Model-free Learning of Regions of Attraction

via Recurrent Sets

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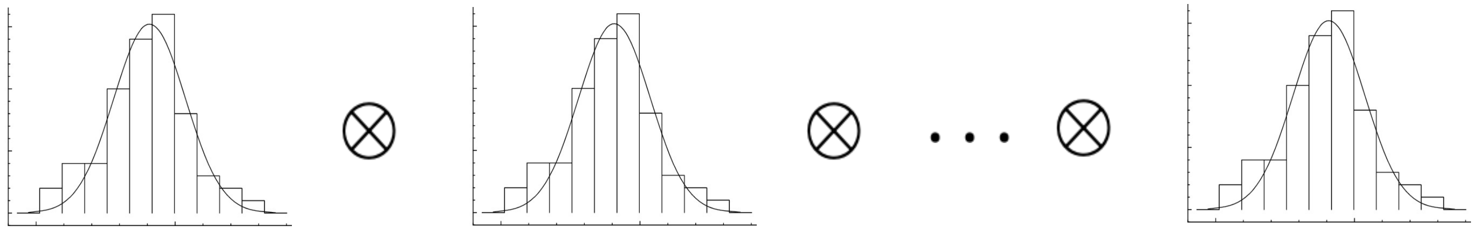
Broad Motivation: The Curse of Dimensionality

- **Sampling in d dimension with resolution ϵ**



Sample complexity:

$$O(\varepsilon^{-d})$$

For $\epsilon = 0.1$ and $d = 100$, we would need 10^{100} points.

- **Verifying non-negativity of polynomials**

Copositive matrices:

$$[x_1^2 \dots x_d^2] A [x_1^2 \dots x_d^2]^{\mathrm{T}} \geq 0$$

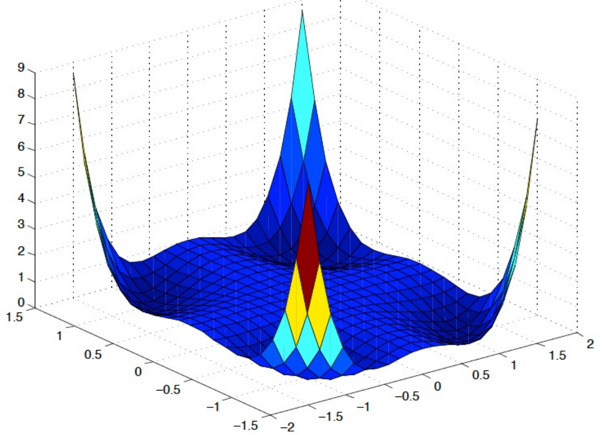
Murty&Kadabi [1987]: Testing co-positivity is NP-Hard

Sum of Squares (SoS):

$$z(x)^T Q z(x) \geq 0, \quad z_i(x) \in \mathbb{R}[x], \ x \in \mathbb{R}^d, Q \succcurlyeq 0$$

Artin [1927] (Hilbert's 17th problem):

Non-negative polynomials are sum of square of *rational* functions



Motzkin [1967]:
$p = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$
is nonnegative,
not a sum of squares,
but $(x^2+y^2)^2 p$ is SoS

Question: Are we asking too much?

- Learnability requires uniform approximation errors across the ***entire domain***

  Q: Can we provide local guarantees, and progressively expand as needed?

[arXiv '22] Shen, Bichuch, M

- Lyapunov functions and control barrier functions require strict and exhaustive notions of ***invariance***

  Q: Can we substitute invariance with less restrictive properties?

[arXiv '22] Shen, Bichuch, M

- Control synthesis usually aims for the ***best*** (optimal) controller

  Q: Can we focus on feasibility, rather than optimality?

[arXiv '21, L4DC 22] Castellano, Min, Bazerque, M

[arXiv 22] Shen, Bichuch, M, *Model-free Learning of Regions of Attraction via Recurrent Sets,* **submitted to CDC 2022,** preprint arXiv:2204.10372.
[L4DC 22] Castellano, Min, Bazerque, M, *Reinforcement Learning with Almost Sure Constraints,* **Learning for Dynamics and Control (L4DC) Conference, 2022**
[arXiv 21] Castellano, Min, Bazerque, M, *Learning to Act Safely with Limited Exposure and Almost Sure Certainty,* **submitted to IEEE TAC, 2021, under review**, preprint arXiv:2105.08748

Model-free Learning of Regions of Attraction via Recurrent Sets

Yue Shen, Maxim Bichuch, and Enrique Mallada

[preprint arXiv:2204.10372v1]

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Region of Attraction

Continuous time dynamical system: $\dot{x}(t) = f(x(t))$

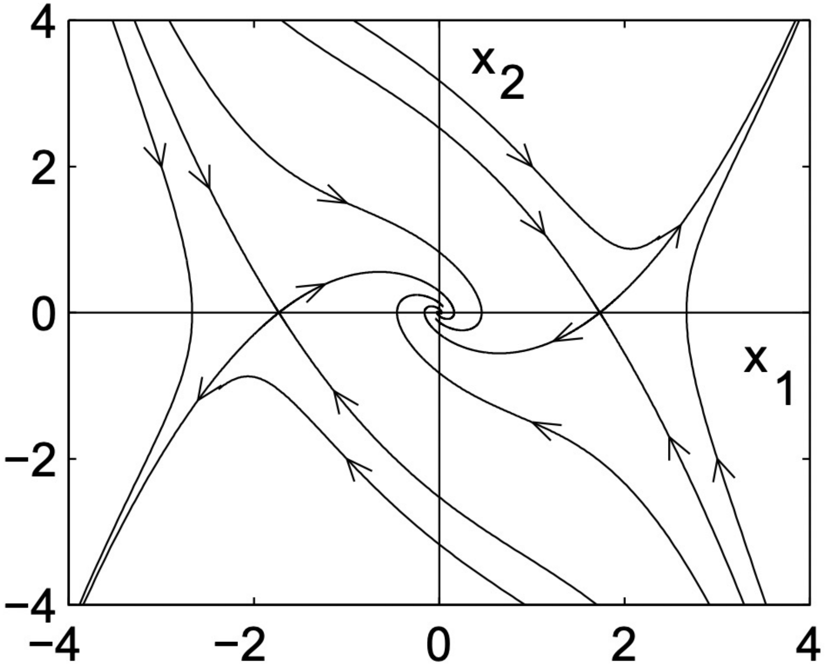
- With initial condition $x_0 = x(0)$, solution at time t: $\phi(t, x_0)$.
- The ω-limit set of the system: $\Omega(f)$

Region of attraction (ROA) of a set $S \subseteq \Omega(f)$:

$$\mathcal{A}(S) := \left\{ x_0 \in \mathbb{R}^d \middle| \lim_{t\to\infty} \phi(t, x_0) \in S \right\}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 + \frac{1}{3}x_1^3 - x_2 \end{bmatrix}$$

$$\Omega(f) = \{(0,0), (-\sqrt{3}, 0), (\sqrt{3}, 0)\}$$



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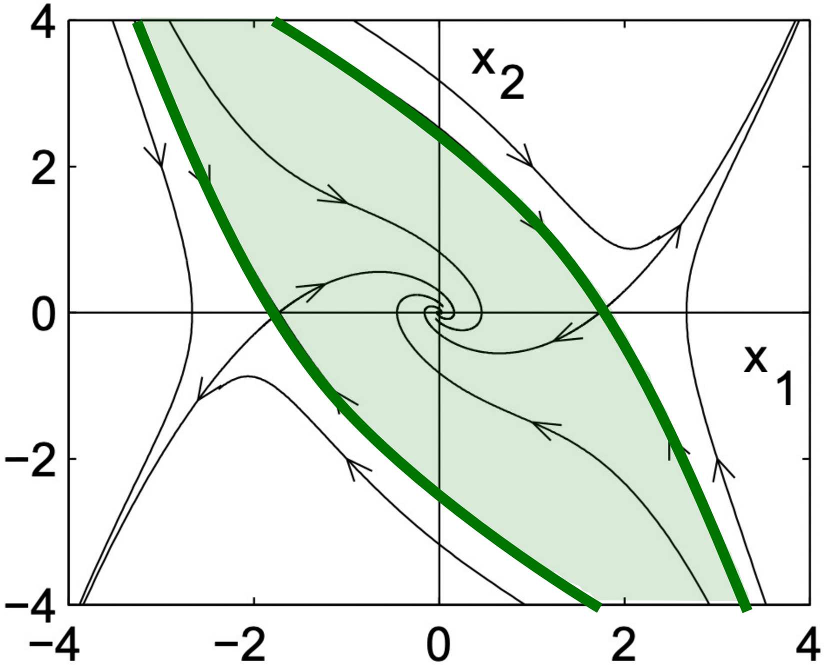
$\mathcal{A}(\Omega(f))$



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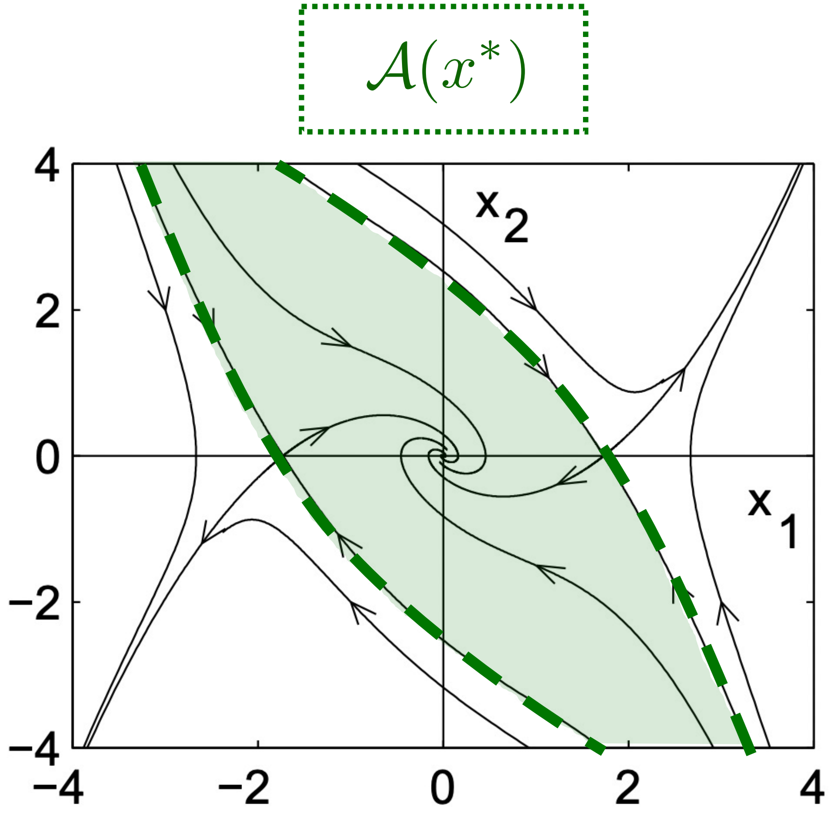
Asymptotically stable equilibrium at $x^* = (0,0)$



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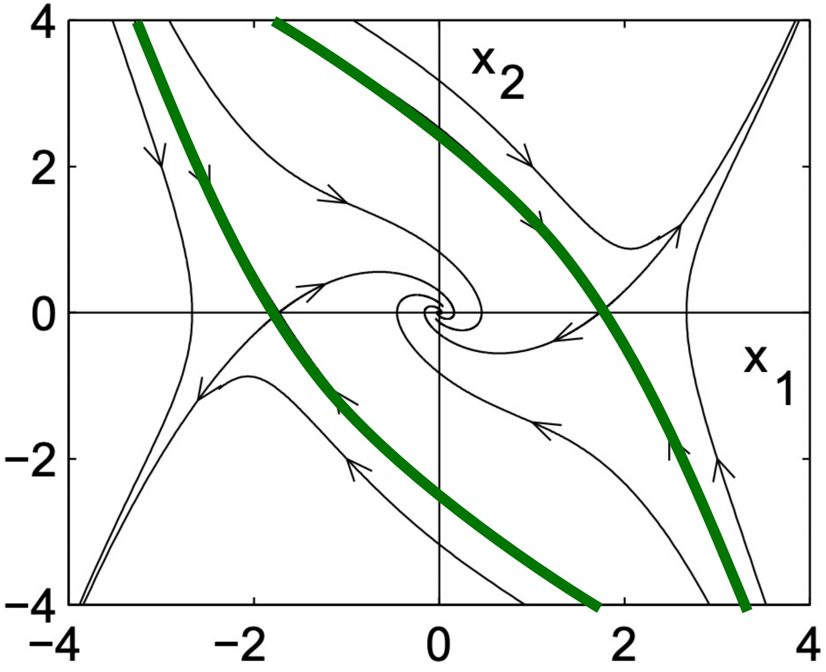
$$\Omega(f) = \{(0,0), (-\sqrt{3}, 0), (\sqrt{3}, 0)\}$$

Unstable equilibria $\{(\sqrt{3}, 0), (-\sqrt{3}, 0)\}$

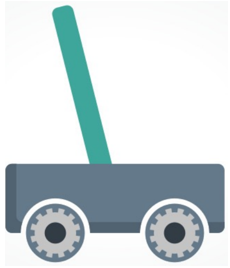
$\mathcal{A}(\{(\sqrt{3}, 0), (-\sqrt{3}, 0)\})$



Motivation: Estimation of Regions of Attraction

Having an approximation of the region of attraction allows us to

- **Test the limits of controller designs**
  especially for those based on (possibly linear) approximations of nonlinear systems

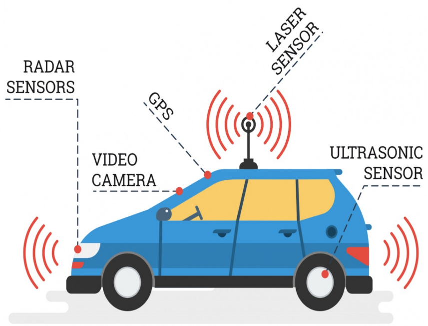


(Cart-pole)

(Quadcopter)

…

- **Verify safety of certain operating condition**



(self-driving)

(HVAC system)

…

Is Invariance Key for Approximating RoAs?

