,\ldots$ **do:** (set updates)
  - ***For*** *iteration* $j = 0,1,\ldots$ **do:** (samples)
    - *Generate random sample* $p_{ij} \in \hat{\mathcal{S}}^{(i)}$ *uniformly*



Sphere approximations of RoA

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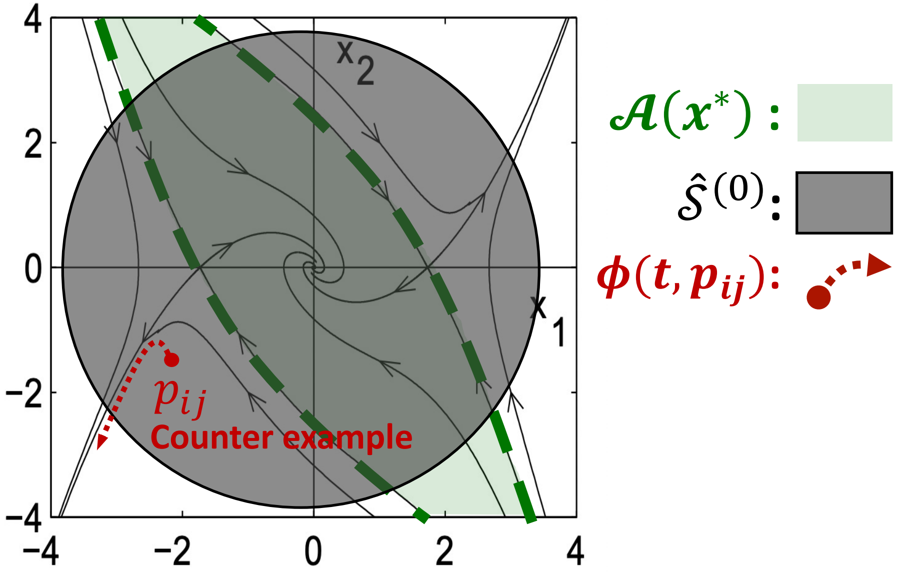

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- **For** *iteration* $i = 0,1, \dots$ **do:**
  - ***For*** *iteration* $j = 0,1, \dots$ **do:**
    - *Generate random sample* $p_{ij} \in \hat{\mathcal{S}}^{(i)}$ *uniformly*
    - **If** p_{ij} *is a counter-example w.r.t* $\hat{\mathcal{S}}^{(i)}$ **do:**

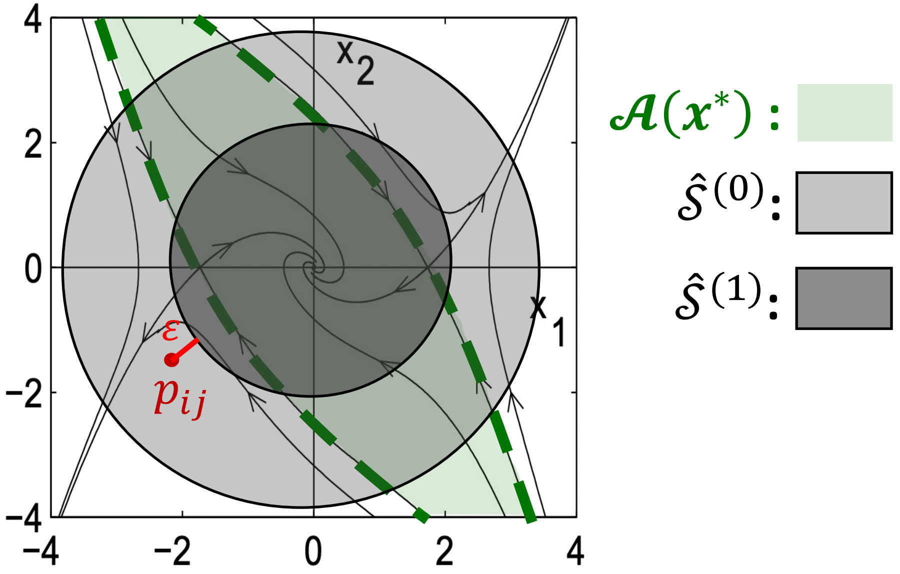
We say sample point p_{ij} *is a valid* k*-recurrent point w.r.t current approximation* $\hat{\mathcal{S}}^{(i)}$ *if starting from* $x_0 = p_{ij}$,

$$\exists\, n \in \{1, \dots, k\}, \text{ s.t. } x_n \in \hat{\mathcal{S}}^{(i)}.$$

Otherwise, we say p_{ij} *is a counter-example.*

Sphere approximations of RoA

Algorithm:



- **Initialize** $\hat{\mathcal{S}}^{(0)}$ ***as*** $\hat{\mathcal{S}}^{(0)} \coloneqq \{x | \|x\|_2 \leq b^{(0)} \coloneqq c\} \supseteq \mathcal{B}_\delta$
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  - ***For*** *iteration* $j = 0,1, \ldots$ **do:**
    - *Generate random sample* $p_{ij} \in \hat{\mathcal{S}}^{(i)}$ *uniformly*
    - **If** p_{ij} *is a counter-example w.r.t* $\hat{\mathcal{S}}^{(i)}$ **do:**
      - *Update* $b^{(i)}$ *to* $b^{(i+1)}$, $\hat{\mathcal{S}}^{(i)}$ *to* $\hat{\mathcal{S}}^{(i+1)}$

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Otherwise, we say p_{ij} *is a counter-example.*

If p_{ij} *is a counter-example*, we update:

$$b^{(i+1)} = \|p_{ij}\|_2 - \varepsilon;$$

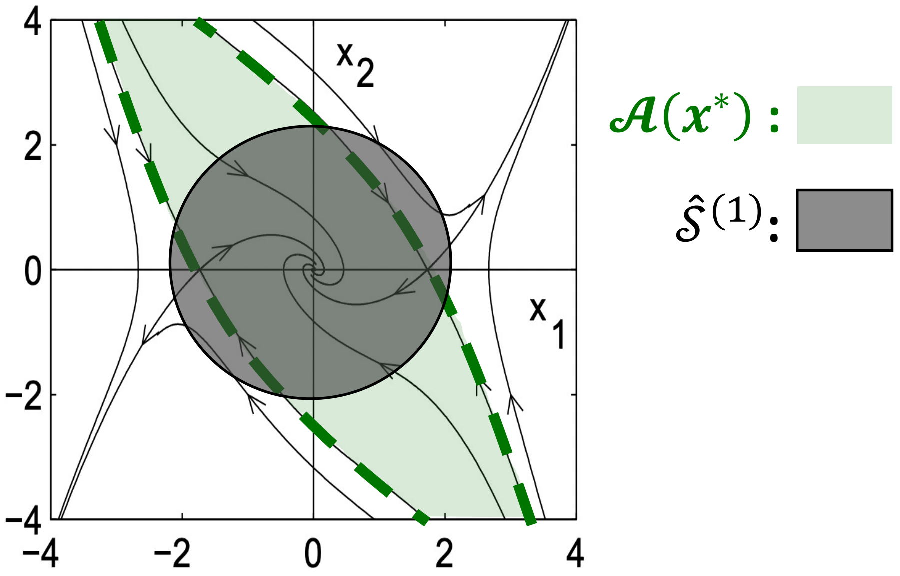
$$\hat{\mathcal{S}}^{(i+1)} = \{x | \|x\|_2 \leq b^{(i+1)}\},$$

where $\varepsilon > 0$ *is an algorithm parameter expressing the level of conservativeness in our update.*