A set $I \subseteq \mathbb{R}^d$ is **positively invariant** if and only if: $x_0 \in \mathcal{I} \implies \phi(t, x_0) \in \mathcal{I}, \quad \forall t \in \mathbb{R}^+$
(Any trajectory starting in the set remains in inside it.)

- **Invariant sets guarantee stability**

(**Lyapunov stability**: solutions starting "close enough" to the equilibrium (within a distance δ) remain "close enough" forever (within a distance ε))

- **Invariant sets further certify asymptotic stability via Lyapunov's direct method**

(**Asymptotic stability**: solutions that start close enough not only remain close enough but also eventually converge to the equilibrium.)

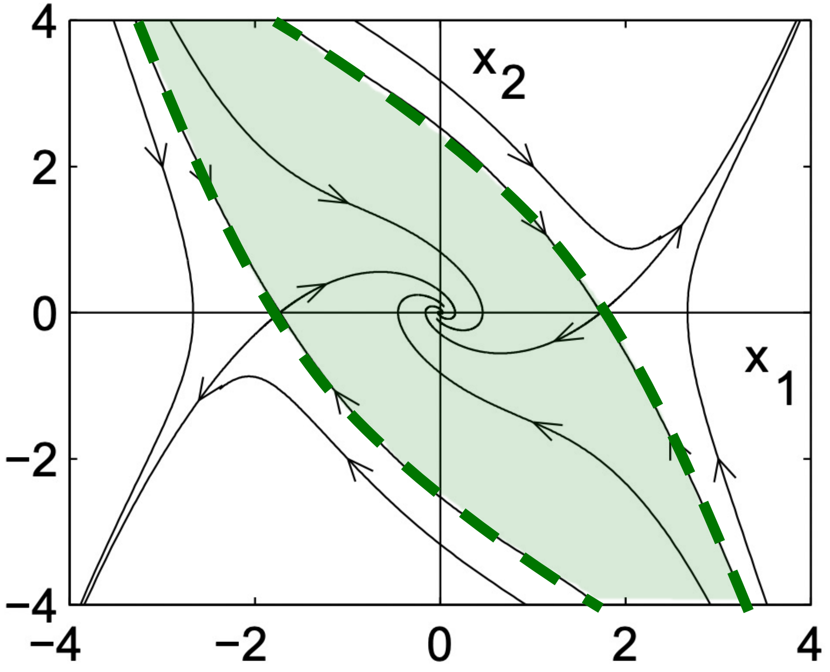
Regions of attraction are invariant sets, and so are the outcome of most approximation methods!

Challenges of Working with Invariant Set

Learning ROA $\mathcal{A}(x^*)$ by finding an invariant set $\mathcal{S} \subseteq \mathcal{A}(x^*)$

***Assumption** 1. The system $\dot{x}(t) = f(x(t))$ has an asymptotically stable equilibrium at x^*.*

***Remark** 1. It follows from Assumption 1 that the (positively invariant) ROA $\mathcal{A}(x^*)$ **is an open contractible set** [Sontag, 2013], i.e., the identity map of $\mathcal{A}(x^*)$ to itself is null-homotopic [Munkres, 2000].*



$\mathcal{A}(x^*)$:

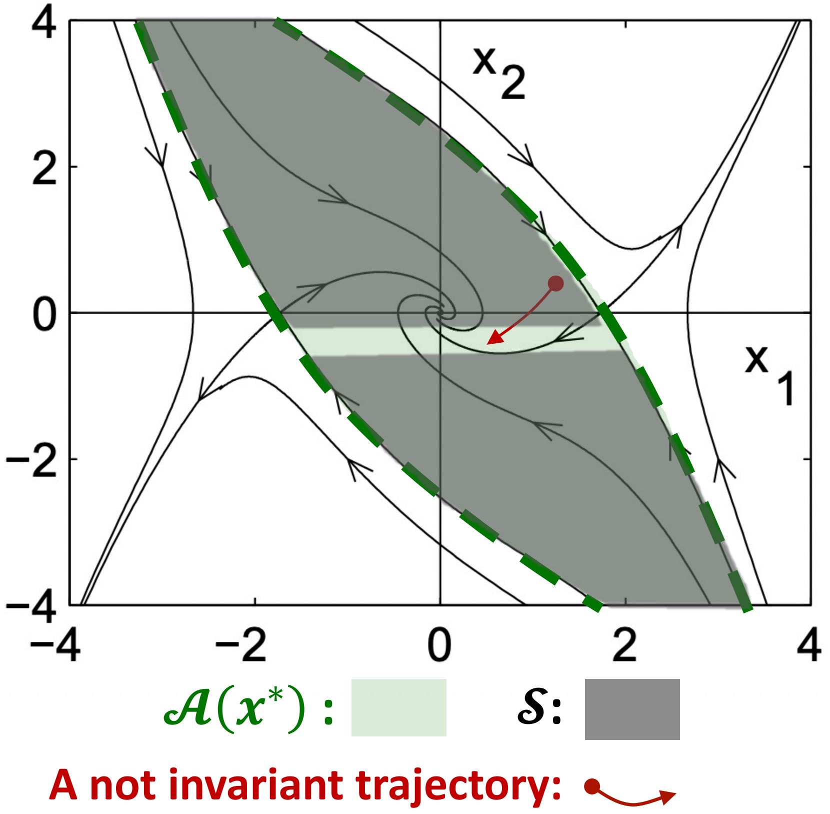
E. Sontag. “Mathematical Control Theory: Deterministic Finite Dimensional Systems.” Springer 2013
J. R. Munkres. “Topology.” Prentice Hall 2000

Challenges of Working with Invariant Set

Learning ROA $\mathcal{A}(x^*)$ by finding an invariant set $\mathcal{S} \subseteq \mathcal{A}(x^*)$

- $\mathcal{S}$ need to be a connected set

Example 1: A good approximation $\mathcal{S} \subseteq \mathcal{A}(x^*)$ is not invariant

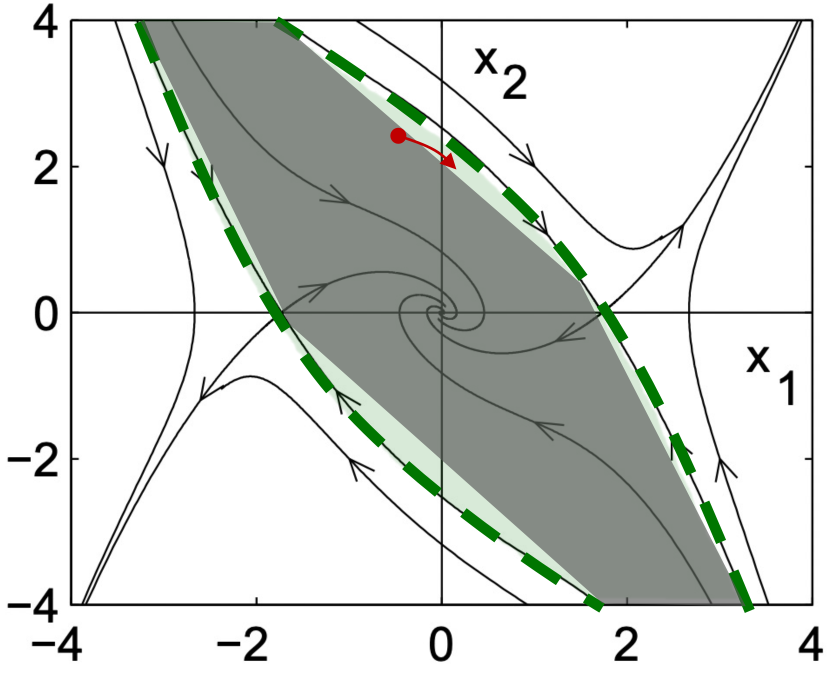


Challenges of Working with Invariant Set

Learning ROA $\mathcal{A}(x^*)$ by finding an invariant set $\mathcal{S} \subseteq \mathcal{A}(x^*)$

- $\mathcal{S}$ need to be a connected set
- $f(x)$ should point inwards for $x \in \partial\mathcal{S}$

Example 2: Another good approximation $\mathcal{S} \subseteq \mathcal{A}(x^*)$ is not invariant



$\mathcal{A}(x^*)$: $\mathcal{S}$:

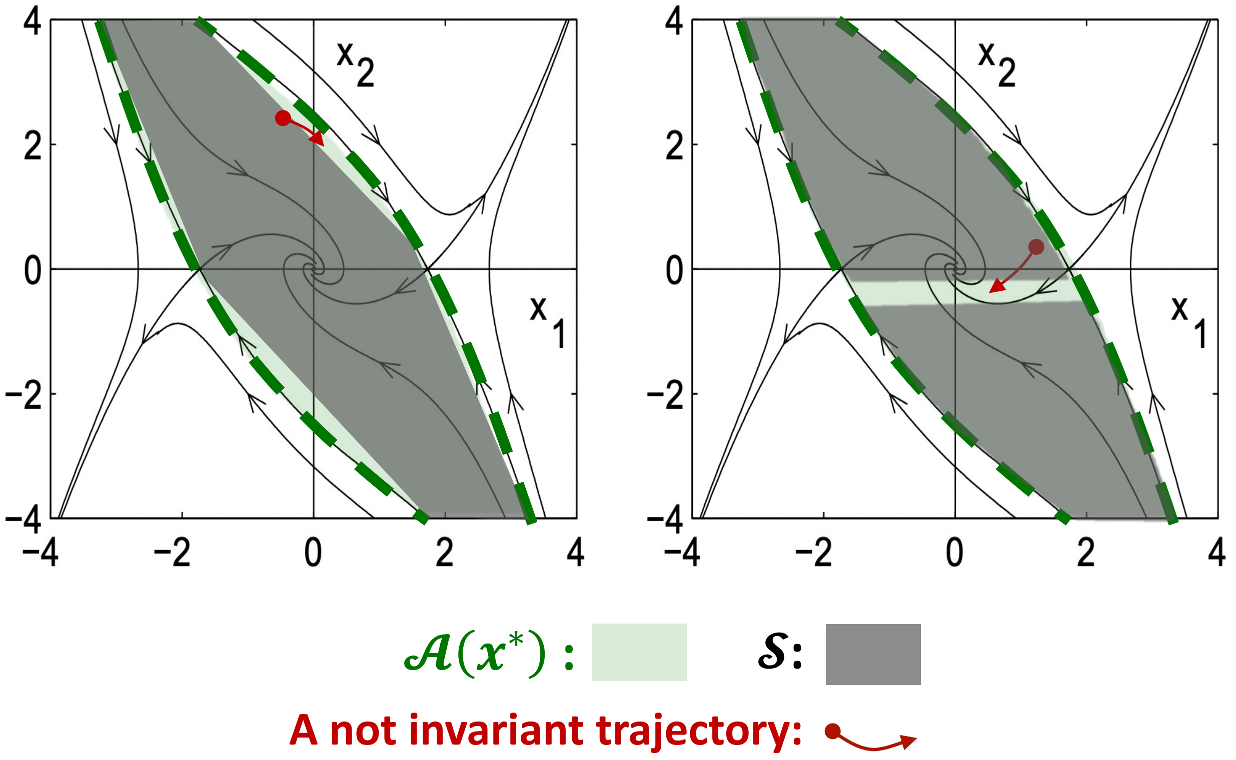
A not invariant trajectory:

Challenges of Working with Invariant Set

Learning ROA $\mathcal{A}(x^*)$ by finding an invariant set $\mathcal{S} \subseteq \mathcal{A}(x^*)$

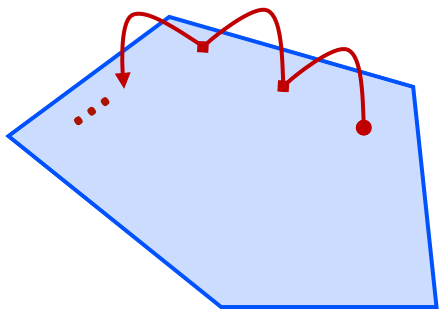
- $\mathcal{S}$ need to be a connected set
- f should point inwards for $x \in \partial\mathcal{S}$

A subset of an invariant set is not necessary an invariant set



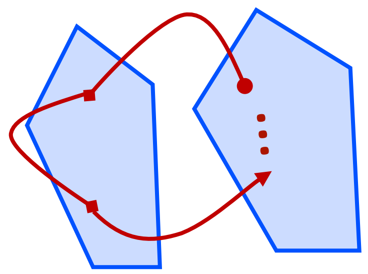
Recurrent Sets: Letting things go, and come back

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if and only if whenever $x_0 \in \mathcal{R}$, $\exists\, t' > 0 \text{ s.t. } \phi(t', x_0) \in \mathcal{R}$



Property of Recurrent Sets

- $\mathcal{R}$ need **not** be **connected**
- $\mathcal{R}$ does **not** require f to **point inwards** on all $\partial\mathcal{R}$



Lemma *1. Consider a compact recurrent set $\mathcal{R}$. Then for any point $x_0 \in \mathcal{R}$ and time $\tau > 0$, there exist a $\tau' > \tau$, such that $\phi(\tau', x_0) \in \mathcal{R}$.*

Recurrent sets, while not invariant, guarantee that solutions that start in this set, will visit it back infinitely often.