Sphere approximations of RoA

Algorithm:

- **Initialize** $\hat{\mathcal{S}}^{(0)}$ ***as*** $\hat{\mathcal{S}}^{(0)} := \{x \mid \|x\|_2 \le b^{(0)} := c\} \supseteq \mathcal{B}_\delta$
- **For** *iteration* $i = 0,1,\ldots$ **do:**
  - ***For*** *iteration* $j = 0,1,\ldots$ **do:**
    - *Generate random sample* $p_{ij} \in \hat{\mathcal{S}}^{(i)}$ *uniformly*
    - **If** p_{ij} *is a counter-example w.r.t* $\hat{\mathcal{S}}^{(i)}$ **do:**
      - *Update* $b^{(i)}$ *to* $b^{(i+1)}$, $\hat{\mathcal{S}}^{(i)}$ *to* $\hat{\mathcal{S}}^{(i+1)}$
      - **Break**
    - **End if**
  - **End for**
- **End for**



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where $\varepsilon > 0$ *is an algorithm parameter expressing the level of conservativeness in our update.*

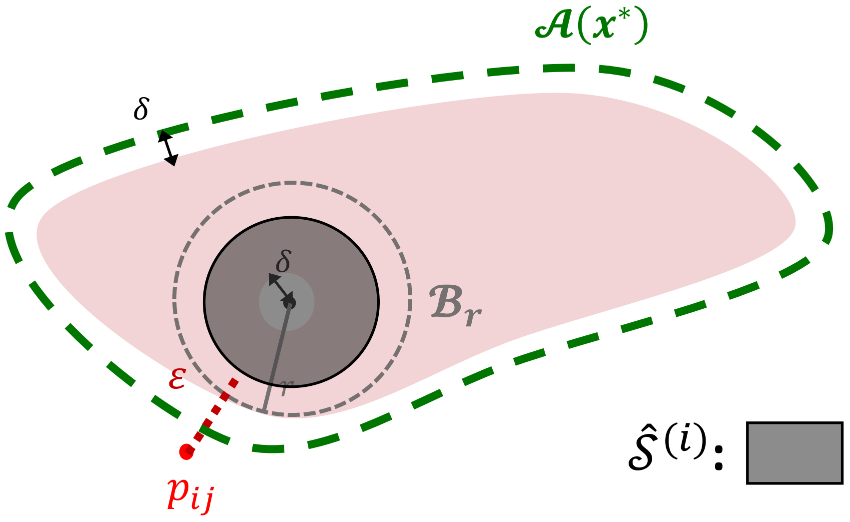
Parameter choice

Choice of ε: $b^{(i+1)} = \left|\left|p_{ij}\right|\right| - \varepsilon$

- Given $k > \bar{k}$, any set $\mathcal{S}^{(i)} = \{x : ||x|| \leq b^{(i)}\}$ satisfying:

  $$\mathcal{B}_\delta \subseteq \mathcal{S}^{(i)} \subseteq \mathcal{A}(0) \backslash \{\partial \mathcal{A}(0) + \text{int}\, \mathcal{B}_\delta\}$$

  is k-recurrent.
- Let $\mathcal{B}_r$ the largest ball inside $\mathcal{A}(0) \backslash \{\partial \mathcal{A}(0) + \text{int}\, \mathcal{B}_\delta\}$
- Then, if $\varepsilon \leq r - \delta$ we always guarantee $\mathcal{B}_\delta \subseteq \mathcal{S}^{(i)}$



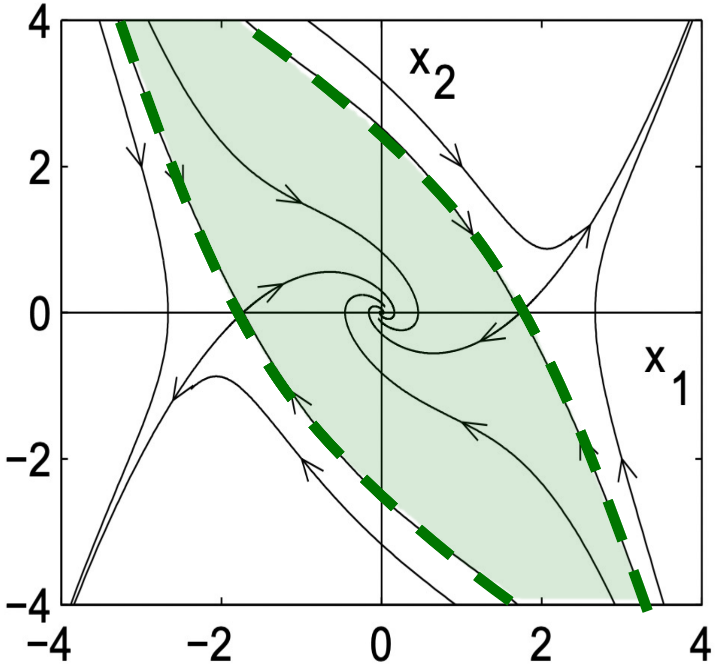
Choice of trajectory length k:

- $\bar{k}(\delta)$ depends highly non-trivially on δ.
- If $k < \bar{k}(\delta)$, we get $b^{(i)} < 0 \Longrightarrow$ Failure!
- **Solution:** doubling the size of k, i.e., $k^+ = 2k$, **every time we fail.**

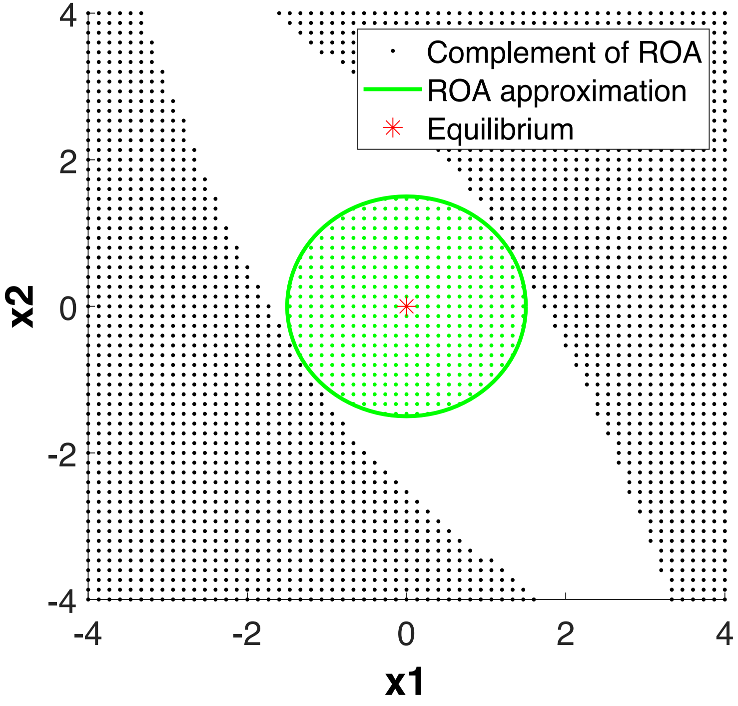
With k-doubling, the total number of counter-examples is bounded by

$$\textbf{\#counter-examples} \leq \frac{b^{(0)}}{\varepsilon} \log_2 \bar{k}(\delta)$$