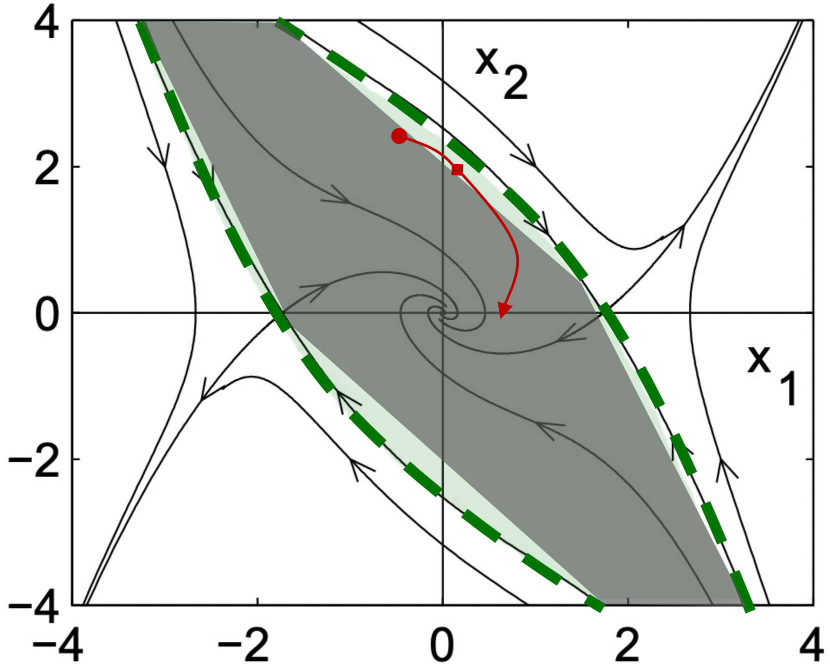
Recurrent set $\mathcal{R}$:

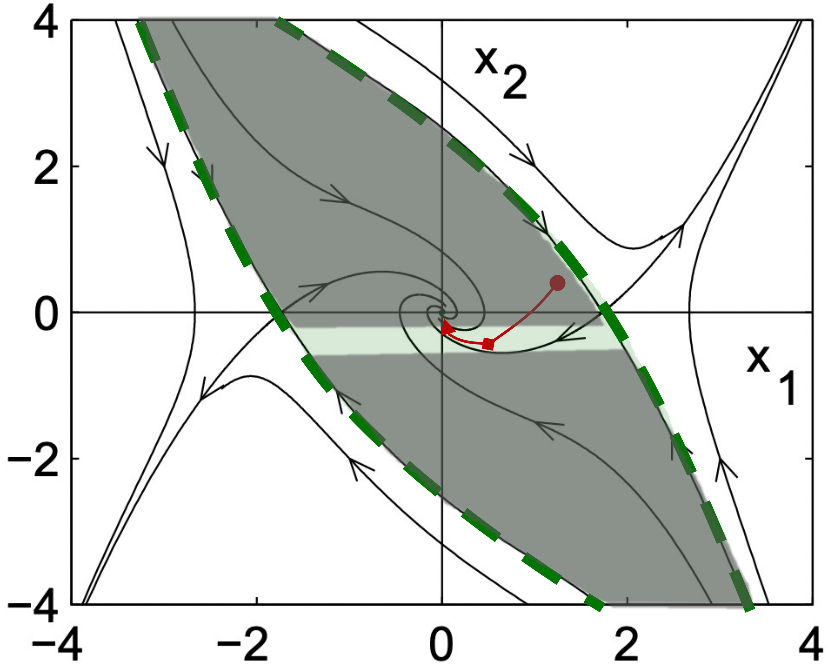
A recurrent trajectory:

Recurrent Sets: Letting things go, and come back

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **recurrent** if and only if whenever $x_0 \in \mathcal{R}$, $\exists\, t' > 0 \text{ s.t. } \phi(t', x_0) \in \mathcal{R}$

Previous two good inner approximations of $\mathcal{A}(x^*)$ are recurrent sets





Recurrent sets *are subsets of the region of attraction

A set $\mathrm{R} \subseteq \mathbb{R}^d$ is **recurrent** if and only if whenever $x_0 \in R$, $\exists\, t' > 0 \text{ s.t. } \phi(t', x_0) \in \mathcal{R}$

***Theorem** 2. Let $\mathcal{R} \subset \mathbb{R}^d$ be a compact set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$. Then:*

$\mathcal{R}$ is recurrent	⬌	$\mathcal{R} \cap \Omega(f) \neq \emptyset$ $\mathcal{R} \subset \mathcal{A}(\mathcal{R} \cap \Omega(f))$

***Proof**: [Sketch]*

(→) If $\mathcal{R}$ is recurrent, solutions that start in $\mathcal{R}$ will visit it back infinitely often. Thus, for each $x_0 \in \mathcal{R}$, we can construct an infinite sequence $\{x_n\}_{n=0}^{\infty} \in \mathcal{R}$. Then Bolzano-Weierstrass theorem implies there exists a sub-sequence $\{x_{n\_i}\}_{i=0}^{\infty}$ that converges to an accumulation point $\bar{\boldsymbol{x}} \in \Omega(f) \cap \mathcal{R} \neq \emptyset$.

(←) Starting from any $x_0 \in \mathcal{R} \subset \mathcal{A}(\mathcal{R} \cap \Omega(f))$, $\phi(t, x_0)$ converges to $\mathcal{R} \cap \Omega(f) \subseteq \operatorname{int} \mathcal{R}$. It then follows from the continuity of ϕ that there always exists some time $t > 0$ such that $\phi(t, x_0) \in \mathcal{R}$. Thus, $\mathcal{R}$ is recurrent.

Recurrent sets *are subsets of the region of attraction

A set $\mathrm{R} \subseteq \mathbb{R}^d$ is **recurrent** if and only if whenever $x_0 \in R$, $\exists\, t' > 0 \text{ s.t. } \phi(t', x_0) \in \mathcal{R}$

Theorem *2. Let $\mathcal{R} \subset \mathbb{R}^d$ be a compact set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$. Then:*

$\mathcal{R}$ *is recurrent*	⬌	$\mathcal{R} \cap \Omega(f) \neq \emptyset$ $\mathcal{R} \subset \mathcal{A}(\mathcal{R} \cap \Omega(f))$

Assumption *2. The ω-limit set $\Omega(f)$ is composed by **hyperbolic equilibrium points**, with only one of them, say x^*, being asymptotically stable.*

Corollary *2. Let Assumption 2 hold and let $\mathcal{R} \subset \mathbb{R}^d$ be a compact set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$. Then:*

$\mathcal{R}$ *is recurrent*	⬌	$\mathcal{R} \cap \Omega(f) = x^*$ $\mathcal{R} \subset \mathcal{A}(x^*)$

Recurrent sets imply a subset of the region of attraction

A set $\mathrm{R} \subseteq \mathbb{R}^d$ is **recurrent** if and only if whenever $x_0 \in R$, $\exists\, t' > 0$? s.t. $\phi(t', x_0) \in \mathcal{R}$

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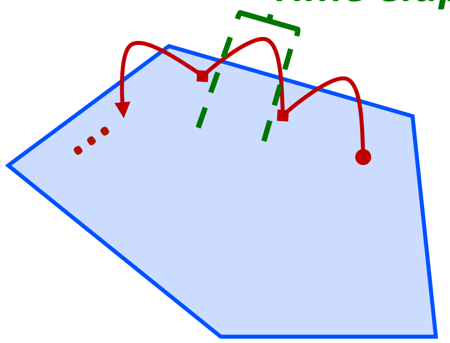
$\mathcal{R}$ is recurrent $\Longleftrightarrow$ $\mathcal{R} \cap \Omega(f) = x^*$, $\mathcal{R} \subset \mathcal{A}(x^*)$

- Corollary 2 suggests that we may use recurrence as a mechanism for practically finding inner approximations for $\mathcal{A}(x^*)$.
- However, we do not know a priori how long it may take for a trajectory to come back to R after it leaves it.

Set dependent bounds on recurrence

A set $\mathcal{R} \subseteq \mathbb{R}^d$ is **τ-recurrent** if whenever $x_0 \in \mathcal{R}$, then $\exists\, t' \in (0, \tau]$ s.t. $\phi(t', x_0) \in \mathcal{R}$



τ-recurrent set $\mathcal{R}$:

A τ-recurrent trajectory:

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Theorem *3. Let Assumption 1 hold and consider a compact set $\mathcal{R} \subset \mathcal{A}(x^*)$* satisfying $x^* \in \operatorname{int} \mathcal{R}$ and $\mathcal{R} \cap \mathcal{A}(x^*) = \emptyset$. Then there exists positive constants $\bar{c}$, $\underline{c}$, and a, depending on $\mathcal{R}$ such that for all

$$\tau \geq \bar{\tau} := \frac{\underline{c} - \bar{c}}{a},$$

the set $\mathcal{R}$ is τ-recurrent . Further, starting from any point $x \in \mathcal{R}$, the solution $\phi(t, x) \in \mathcal{R}$ for all $t \geq \bar{\tau}$.

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Proof: *[Sketch]*

- *Under Assumption 1, it follows from [Driver, 1964] that there exists a Lyapunov function* $V(x)$ with domain on $\mathcal{A}(x^*)$ satisfying :

- $V(x^*) = 0$, $0 < V(x) < 1$ for all $x \in \mathcal{A}(x^*)\backslash x^*$
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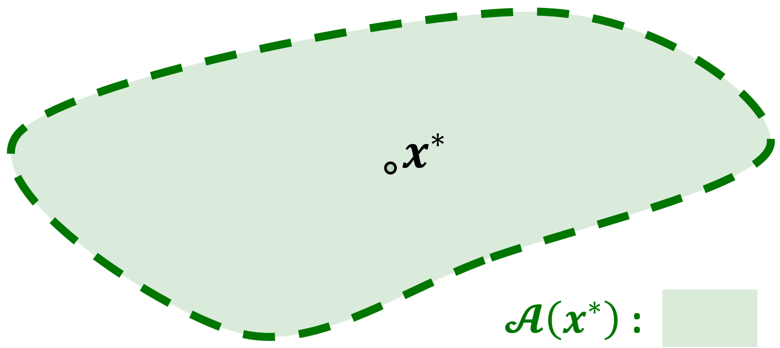

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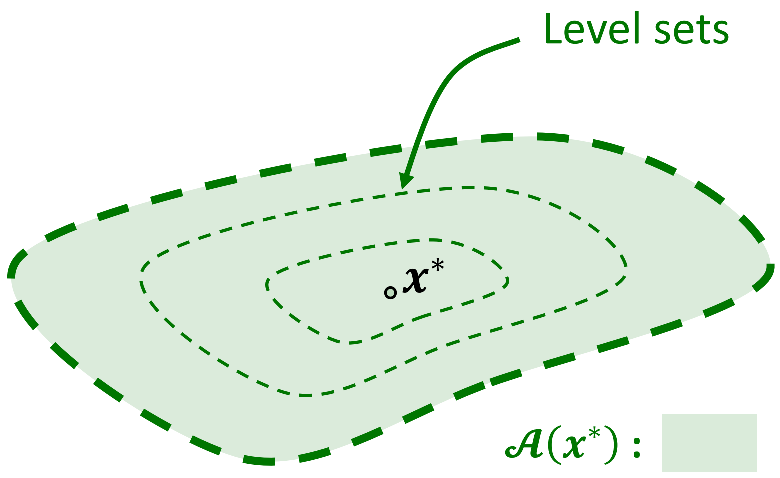
- For any $c \in (0,1)$, *we can further show the level set* $V_{<c} := \{x : V(x) < c\}$ *is a contractible invariant subsets of* $\mathcal{A}(x^*)$.