Algorithm Result - Sphere Approximations

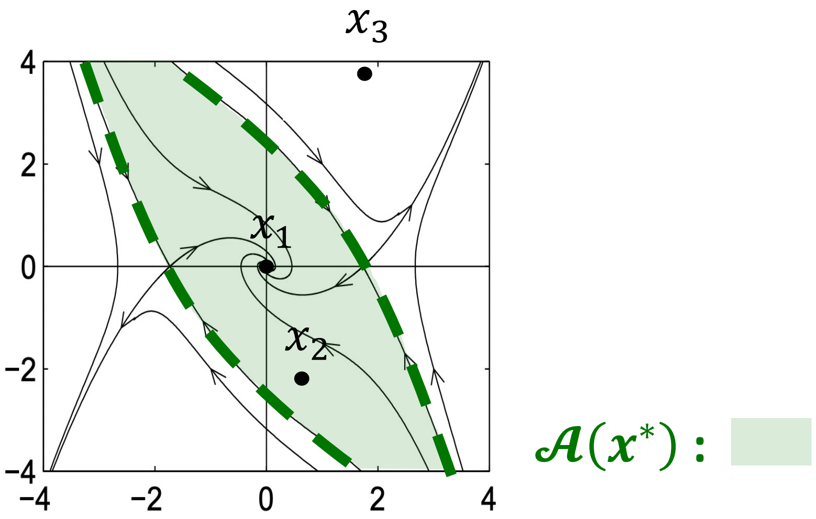


$\mathcal{A}(0)$:



Multi-center approximation

- **Consider $\boldsymbol{h} \in \mathbb{N}^{+}$ center points $\boldsymbol{x_q}$ indexed by $\boldsymbol{q} \in \{\mathbf{1}, \dots, \boldsymbol{h}\}$.**
  - Let the first center point $x_1 = x^* = 0$
  - Additional center point $x_2, \dots, x_h$ can be designed chosen uniformly.



Multi-center approximation

- **Consider $h \in \mathbb{N}^+$ center points x_q indexed by $q \in \{1, \dots, h\}$.**
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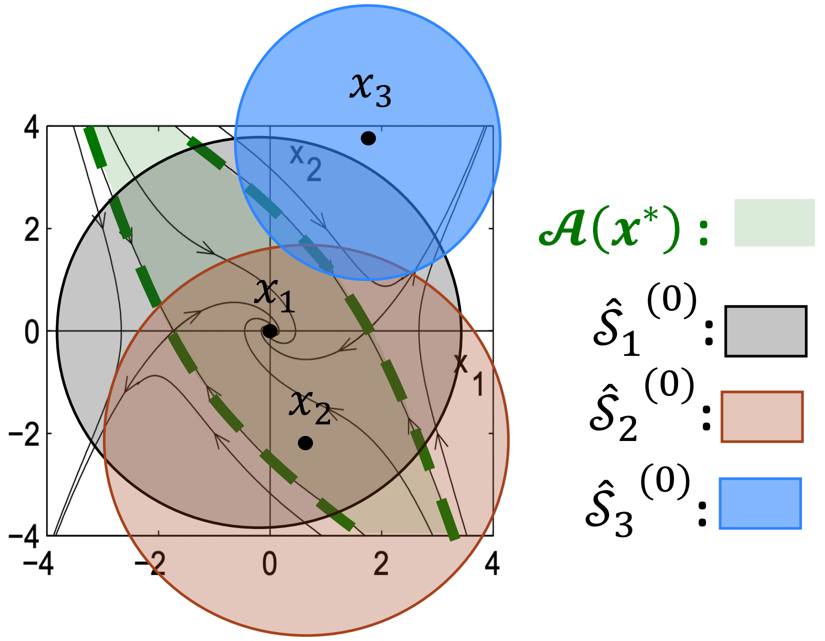
- **Respectively defined approximations centered at each x_q**
  - (Sphere case) $\hat{\mathcal{S}}_q^{(i)} := \{x | \|x - x_q\|_2 \le b_q^{(i)}\}$
  - (Polytope case) $\hat{\mathcal{S}}_q^{(i)} := \{x | A(x - x_q) \le b_q^{(i)}\}$



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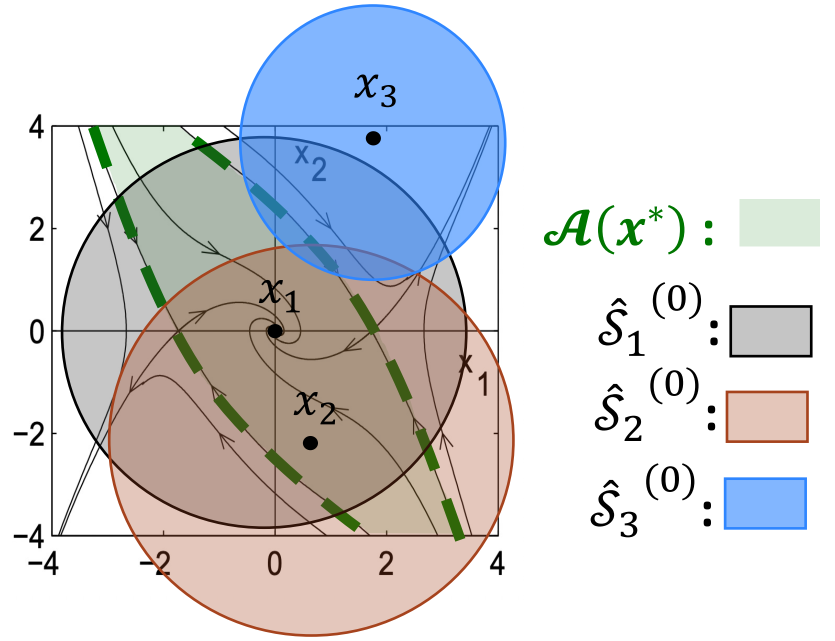
- **Multiple centers approximation** $\hat{\mathcal{S}}_{\text{multi}}^{(i)} := \cup_{q=1}^{h} \hat{\mathcal{S}}_q^{(i)}$