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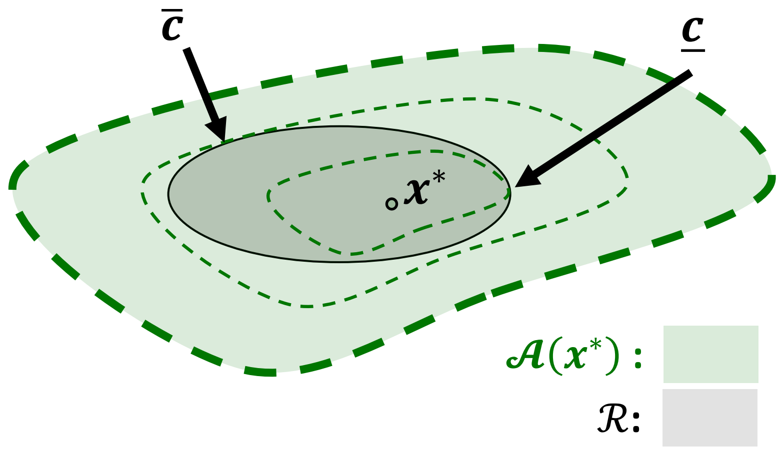
- Given compact set $\mathcal{R}$*, let us now define*

$$\underline{c} := \min_{x \in \partial \mathcal{R}} V(x),\ \bar{c} := \max_{x \in \partial \mathcal{R}} V(x),\ \text{and}\ a := \max_{x \in C} \nabla V(x)^T f(x),$$

where $C = \{x \in \mathbb{R}^d : \underline{c} \leq V(x) \leq \bar{c}\}$.



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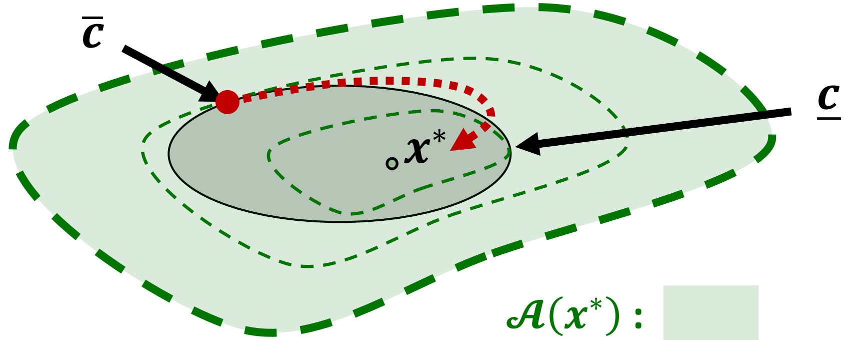
- For any $c \in (0,1)$, we can further show the level set $V_{<c} := \{x : V(x) < c\}$ is a contractible invariant subsets of $\mathcal{A}(x^*)$.

- Given compact set $\mathcal{R}$, let us now define

$$\underline{c} := \min_{x \in \partial \mathcal{R}} V(x),\ \bar{c} := \max_{x \in \partial \mathcal{R}} V(x),\ \text{and}\ a := \max_{x \in C} \nabla V(x)^T f(x),$$

where $C = \{x \in \mathbb{R}^d : \underline{c} \leq V(x) \leq \bar{c}\}$.

- We can then conclude: $V_{<\underline{c}} \subseteq \mathcal{R} \subseteq V_{<\bar{c}}$. Moreover, for any point $x \in \mathcal{R}$, the Lyapunov value $V(\phi(t,x)) \leq \underline{c}$ after $t \geq \bar{t} := \frac{c-\bar{c}}{a}$, which implies $\phi(t,x) \in \mathcal{R}$.



$\mathcal{A}(x^*)$:

An example trajectory:

$\mathcal{R}$:

Beyond set dependent bounds

Theorem *3. Let Assumption 1 hold and consider a compact set* $\mathcal{R} \subset \mathcal{A}(x^*)$ satisfying $x^* \in \text{int}\,\mathcal{R}$ and $\mathcal{R} \cap \mathcal{A}(x^*) = \emptyset$. Then there exists positive constants $\bar{c}$, $\underline{c}$, and a, depending on $\mathcal{R}$ such that for all

$$\tau \geq \bar{\tau} := \frac{\underline{c} - \bar{c}}{a},$$

the set $\mathcal{R}$ *is* $\boldsymbol{\tau}$*-recurrent . Further, starting from any point* $x \in \mathcal{R}$*, the solution* $\phi(t, x) \in \mathcal{R}$ *for all* $t \geq \bar{\tau}$*.*

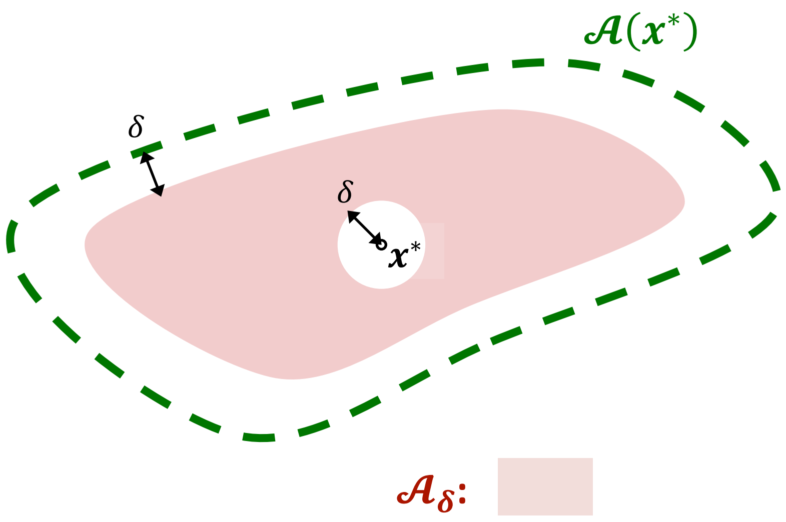
Recurrent bound depends on the set

Beyond set dependent bounds

To eliminate this dependence, we consider the set:

$$\mathcal{A}_\delta := \mathcal{A}(x^*)\setminus\left(\{\partial\mathcal{A}(x^*) + \mathrm{int}\mathcal{B}_\delta\}\cup\{\mathrm{int}\mathcal{B}_\delta + x^*\}\right),$$

where $\mathcal{B}_\delta := \{x\colon \|x\|_2 \le \delta\}$ and δ is chosen to be small enough such that $\mathcal{B}_\delta + x^* \subseteq \boldsymbol{\mathcal{A}}(\boldsymbol{x}^*)\setminus\{\partial\boldsymbol{\mathcal{A}}(\boldsymbol{x}^*) + \mathcal{B}_\delta\}$.



Beyond set dependent bounds

To eliminate this dependence, we consider the set:

$$\mathcal{A}_\delta := \mathcal{A}(x^*) \setminus \left(\{\partial\mathcal{A}(x^*) + \mathrm{int}\mathcal{B}_\delta\} \cup \{\mathrm{int}\mathcal{B}_\delta + x^*\}\right),$$

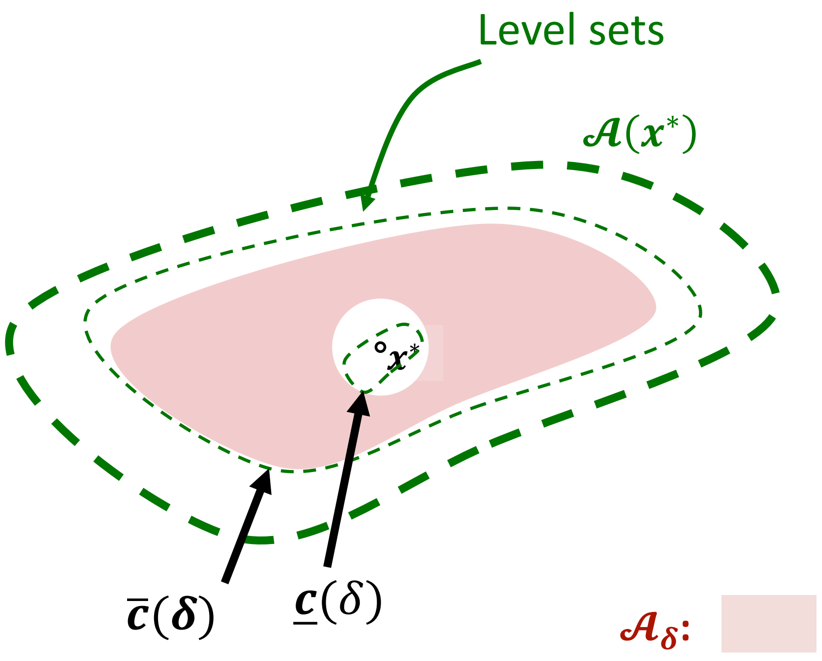
where $\mathcal{B}_\delta \coloneqq \{x\colon \|x\|_2 \leq \delta\}$ and δ is chosen to be small enough such that $\mathcal{B}_\delta + x^* \subseteq \boldsymbol{\mathcal{A}}(\boldsymbol{x}^*)\setminus\{\partial\boldsymbol{\mathcal{A}}(\boldsymbol{x}^*) + \mathcal{B}_\delta\}$.

Then , we define:

$$\bar{c}(\delta) := \max_{x\in\mathcal{A}_\delta} V(x), \quad \underline{c}(\delta) := \min_{x\in\mathcal{A}_\delta} V(x),$$

$$\text{and} \quad a(\delta) := \max_{x\in C_\delta} \nabla V(x)^T f(x),$$

where $C_\delta = \left\{x \in \mathbb{R}^d\colon \underline{c}(\delta) \leq V(x) \leq \bar{c}(\delta)\right\}$.



Beyond set dependent bounds

To eliminate this dependence, we consider the set:

$$\mathcal{A}_\delta := \mathcal{A}(x^*)\setminus\left(\{\partial\mathcal{A}(x^*) + \mathrm{int}\mathcal{B}_\delta\}\cup\{\mathrm{int}\mathcal{B}_\delta + x^*\}\right),$$

where $\mathcal{B}_\delta \coloneqq \{x: \|x\|_2 \leq \delta\}$ and δ is chosen to be small enough such that $\mathcal{B}_\delta + x^* \subseteq \boldsymbol{\mathcal{A}}(\boldsymbol{x}^*)\setminus\{\partial\boldsymbol{\mathcal{A}}(\boldsymbol{x}^*) + \mathcal{B}_\delta\}$.

Then , we define:

$$\bar{c}(\delta) := \max_{x\in\mathcal{A}_\delta} V(x), \quad \underline{c}(\delta) := \min_{x\in\mathcal{A}_\delta} V(x),$$

$$\text{and} \quad a(\delta) := \max_{x\in C_\delta} \nabla V(x)^T f(x),$$

where $C_\delta = \{x \in \mathbb{R}^d: \underline{c}(\delta) \leq V(x) \leq \bar{c}(\delta)\}$.

Uniform bounds on recurrence

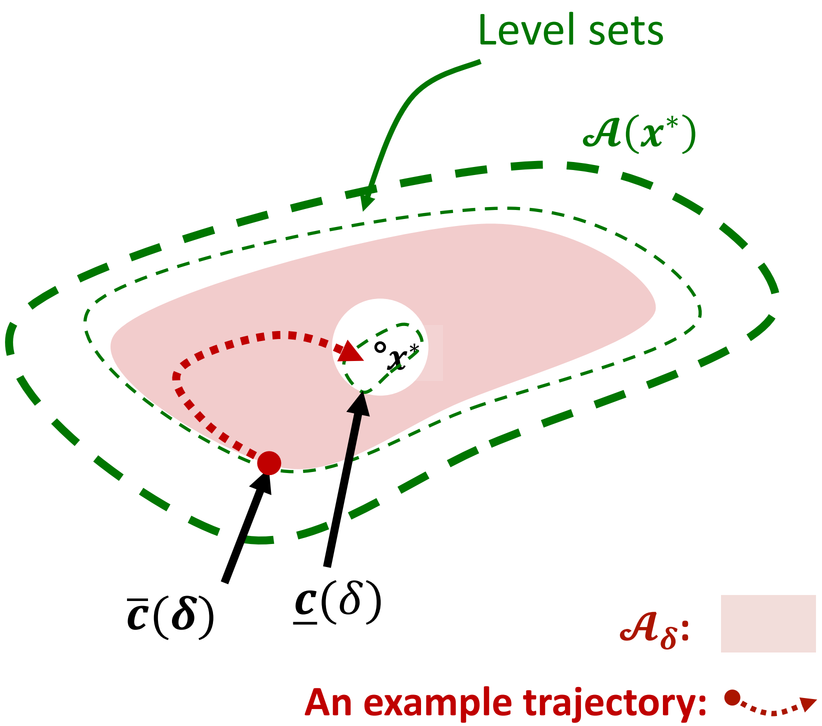
***Theorem** 4. Under Assumption 1, any compact set $\mathcal{R}$* satisfying:

$$\mathcal{B}_\delta + x^* \subseteq \mathcal{R} \subseteq \mathcal{A}(x^*)\setminus\{\partial\mathcal{A}(x^*) + \mathrm{int}\,\mathcal{B}_\delta\}$$

is $\boldsymbol{\tau}$-recurrent for any

$$\tau \geq \bar{\tau}(\delta) \coloneqq \frac{\underline{c}(\delta) - \bar{c}(\delta)}{a(\delta)}.$$

Moreover, when $t \geq \bar{\tau}$, $\phi(t,x) \in \mathcal{R}$ for any point $x \in \mathcal{R}$.



Learning Recurrent Sets

- We now propose a method to compute inner-approximations of the region of attraction $\mathcal{A}(x^*)$ based on checking the recurrence property on finite-length trajectory.

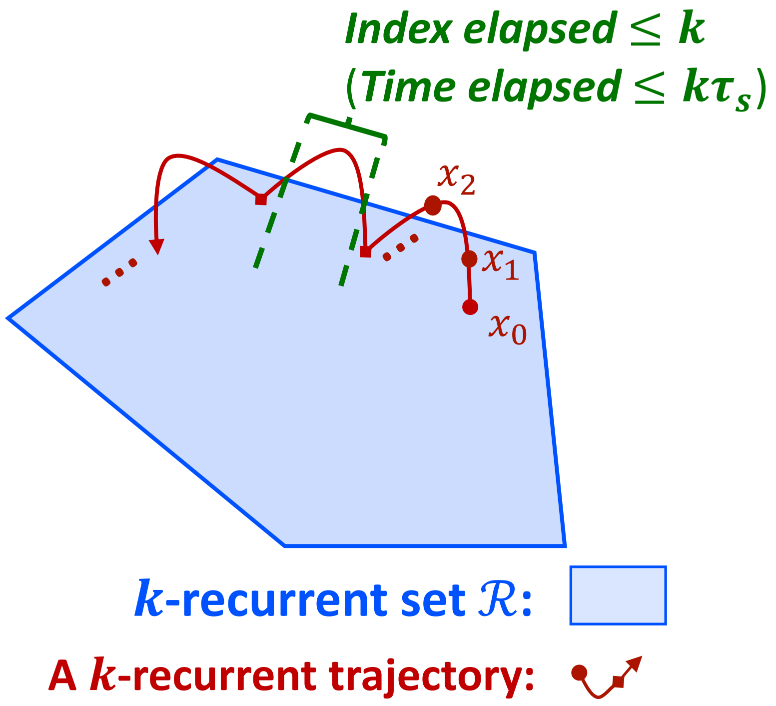
- Consider the following type of sample trajectories:

  $$x_n = \phi(n\tau_s, x_0), \quad x_0 \in \mathbb{R}^d, \; n \in \mathbb{N},$$

  where $\tau_s > 0$ is the sampling period.

- Notion of discrete recurrence w.r.t. a length k trajectory:

  A set $\mathcal{R} \subseteq \mathbb{R}^d$ is k-**recurrent** if whenever $x_0 \in \mathcal{R}$, then $\exists\, n \in \{1, \ldots, k\}$ s.t. $x_n \in \mathcal{R}$.



Learning Recurrent Sets

- A set $\mathcal{R}$ being k-recurrent implies $\mathcal{R}$ is τ-recurrent with $\tau = k\tau_s$. We can further conclude $\mathcal{R} \subset \mathcal{A}(x^*)$ under the assumptions of Corollary 2.

- To ensure one can find such a k-recurrent set, we provide the following sufficient condition for a set $\mathcal{R}$ to be k-recurrent .

*(**Theorem** 4). Under Assumption 1, any compact set $\mathcal{R}$ satisfying:*

$$\mathcal{B}_\delta + x^* \subseteq \mathcal{R} \subseteq \mathcal{A}(x^*) \setminus \{\partial\mathcal{A}(x^*) + \text{int}\, \mathcal{B}_\delta\}$$

is $\boldsymbol{\tau}$-recurrent for any

$$\tau \geq \bar{\tau}(\delta) := \frac{\underline{c}(\delta) - \bar{c}(\delta)}{a(\delta)}.$$

Moreover, when $t \geq \bar{\tau}$, $\phi(t, x) \in \mathcal{R}$ for any point $x \in \mathcal{R}$.

***Theorem** 5. Under Assumption 1, any compact set $\mathcal{R}$* satisfying:

$$\mathcal{B}_\delta + x^* \subseteq \mathcal{R} \subseteq \mathcal{A}(x^*) \setminus \{\partial\mathcal{A}(x^*) + \text{int}\, \mathcal{B}_\delta\}$$

is k-recurrent for any $k > \bar{k} := \bar{\tau}(\delta)/\tau_s$.

Algorithm Explanation

Assumptions:

- ***(Assumption** 2). The ω-limit set $\Omega(f)$ is composed by hyperbolic equilibrium points, with only one of them, say x^*, being asymptotically stable.*
- *We further assume w.o.l.g $x^* = 0$ for simplicity.*

Goal:

*With a compact initial approximation $\hat{\mathcal{S}}^{(0)} \subset \mathbb{R}^d$ of the ROA satisfying $\hat{\mathcal{S}}^{(0)} \supseteq \mathcal{B}_\delta$, we seek to find a subset of ROA within $\mathcal{A}(x^*) \cap \hat{\mathcal{S}}^{(0)}$ by **learning a k-recurrent set** $\mathcal{R}$.*

Sufficiency:

$\mathcal{R}$ is k-recurrent ➡ $\mathcal{R}$ is τ-recurrent with $\tau = k\tau_s$ ✚ $\mathcal{R}$ is compact $\partial\mathcal{R} \cap \Omega(f) = \emptyset$ ➡ $\mathcal{R} \subset \mathcal{A}(x^*)$

(Corollary 2, under Assumption 2)

Necessity:

*(**Theorem** 5). Under Assumption 1, any compact set $\mathcal{R}$* satisfying:

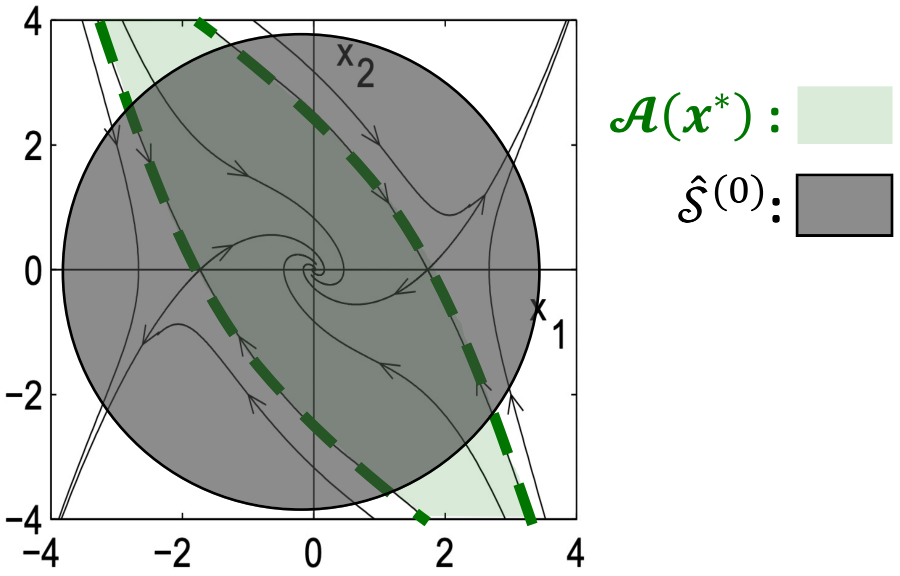
$$\mathcal{B}_\delta + x^* \subseteq \mathcal{R} \subseteq \mathcal{A}(x^*) \backslash \{\partial\mathcal{A}(x^*) + \text{int}\, \mathcal{B}_\delta\}$$

is k-recurrent for any $k > \bar{k} := \bar{\tau}(\delta)/\tau_s$.

Algorithm Explanation

Algorithm (sphere approximation):

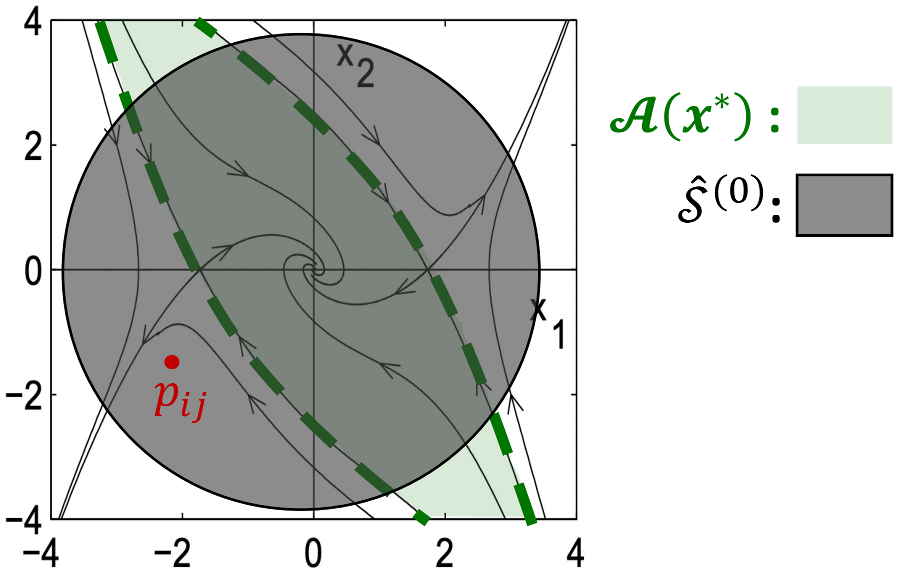
- **Initialize** $\hat{\mathcal{S}}^{(0)}$ ***as*** $\hat{\mathcal{S}}^{(0)} := \{x \mid \|x\|_2 \leq b^{(0)} := c\} \supseteq \mathcal{B}_\delta$



Algorithm Explanation

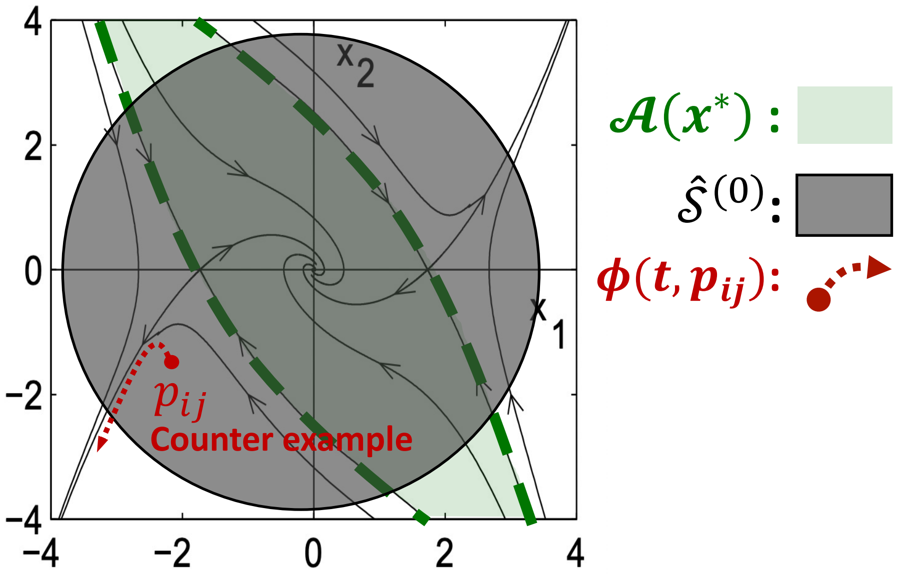
Algorithm (sphere approximation):

- **Initialize** $\hat{\mathcal{S}}^{(0)}$ ***as*** $\hat{\mathcal{S}}^{(0)} := \{x | \|x\|_2 \le b^{(0)} := c\} \supseteq \mathcal{B}_\delta$
- **For** *iteration* $i = 0,1,\ldots$ **do:** (set updates)
    - ***For*** *iteration* $j = 0,1,\ldots$ **do:** (samples)
        - *Generate random sample* $p_{ij} \in \hat{\mathcal{S}}^{(i)}$ *uniformly*



Algorithm Explanation

Algorithm (sphere approximation):



- **Initialize** $\hat{\mathcal{S}}^{(0)}$ ***as*** $\hat{\mathcal{S}}^{(0)} \coloneqq \{x \mid \|x\|_2 \leq b^{(0)} \coloneqq c\} \supseteq \mathcal{B}_\delta$
- **For** *iteration* $i = 0,1,\ldots$ **do:**
  - ***For*** *iteration* $j = 0,1,\ldots$ **do:**
    - *Generate random sample* $p_{ij} \in \hat{\mathcal{S}}^{(i)}$ *uniformly*
    - **If** p_{ij} *is a counter-example w.r.t* $\hat{\mathcal{S}}^{(i)}$ **do:**

We say sample point p_{ij} *is a valid* k*-recurrent point w.r.t current approximation* $\hat{\mathcal{S}}^{(i)}$ *if starting from* $x_0 = p_{ij}$,