Multi-center approximation

- **Consider $h \in \mathbb{N}^+$ center points x_q indexed by $q \in \{1, \dots, h\}$.**
  - Let the first center point $x_1 = x^* = 0$
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- **Respectively defined approximations centered at each x_q**
  - (Sphere case) $\hat{S}_q^{(i)} := \{x \mid \|x - x_q\|_2 \le b_q^{(i)}\}$
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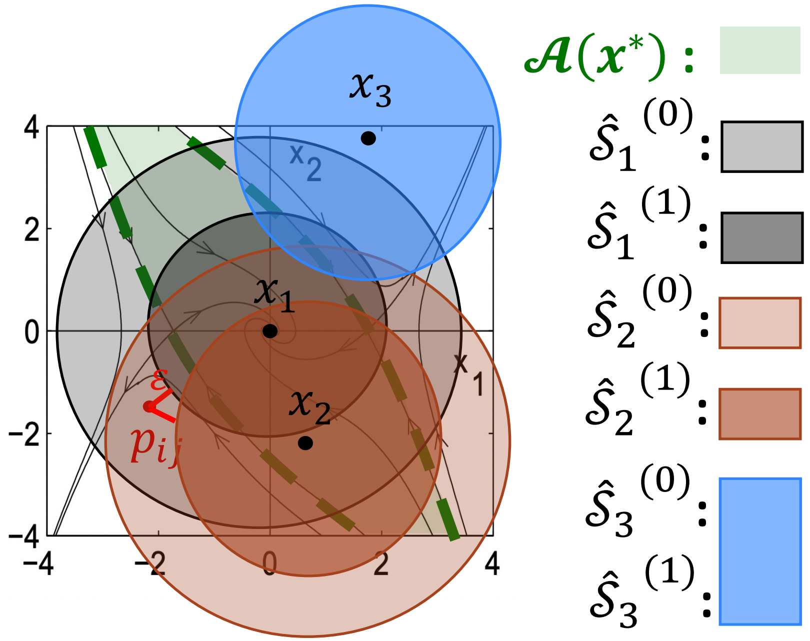
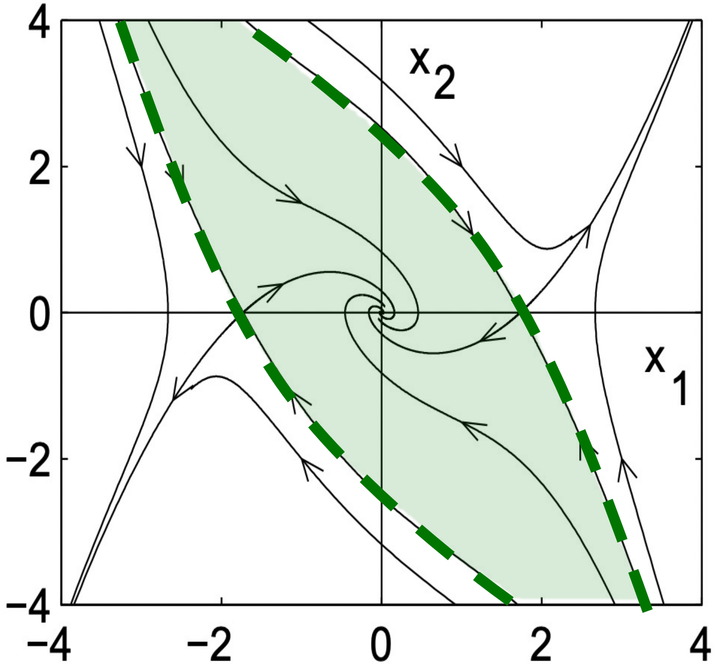
- **Multiple centers approximation** $\hat{S}_{\mathrm{multi}}^{(i)} := \cup_{q=1}^{h} \hat{S}_q^{(i)}$

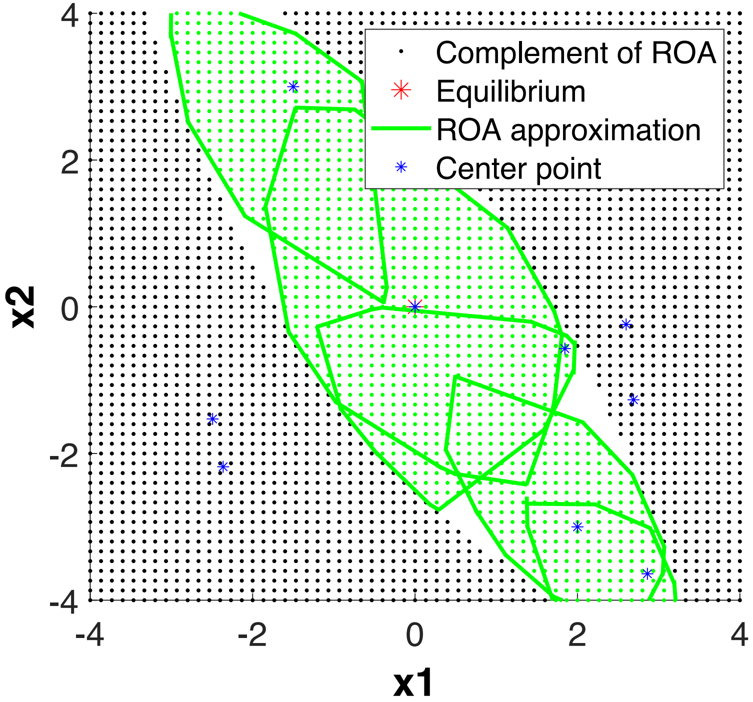
- **If p_{ij} is a counter-example w.r.t** $\hat{S}_{\mathrm{multi}}^{(i)}$
  - We shrink every $\hat{S}_q^{(i)}$ satisfying $p_{ij} \in \hat{S}_q^{(i)}$
  - For the rest approximations, we simply let $\hat{S}_q^{(i+1)} = \hat{S}_q^{(i)}$



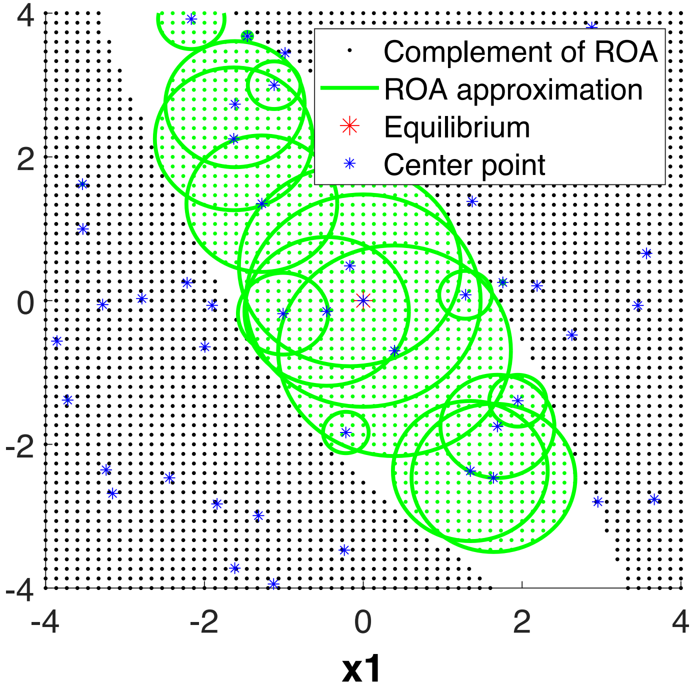
Algorithm results – Multi-center approximation



$\mathcal{A}(0)$:



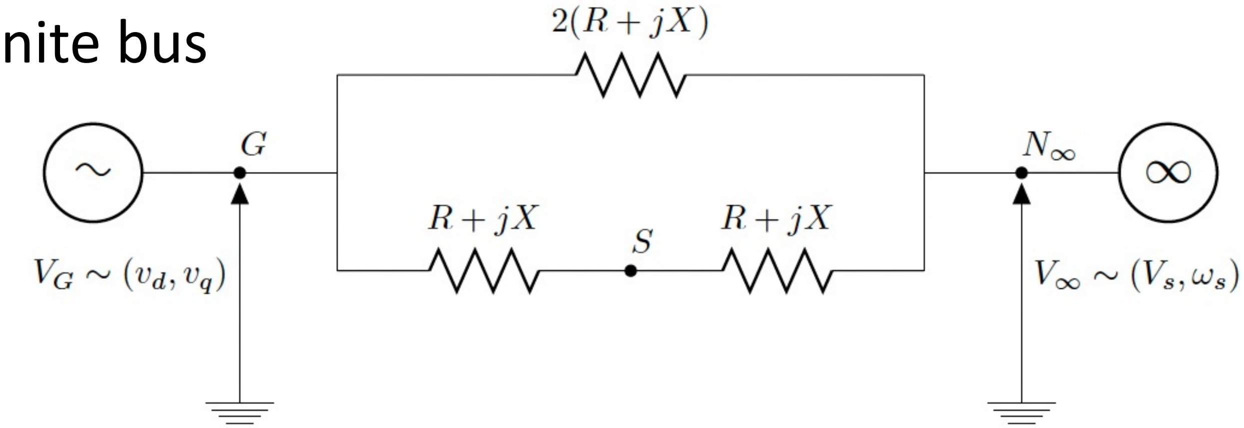
(10 polytope approximations)



(50 sphere approximations)

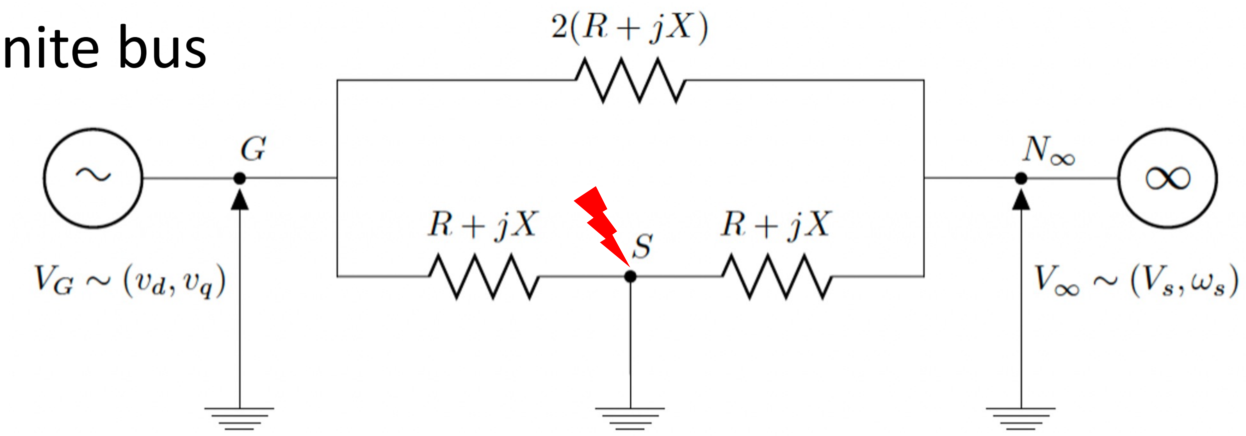
Transient Stability Analysis

- Synchronous machine connected to infinite bus



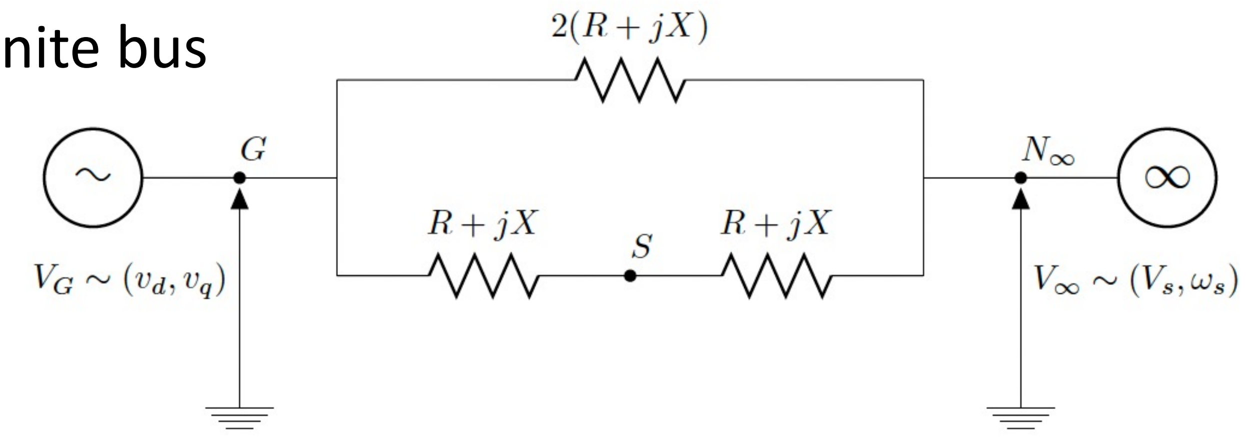
Transient Stability Analysis

- Synchronous machine connected to infinite bus
- t_1 lower line is short-circuited



Transient Stability Analysis

- Synchronous machine connected to infinite bus
- t_1 lower line is short-circuited
- t_2 fault is cleared



Transient Stability Analysis

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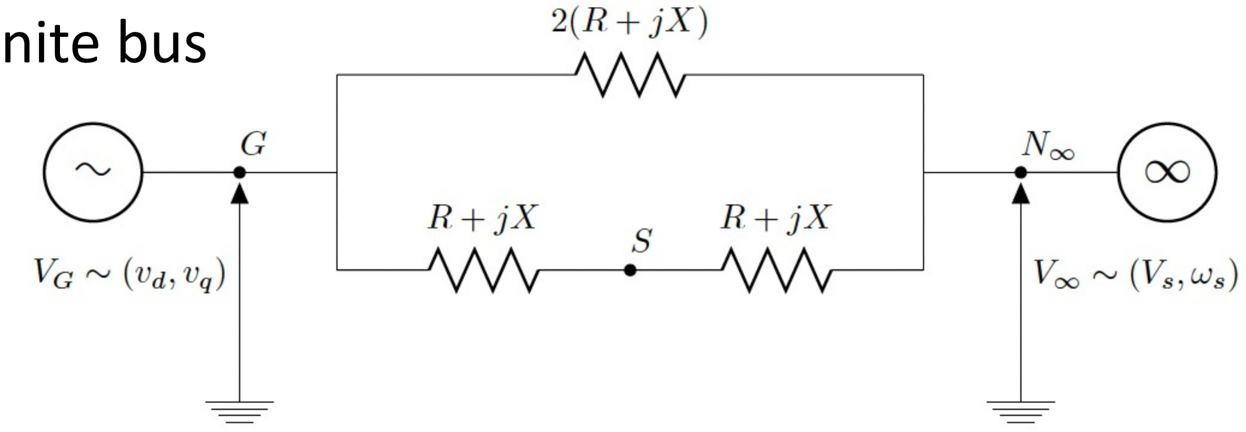

$$\frac{d\delta}{dt} = \omega - \omega_s$$
$$2H\frac{d\omega}{dt} = P_m - (v_d i_d + v_q i_q + e i_d^2 + r i_q^2)$$
$$T'_{d_0}\frac{de'_q}{dt} = -e'_q - (x_d - x'_d)i_d + E_{fd}$$
$$T_a\frac{dE_{fd}}{dt} = -E_{fd} + K_a(V_{ref} - V_t)$$
$$T_g\frac{dP_m}{dt} = -P_m + P_{ref} + K_g(\omega_{ref} - \omega)$$
$$i_q = \frac{(X - x'_d)V_s\sin(\delta) - (R+r)(V_s\cos(\delta) - e'_q)}{(R+r)^2 + (X + x'_d)(X + x_q)}$$

$$i_d = \frac{X - x_q}{R+r}i_q - \frac{1}{R+r}V_s\sin(\delta)$$
$$v_d = x_q i_q - r - i_d$$
$$v_q = R i_q + X i_d + V_s\cos(\delta)$$
$$V_t = \sqrt{v_d^2 + v_q^2}$$