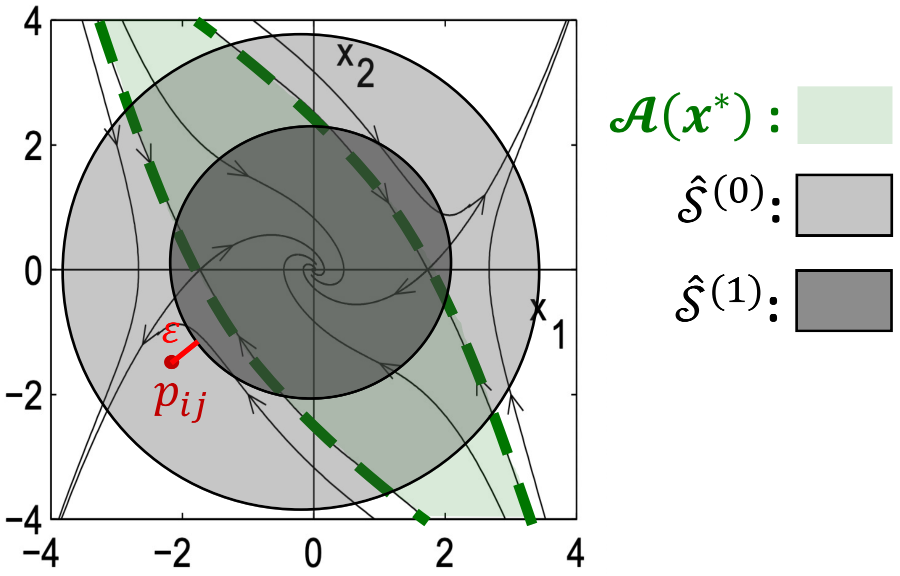
$$\exists\, n \in \{1, \ldots, k\},\ \text{s.t.}\ \ x_n \in \hat{\mathcal{S}}^{(i)}.$$

Otherwise, we say p_{ij} *is a counter-example.*

Algorithm Explanation

Algorithm (sphere approximation):

- **Initialize** $\hat{\mathcal{S}}^{(0)}$ ***as*** $\hat{\mathcal{S}}^{(0)} := \{x \mid \|x\|_2 \leq b^{(0)} := c\} \supseteq \mathcal{B}_\delta$
- **For** *iteration* $i = 0,1,\ldots$ **do:**
  - ***For*** *iteration* $j = 0,1,\ldots$ **do:**
    - *Generate random sample* $p_{ij} \in \hat{\mathcal{S}}^{(i)}$ *uniformly*
    - **If** p_{ij} *is a counter-example w.r.t* $\hat{\mathcal{S}}^{(i)}$ **do:**
      - *Update* $b^{(i)}$ *to* $b^{(i+1)}$, $\hat{\mathcal{S}}^{(i)}$ *to* $\hat{\mathcal{S}}^{(i+1)}$



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$$\exists\, n \in \{1,\ldots,k\}, \text{ s.t. } x_n \in \hat{\mathcal{S}}^{(i)}.$$

Otherwise, we say p_{ij} *is a counter-example.*

If p_{ij} *is a counter-example*, we update:

$$b^{(i+1)} = \|p_{ij}\|_2 - \varepsilon;$$

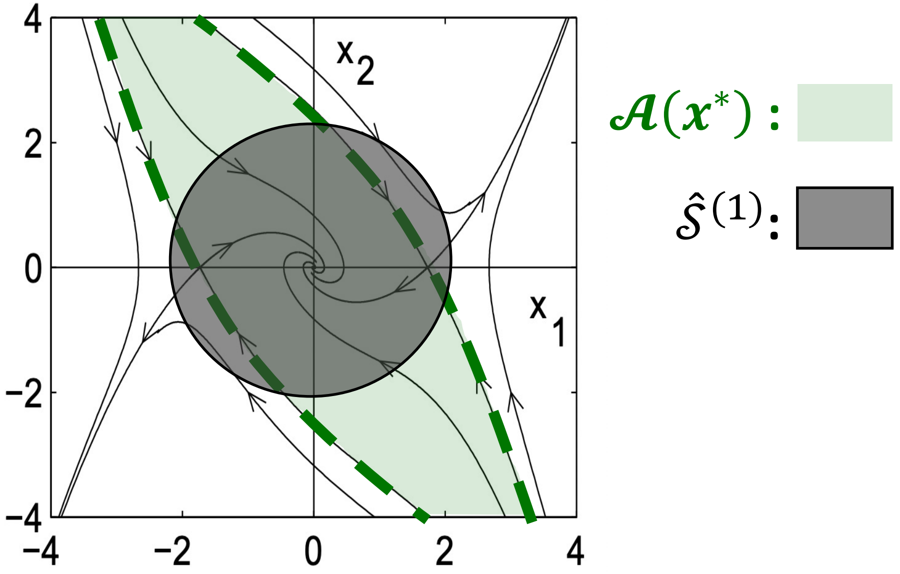
$$\hat{\mathcal{S}}^{(i+1)} = \{x \mid \|x\|_2 \leq b^{(i+1)}\},$$

where $\varepsilon > 0$ *is an algorithm parameter expressing the level of conservativeness in our update.*

Algorithm Explanation

Algorithm (sphere approximation):

- **Initialize** $\hat{\mathcal{S}}^{(0)}$ ***as*** $\hat{\mathcal{S}}^{(0)} := \{x \mid \|x\|_2 \leq b^{(0)} := c\} \supseteq \mathcal{B}_\delta$
- **For** *iteration* $i = 0,1,\ldots$ **do:**
  - ***For*** *iteration* $j = 0,1,\ldots$ **do:**
    - *Generate random sample* $p_{ij} \in \hat{\mathcal{S}}^{(i)}$ *uniformly*
    - **If** p_{ij} *is a counter-example w.r.t* $\hat{\mathcal{S}}^{(i)}$ **do:**
      - *Update* $b^{(i)}$ *to* $b^{(i+1)}$, $\hat{\mathcal{S}}^{(i)}$ *to* $\hat{\mathcal{S}}^{(i+1)}$
      - **Break**
    - **End if**
  - **End for**
- **End for**



We say sample point p_{ij} *is a valid* k*-recurrent point w.r.t current approximation* $\hat{\mathcal{S}}^{(i)}$ *if starting from* $x_0 = p_{ij}$,

$$\exists\, n \in \{1,\ldots,k\}, \text{ s.t. } x_n \in \hat{\mathcal{S}}^{(i)}.$$

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where $\varepsilon > 0$ *is an algorithm parameter expressing the level of conservativeness in our update.*

Algorithm Explanation

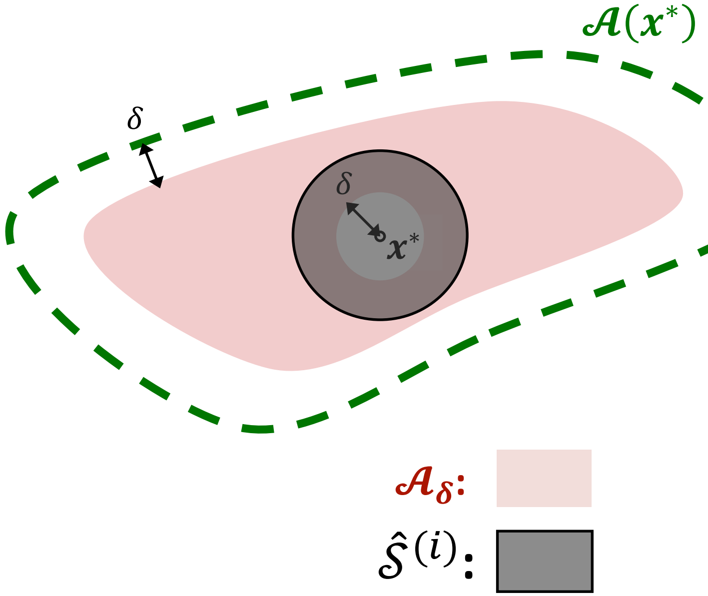
Choice of ε:

*(**Theorem** 5). Under Assumption 1, any compact set* $\mathcal{R}$ satisfying:

$$\mathcal{B}_\delta + x^* \subseteq \mathcal{R} \subseteq \mathcal{A}(0)\backslash\{\partial\mathcal{A}(0) + \text{int}\,\mathcal{B}_\delta\}$$

is k-recurrent for any $k > \bar{k} := \bar{\tau}(\delta)/\tau_s$.

- *Given $k > \bar{k}$, and an* arbitrary approximation $\hat{\mathcal{S}}^{(i)}$ satisfying $\mathcal{B}_\delta \subseteq \hat{\mathcal{S}}^{(i)} \subseteq \mathcal{A}(0)\backslash\{\partial\mathcal{A}(0) + \text{int}\,\mathcal{B}_\delta\}$, *then any sample $p_{ij} \in \hat{\mathcal{S}}^{(i)}$ will be classified as a valid k-recurrent point.*
- *As a result, the algorithm will stop updating at this point since we cannot find further counter-examples within $\hat{\mathcal{S}}^{(i)}$.*
- This means that, if it is possible for $\hat{\mathcal{S}}^{(\mathrm{i})}$ to become a subset of $\mathcal{A}(0)\backslash\{\partial\mathcal{A}(0) + \text{int}\,\mathcal{B}_\delta\}$, without violating the condition $\mathcal{B}_\delta \subseteq \hat{\mathcal{S}}^{(\mathrm{i})}$, then the algorithm will stop updating and never fail.



Algorithm Explanation

Choice of ε:

*(**Theorem** 5). Under Assumption 1, any compact set* $\mathcal{R}$ satisfying:

$$\mathcal{B}_\delta + x^* \subseteq \mathcal{R} \subseteq \mathcal{A}(0)\backslash\{\partial\mathcal{A}(0) + \text{int } \mathcal{B}_\delta\}$$

is k*-recurrent for any* $k > \bar{k} := \bar{\tau}(\delta)/\tau_s$.

- *Given* $k > \bar{k}$*, and an* arbitrary approximation $\hat{\mathcal{S}}^{(i)}$ satisfying $\mathcal{B}_\delta \subseteq \hat{\mathcal{S}}^{(i)} \subseteq \mathcal{A}(0)\backslash\{\partial\mathcal{A}(0) + \text{int } \mathcal{B}_\delta\}$, *then any sample* $p_{ij} \in \hat{\mathcal{S}}^{(i)}$ *will be classified as a valid* k*-recurrent point.*
- *As a result, the algorithm will stop updating at this point since we cannot find further counter-examples within* $\hat{\mathcal{S}}^{(i)}$.
- This means that, if it is possible for $\hat{\mathcal{S}}^{(\mathrm{i})}$ to become a subset of $\mathcal{A}(0)\backslash\{\partial\mathcal{A}(0) + \text{int } \mathcal{B}_\delta\}$, without violating the condition $\mathcal{B}_\delta \subseteq \hat{\mathcal{S}}^{(\mathrm{i})}$, then the algorithm will stop updating and never fail.

?

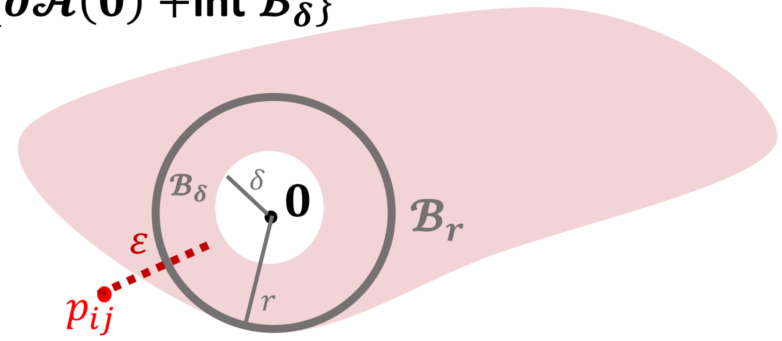
- **It is possible, whenever ε and k are properly chosen.**

Theorem *6. Let the initial approximation satisfy* $\mathcal{B}_\delta \subseteq \hat{\mathcal{S}}^{(i)}$ *and trajectory length* $k > \bar{k}$. *Then, given a counter-example* p_{ij}, *the resulting updated set satisfies* $\mathcal{B}_\delta \subseteq \hat{\mathcal{S}}^{(i+1)}$ *whenever*

$$\varepsilon \leq r - \delta,$$

where r is the smallest distance between the origin (equilibrium) and the boundary $\partial\{\mathcal{A}(0)\backslash\{\partial\mathcal{A}(0) + \text{int } \mathcal{B}_\delta\}\}$.

$\boldsymbol{\mathcal{A}(0)\backslash\{\partial\mathcal{A}(0)} + \textbf{int } \boldsymbol{\mathcal{B}_\delta\}}$



Algorithm Explanation

Choice of k:

- $\bar{k}$ depends in a highly non-trivial way on δ.

- *If* $k < \bar{k}$, *algorithm may reach a* $b^{(i)} < 0$. We declare the search a failure.