$T'_{d_0} = 9.67$	$x_d = 2.38$	$x'_d = 0.336$	$x_q = 1.21$
$H = 3$	$r = 0.002$	$\omega_s = \omega_{ref} = 1$	$R = 0.01$
$X = 1.185$	$V_s = 1$	$T_a = 1$	$K_a = 70$
$V_{ref} = 1$	$T_g = 0.4$	$K_g = 0.5$	$P_{ref} = 0.7$

Transient Stability Analysis

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- t_1 lower line is short-circuited
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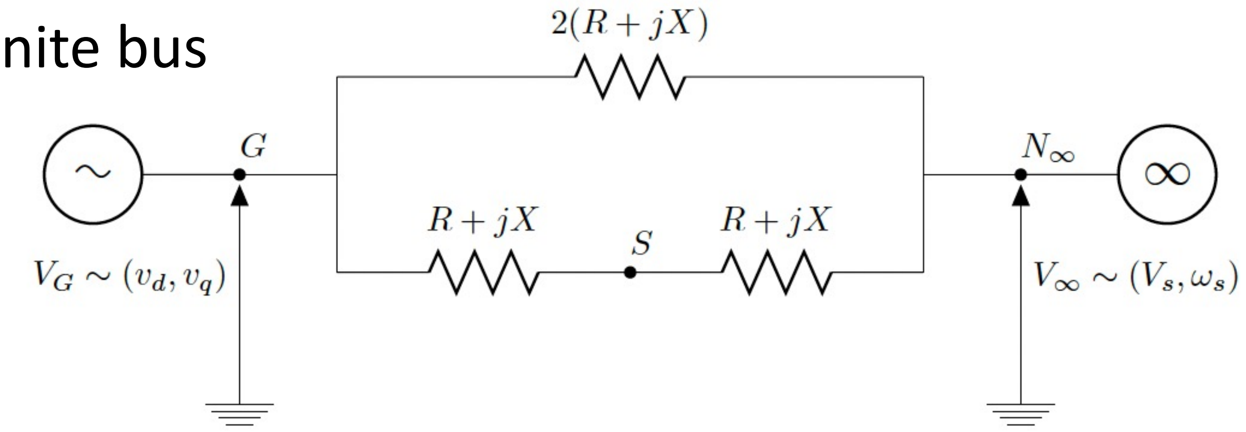
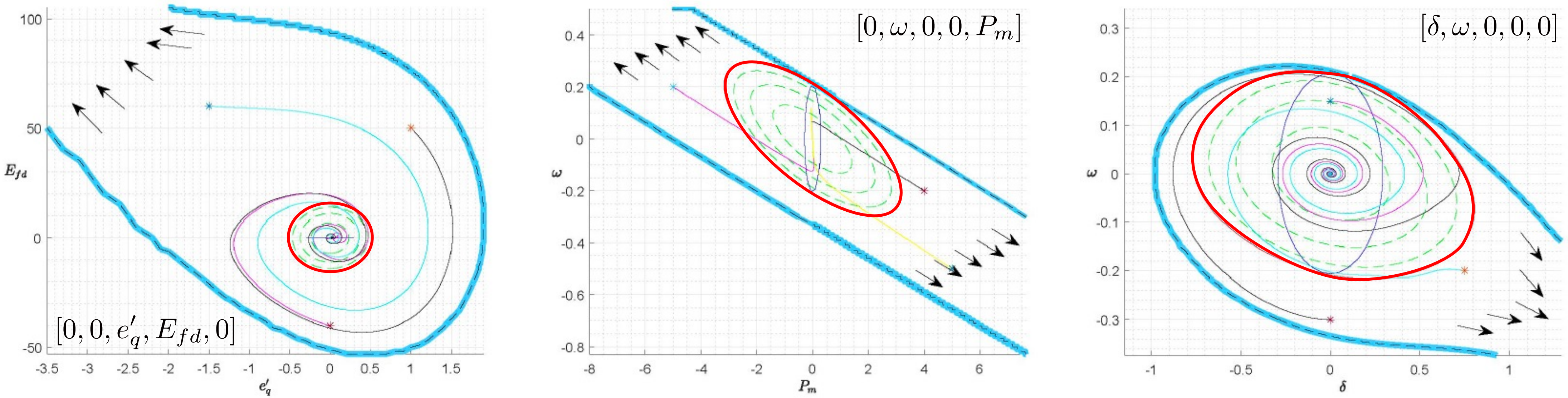


SoS approx. in red (2d-sections)



(a) (b) (c)

M. Tacchi et al "Power system transient stability analysis using SoS programming" Power System Computation Conference (PSCC) 2018

Transient Stability Analysis

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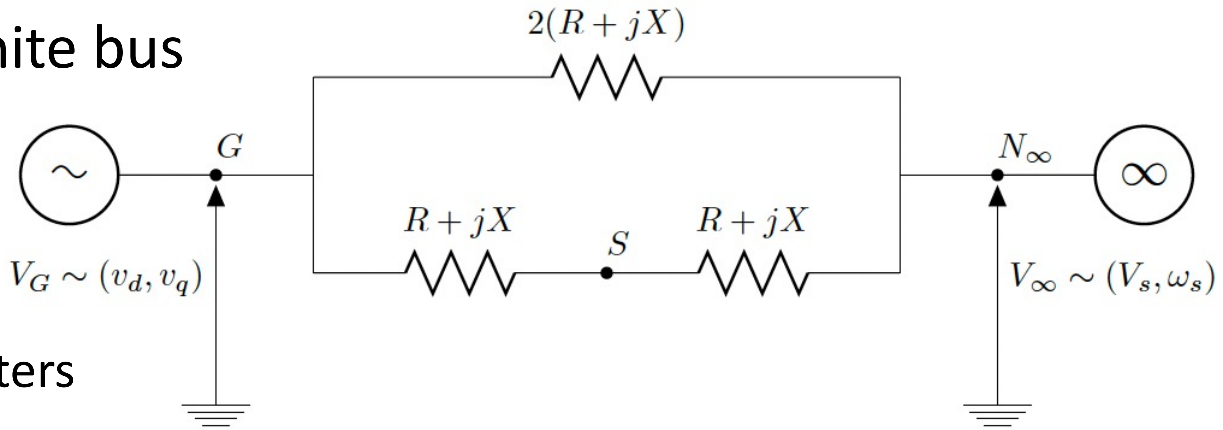
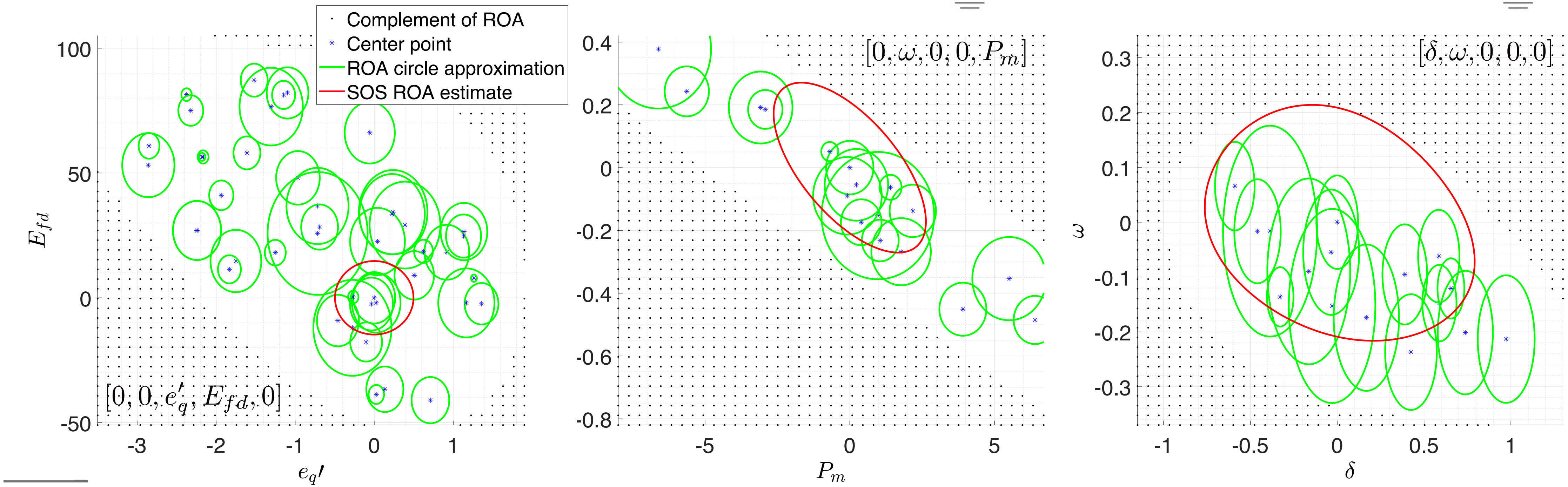
- t_1 lower line is short-circuited
- t_2 fault is cleared



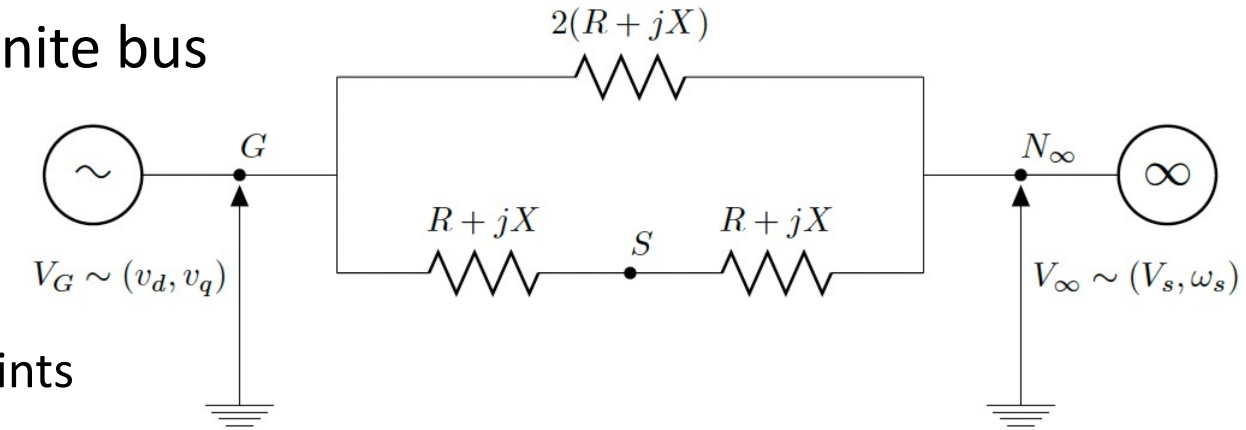
Multi-center in **green**: $\tau_s = 1$, $k = 40$, 2.5K centers



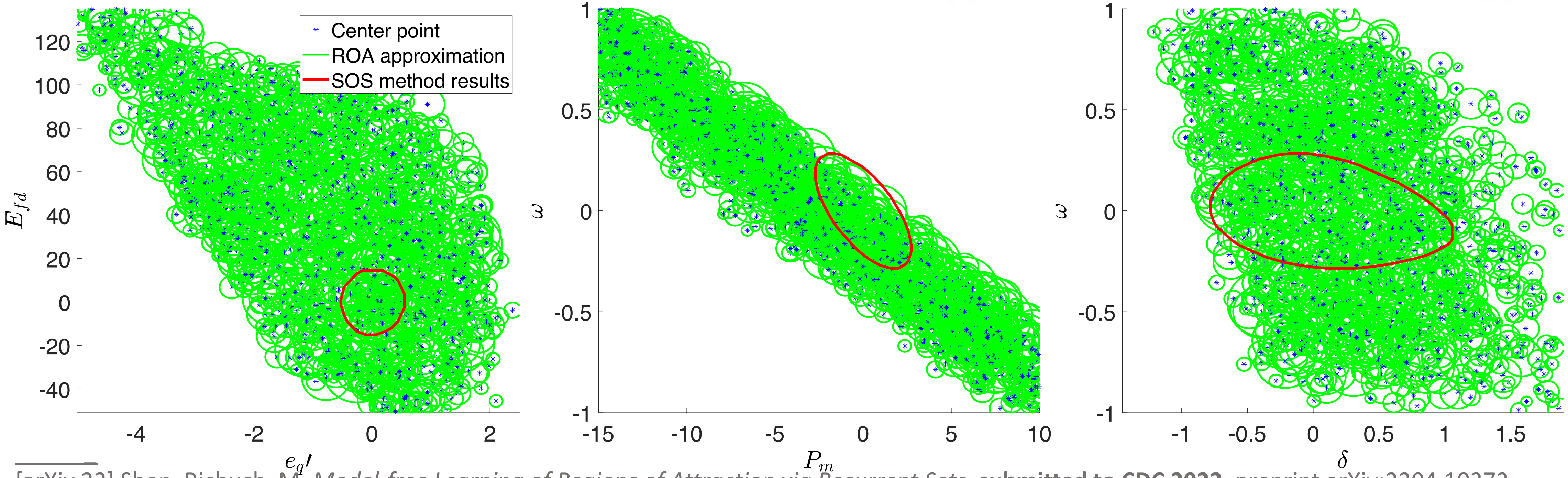
[arXiv 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* **submitted to CDC 2022,** preprint arXiv:2204.10372.

Transient Stability Analysis

- Synchronous machine connected to infinite bus
- t_1 lower line is short-circuited
- t_2 fault is cleared



Multi-center in green: $\tau_s = 1$, $k = 40$, 1.5K points



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Recurrently Decreasing Lyapunov Functions

E Mallada and R. Siegelman, “Stability analysis via recurrently decreasing Lyapunov functions.” in preparation.