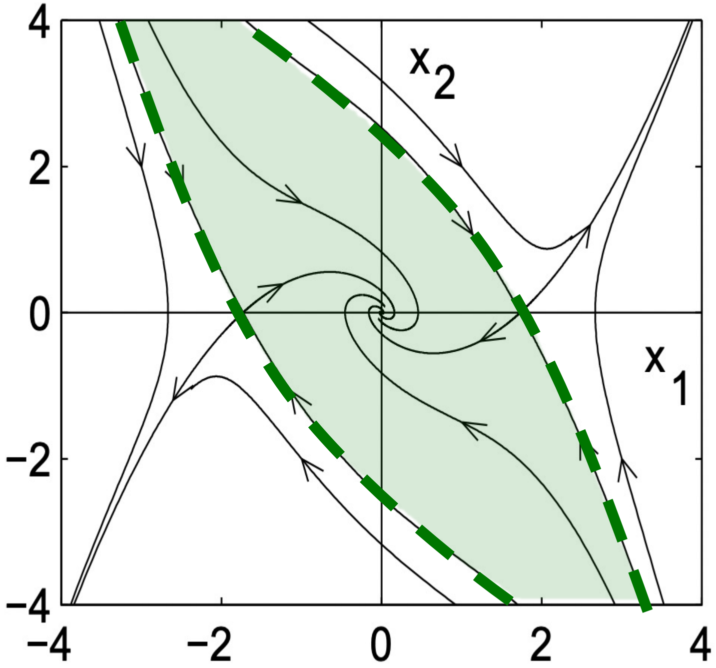
- **We solve this issue by, doubling the size of k, i.e., $k^+ = 2k$, every time we fail.**

With k-doubling after each failure, the total number of counter-examples we encountered is bounded by

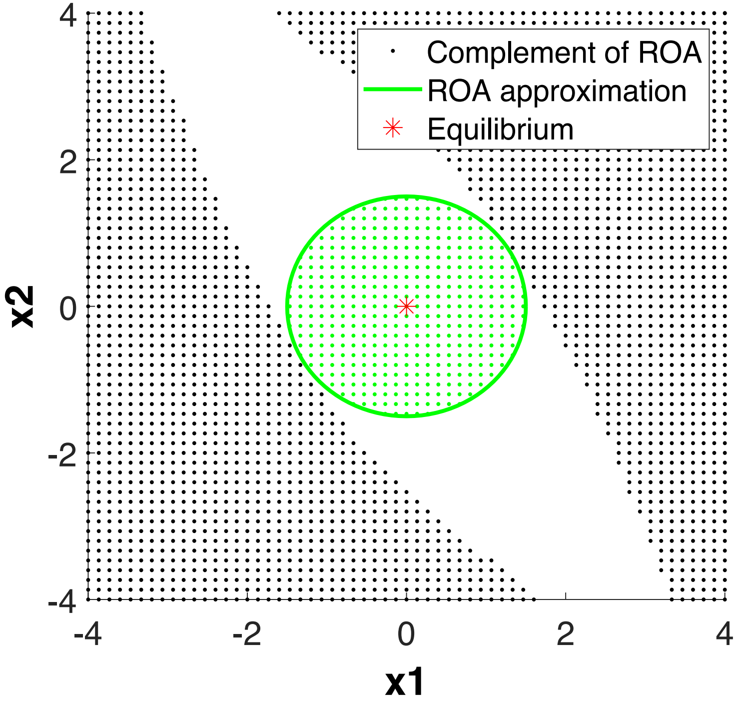
$$\#\text{counter-examples} \leq \frac{c}{\varepsilon} \log_2 \bar{k}$$

Algorithm Result

Result (sphere approximation):



$\mathcal{A}(0)$:

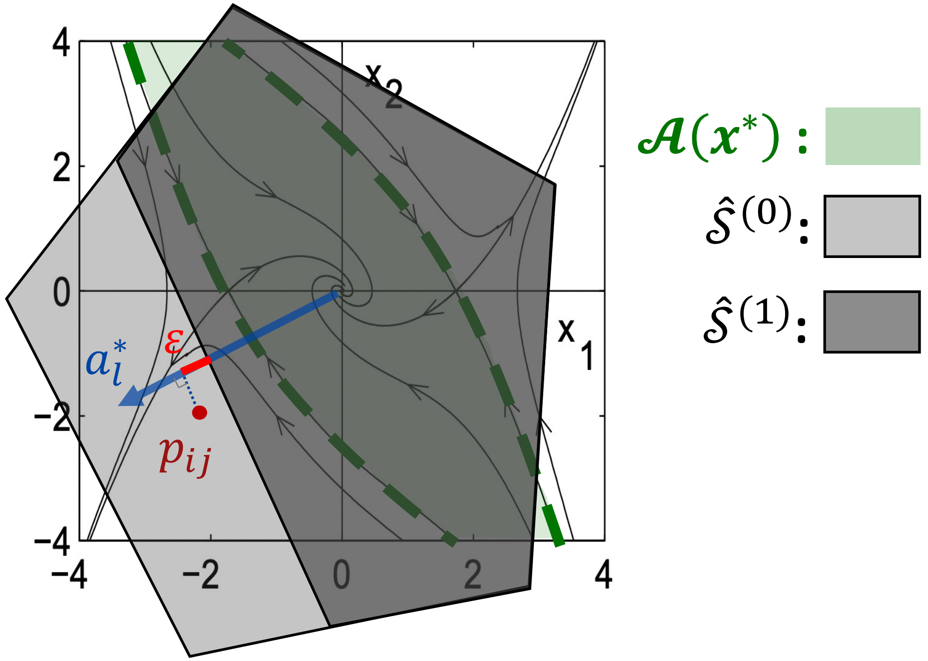


Algorithm Explanation

Algorithm (Polytope approximation):

- **Initialize** $\hat{\mathcal{S}}^{(0)}$ ***as*** $\hat{\mathcal{S}}^{(0)} := \{x | Ax \le b^{(0)} := c\mathbb{I}_n\} \supseteq \mathcal{B}_\delta$

Exploration directions matrix $A := [a_1, \dots, a_n] \subseteq \mathbb{R}^{n \times d}$, where each row vector a_l is a normalized exploration direction indexed by $l \in \{1, \dots, n\}$.



- **For** *iteration* $i = 0,1, \dots$ **do:**
  - ***For*** *iteration* $j = 0,1, \dots$ **do:**
    - *Generate random sample* $p_{ij} \in \hat{\mathcal{S}}^{(i)}$ *uniformly*
    - **If** p_{ij} *is a counter-example w.r.t* $\hat{\mathcal{S}}^{(i)}$ **do:**
      - *Update* $b^{(i)}$ *to* $b^{(i+1)}$, $\hat{\mathcal{S}}^{(i)}$ *to* $\hat{\mathcal{S}}^{(i+1)}$
      - **Break**
    - **End if**
  - **End for**
- **End for**

If p_{ij} *is a counter-example*, we update:

$$b^{(i+1)} = \begin{cases} b_{l^*}^{(i+1)} = a_{l^*} p_{ij} - \varepsilon \\ b_l^{(i+1)} = b_l^{(i)}; \end{cases}$$

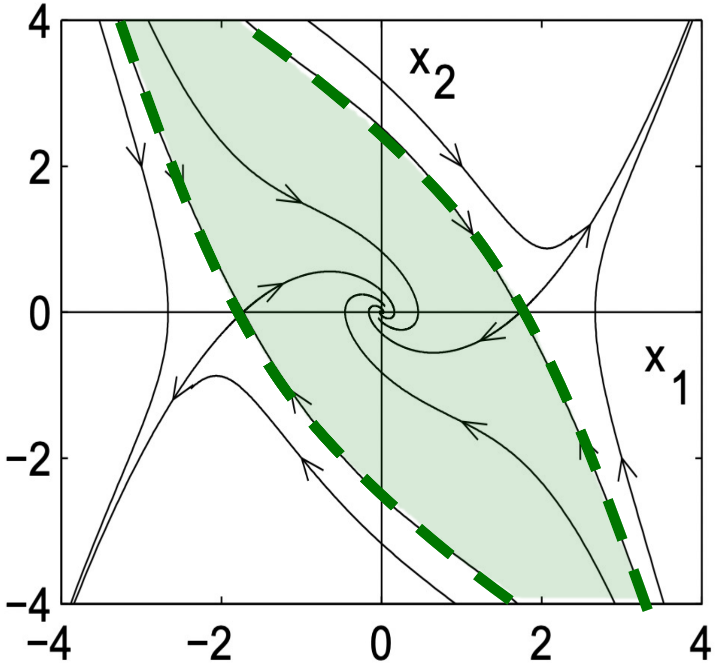
$$\hat{\mathcal{S}}^{(i+1)} = \{x | Ax \le b^{(i+1)}\},$$

where $\varepsilon > 0$ *is fixed and* $l^* = \mathrm{argmax}_{l \in \{1,\dots,n\}} \frac{a_l^T p_{ij}}{\|a_l\| \|p_{ij}\|}$,

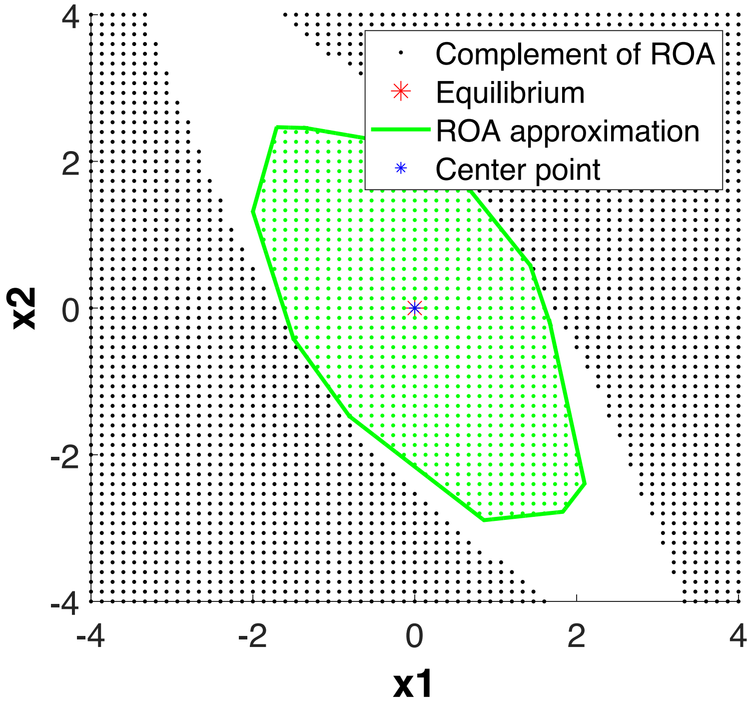
is the index of exploration direction that minimizes the angle between p_{ij} *and* a_l.

Algorithm Result

Result (Polytope approximation):



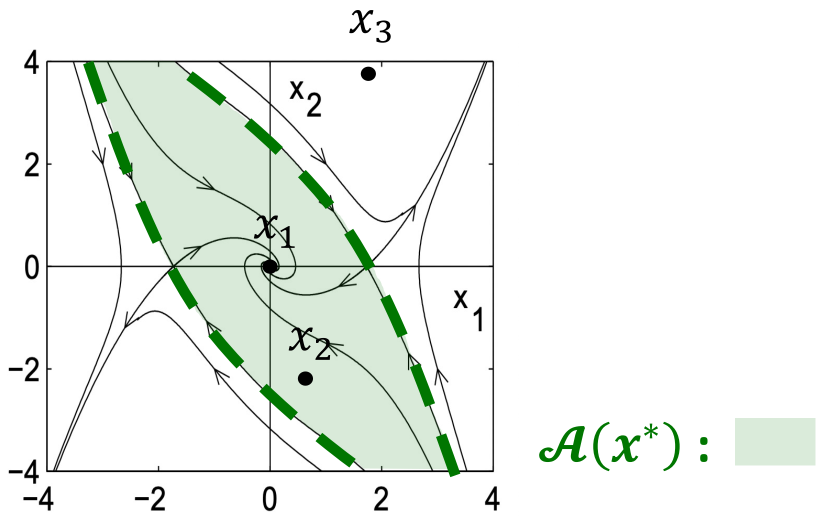
$\mathcal{A}(0)$:



Algorithm Explanation

Multiple center points approximation:

- Consider $h \in \mathbb{N}^+$ center points x_q indexed by $q \in \{1, \dots, h\}$.
  - Let the first center point $x_1 = x^* = 0$
  - Additional center point $x_2, \dots, x_h$ can be designed chosen uniformly.