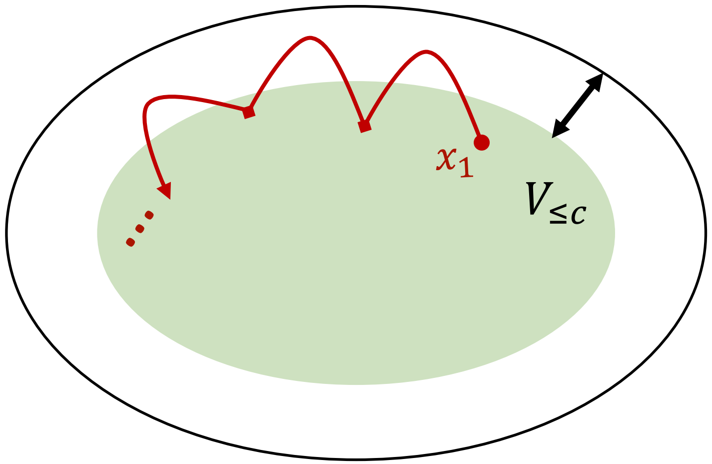
Recurrently Non-Increasing/Decreasing Lyapunov Functions

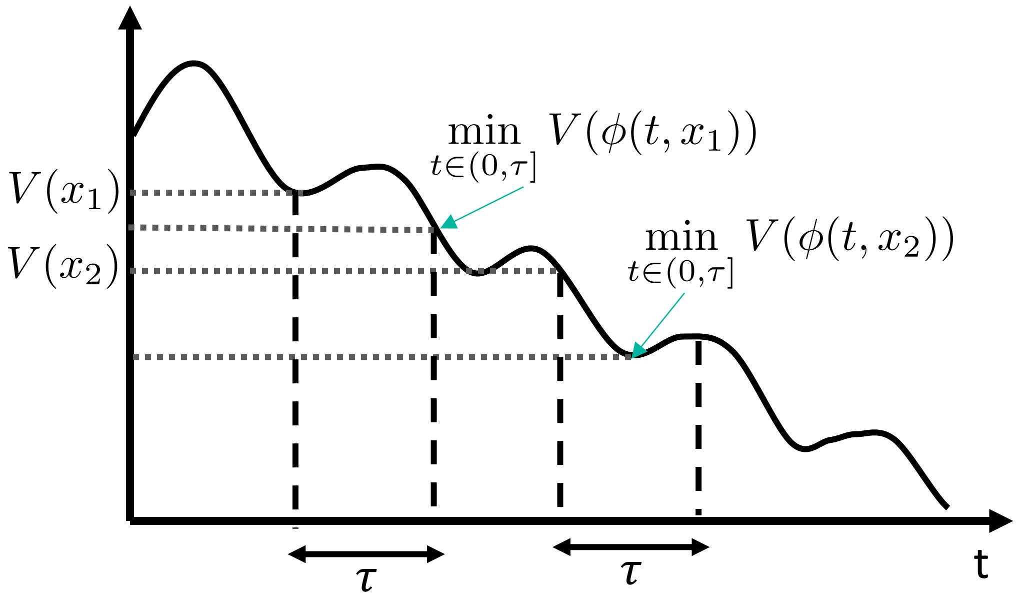
A continuously differentiable function $V: \mathbb{R}^d \to \mathbb{R}_+$ is a **recurrently non-increasing Lyapunov function** over interval of length $\boldsymbol{\tau}$ if

$$\mathcal{L}_f^{(0,\tau]} V(x) := \min_{t \in (0,\tau]} V(\phi(t,x)) - V(x) \leq 0 \quad \forall x \in \mathbb{R}^d$$

Remarks:

- Sub-level sets $\{V(x) \leq c\}$ are $\boldsymbol{\tau}$-recurrent sets.
- When f is globally L-Lipschitz, one can trap trajectories, when $\boldsymbol{\tau L < 1}$.





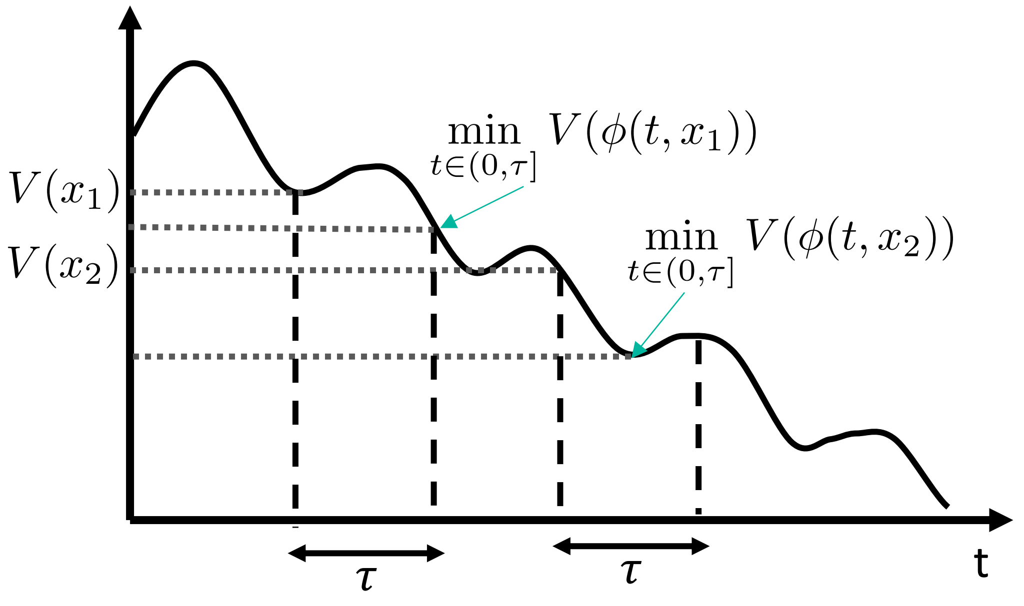
Recurrently Non-Increasing/Decreasing Lyapunov Functions

A continuously differentiable function $V: \mathbb{R}^d \to \mathbb{R}_+$ is a **recurrently non-increasing Lyapunov function** over interval of length $\boldsymbol{\tau}$ if

$$\mathcal{L}_f^{(0,\tau]} V(x) := \min_{t \in (0,\tau]} V(\phi(t,x)) - V(x) \leq 0 \quad \forall x \in \mathbb{R}^d \qquad \textbf{(*)}$$

***Theorem** 4. Let $V: \mathbb{R}^d \to \mathbb{R}_+$ be a recurrently non-increasing Lyapunov function over intervals of length τ, and Assumption 1+2 hold.*

- *Then when f is L-Lipschitz with $L\tau < 1$, the equilibrium x^* is stable.*
- *Further, if the **inequality is strict**, then x^* is asymptotically stable!*



Conclusions and Future work

- **Take-aways**
  - Proposed a **relaxed notion of invariance** known as **recurrence.**
  - Provide **necessary and sufficient conditions** for a recurrent set to be an **inner-approximation** of the ROA.
  - **Our algorithms are sequential, and only incur a limited number of counter-examples.**
  - Provide a **generalized Lyapunov Theorem** based on Recurrence.

- **Ongoing work**
  - Sample complexity bounds, smart choice of multi-points, control recurrent sets, GPU implementation
  - Generalized other Lyapunov notions, Control Lyapunov Functions, Barrier Functions, Control Barrier Functions, etc.

Thanks!

Related Publication:

[arXiv 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* **submitted to CDC 2022,** preprint arXiv:2204.10372.

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