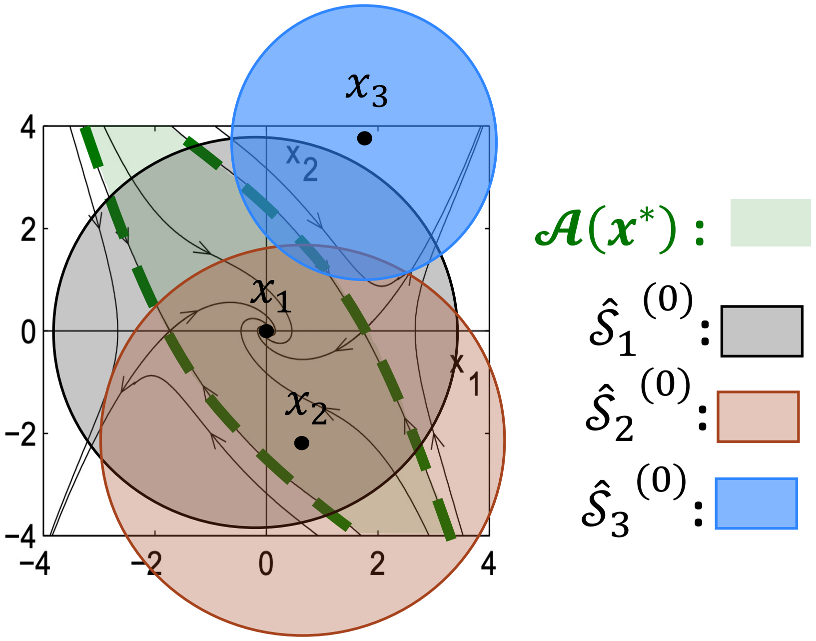


Algorithm Explanation

Multiple center points approximation:

- Consider $\boldsymbol{h} \in \mathbb{N}^{+}$ center points $\boldsymbol{x_q}$ indexed by $\boldsymbol{q} \in \{\mathbf{1}, \dots, \boldsymbol{h}\}$.
  - Let the first center point $\boldsymbol{x_1} = \boldsymbol{x^*} = \mathbf{0}$
  - Additional center point $\boldsymbol{x_2}, \dots, \boldsymbol{x_h}$ can be designed chosen uniformly.

- Respectively defined approximations centered at each $\boldsymbol{x_q}$
  - (Sphere case) $\hat{\mathcal{S}}_q^{(i)} := \{x \mid \|x - x_q\|_2 \leq b_q^{(i)}\}$
  - (Polytope case) $\hat{\mathcal{S}}_q^{(i)} := \{x \mid A(x - x_q) \leq b_q^{(i)}\}$



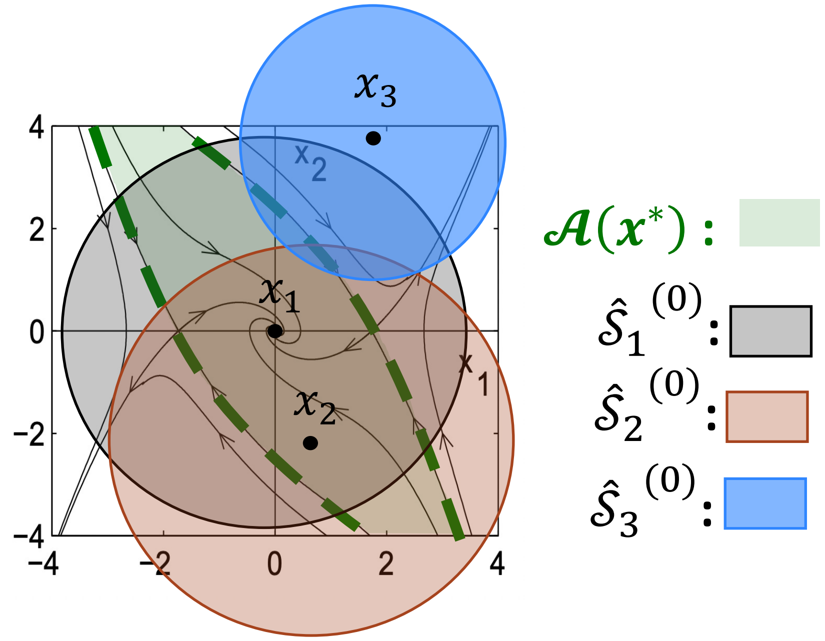
Algorithm Explanation

Multiple center points approximation:

- Consider $\boldsymbol{h} \in \mathbb{N}^+$ center points $\boldsymbol{x_q}$ indexed by $\boldsymbol{q} \in \{\mathbf{1}, \dots, \boldsymbol{h}\}$.
  - Let the first center point $\boldsymbol{x_1} = \boldsymbol{x}^* = \mathbf{0}$
  - Additional center point $\boldsymbol{x_2}, \dots, \boldsymbol{x_h}$ can be designed chosen uniformly.

- Respectively defined approximations centered at each $\boldsymbol{x_q}$
  - (Sphere case) $\hat{\mathcal{S}}_q^{(i)} \coloneqq \{x \mid \|x - x_q\|_2 \leq b_q^{(i)}\}$
  - (Polytope case) $\hat{\mathcal{S}}_q^{(i)} \coloneqq \{x \mid A(x - x_q) \leq b_q^{(i)}\}$

- Multiple centers approximation $\hat{\mathcal{S}}_{\text{multi}}^{(i)} := \cup_{q=1}^{h} \hat{\mathcal{S}}_q^{(i)}$



Algorithm Explanation

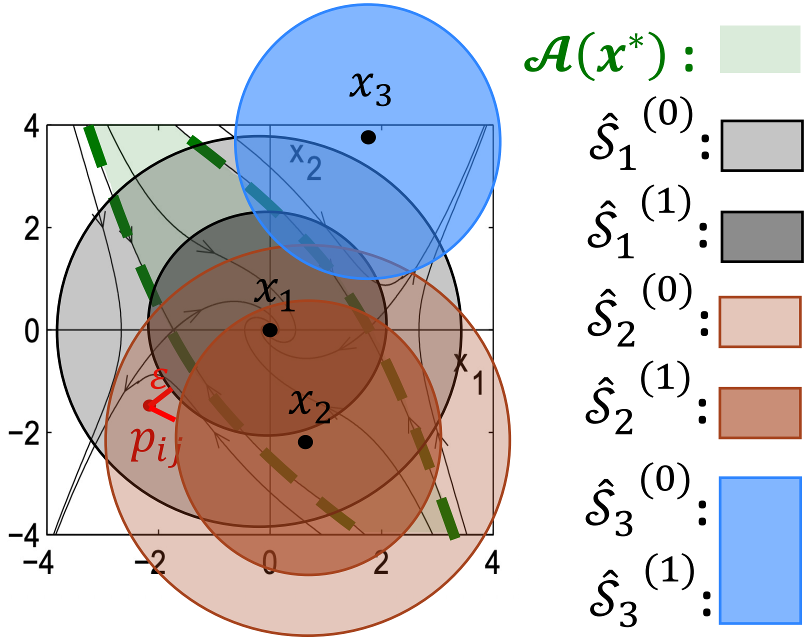
Multiple center points approximation:

- Consider $h \in \mathbb{N}^+$ center points x_q indexed by $q \in \{1, \dots, h\}$.
  - Let the first center point $x_1 = x^* = 0$
  - Additional center point $x_2, \dots, x_h$ can be designed chosen uniformly.

- Respectively defined approximations centered at each x_q
  - (Sphere case) $\hat{S}_q^{(i)} := \{x \mid \|x - x_q\|_2 \leq b_q^{(i)}\}$
  - (Polytope case) $\hat{S}_q^{(i)} := \{x \mid A(x - x_q) \leq b_q^{(i)}\}$

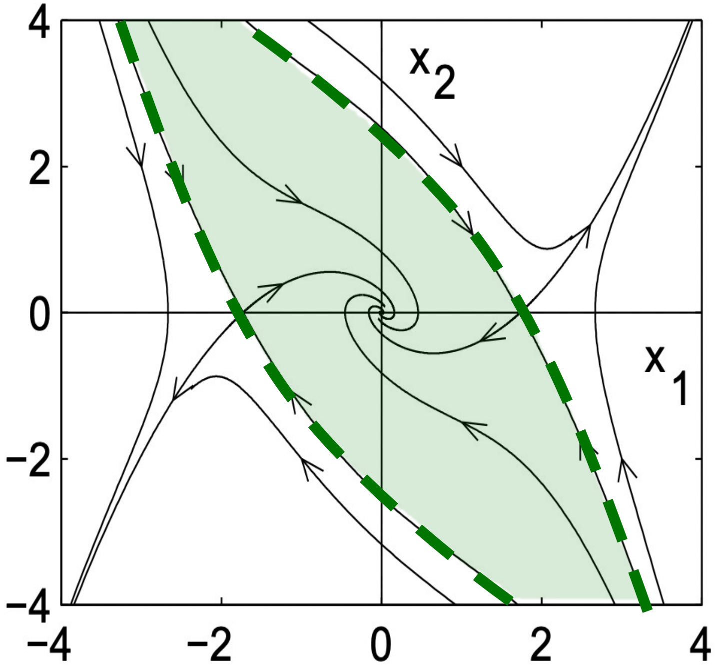
- Multiple centers approximation $\hat{\mathcal{S}}_{\text{multi}}^{(i)} := \cup_{q=1}^{h} \hat{S}_q^{(i)}$

- If p_{ij} is a counter-example w.r.t $\hat{\mathcal{S}}_{\text{multi}}^{(i)}$
  - We shrink every $\hat{S}_q^{(i)}$ satisfying $p_{ij} \in \hat{S}_q^{(i)}$
  - For the rest approximations, we simply let $\hat{S}_q^{(i+1)} = \hat{S}_q^{(i)}$

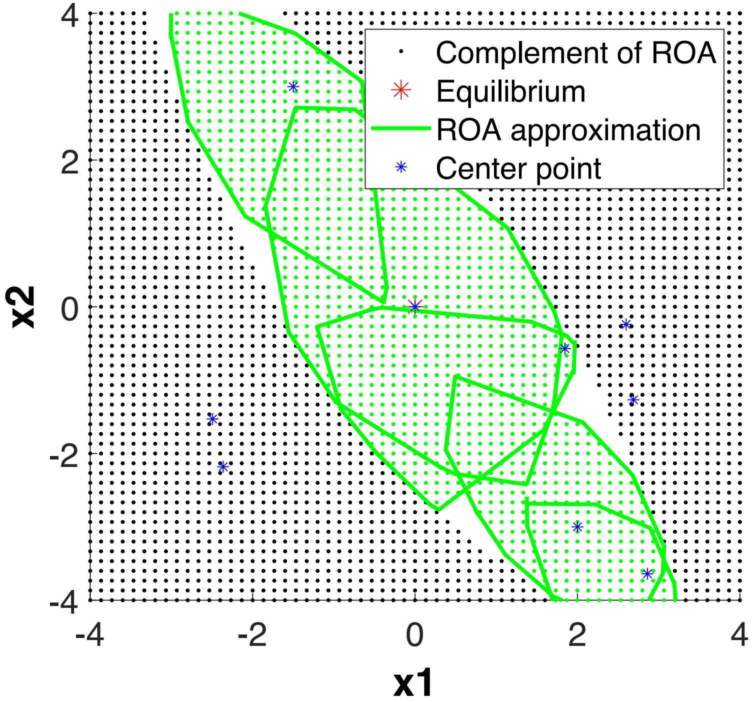


Algorithm Result

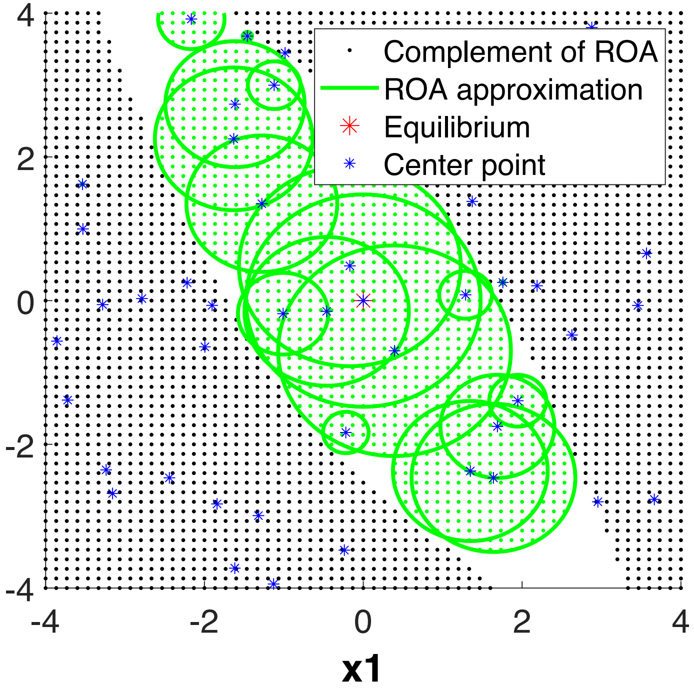
Result (Multiple centers approximation):



$\mathcal{A}(0)$:



(10 polytope approximations)



(50 sphere approximations)

Conclusions

- We propose the use of a more flexible notion of invariance known as recurrence.
- We provide necessary and sufficient conditions for a recurrent set to be an inner-approximation of the ROA.
- Our algorithms are sequential, and only incur on a limited number of counter-examples.
- Future work includes
  - Sample complexity bounds
  - Smart choice of multi-points
  - Control recurrent sets