

# Safety-Critical Control via Recurrent Tracking Functions

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**Abstract**—This paper addresses the challenge of synthesizing safety-critical controllers for high-order nonlinear systems, where constructing valid Control Barrier Functions (CBFs) remains computationally intractable. Leveraging layered control, we design CBFs in reduced-order models (RoMs) while regulating full-order models’ (FoMs) dynamics at the same time. Traditional Lyapunov tracking functions are required to decrease monotonically, and systematic synthesis methods for such functions exist only for fully-actuated systems. To overcome this limitation, we introduce Recurrent Tracking Functions (RTFs), which replace the monotonic decay requirement with a weaker finite-time recurrence condition. This relaxation permits transient deviations of tracking errors while ensuring safety. By integrating CBFs for RoMs with RTFs, we construct recurrent CBFs (RCBFs) whose zero-superlevel set is control  $\tau$ -recurrent, and guarantee safety for all initial states in such a set when RTFs are satisfied. We establish theoretical safety guarantees and validate the approach through a proof-of-concept numerical experiment, demonstrating RTFs’ effectiveness and the safety of FoMs.

## I. INTRODUCTION

Safety-Critical Control in autonomous systems requires controllers that can guarantee constraint satisfaction while achieving performance objectives. Control Barrier Functions (CBFs) provide a principled framework for certifying safety by rendering desired safe sets forward invariant through appropriate control actions. However, the practical deployment of CBF-based methods hinges on the ability to construct valid barrier functions for the system at hand—a challenge that grows increasingly difficult as system complexity and dimensionality increase [1].

Existing CBF synthesis methods typically leverage certain relative degree properties [2], [3], learning-based approaches [4], [5], sum-of-squares programs for polynomial dynamics [6], [7], or reachability-based constructions from Hamilton–Jacobi value functions [8]. These approaches often rely on assumptions that are difficult to satisfy in practice or are hampered by the curse of dimensionality, which limits their applicability to complex systems. As a result, scalable CBF synthesis for systems with higher-order dynamics, such as aerial vehicles, legged robots, and complex power grids, remains an important open problem [9].

A promising alternative is layered control: a reduced-order model (RoM) captures safety-critical states (e.g., collision-relevant kinematics or center-of-mass motion) while a low-level controller regulates the full-order model (FoM) [10],

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[11]. Designing CBFs in the lower-dimensional RoM reduces computation, and safety transfers to the FoM when tracking errors between the true signals and the safe ones admit exponential convergence certificates [12]. However, beyond fully actuated systems, there is no systematic procedure for constructing such tracking functions to certify the Lyapunov condition for exponential convergence, which hinders practice. Robust RoM-based CBFs with predictive error margins [1] mitigate RoM–FoM mismatch, but the conservatism term  $\delta$  discards useful system information.

In this paper, we address these issues by relaxing the Lyapunov condition to a recurrent one [13]–[16]: instead of enforcing monotone decay of the tracking function, we only require the decrease condition to recurrently hold on time instances that are separated by at most a uniform bound  $\tau$ .

Concretely, we introduce the concept of **Recurrent Tracking Functions (RTFs)**, for which any norm is a valid choice, thereby providing a flexible RoM–FoM linkage for general nonlinear systems. Then augmenting a RoM CBF with an RTF yields a recurrent CBF  $h_V$  [16], whose zero-superlevel set  $S_V$  is control  $\tau$ -recurrent: trajectories may temporarily leave  $S_V$  for at most  $\tau$  time units, but never enter the unsafe region. Our main contributions are as follows:

- 1) **RTF framework:** We formulate RTFs as a relaxation of standard Lyapunov decrease conditions. For Lipschitz continuous systems with exponentially stable tracking error, any norm of the tracking error is a valid RTF [17].
- 2) **Safety analysis:** We prove that if the initial state lies in  $S_V$ , then any controller satisfying the RTF condition guarantees safety for all time.
- 3) **Numerical validation:** We demonstrate the effectiveness of the proposed approach on the FoM through numerical experiments.

**Organization.** Section II reviews layered control, CBFs, and recurrence-based analysis. Section III presents the proposed RTF-based framework and its associated safety analysis. Section IV presents numerical results. Section V concludes the paper and outlines future directions.

## II. PRELIMINARIES AND PROBLEM FORMULATION

### A. Layered Control System

We begin this paper by recalling the setting of a layered-control system, consisting of a **full-order model (FoM)** and a **reduced-order model (RoM)**, which interacts with the FoM through the projection of the full state onto the RoM [10].

**Definition 1 (Full-Order Model).** *The FoM represents the physical system, typically high-dimensional, nonlinear, and*

only partially known. Its dynamics are assumed to evolve as

$$\dot{x} = F(x, u), \quad (1)$$

where  $x \in \mathcal{X} \subseteq \mathbb{R}^N$  is the state and  $u \in U \subseteq \mathbb{R}^M$  is the control input.

Given an interval  $\mathcal{I} \subseteq \mathbb{R}_{\geq 0}$ , let  $\mathcal{U}^{\mathcal{I}} := \{u : \mathcal{I} \rightarrow U \mid u \text{ measurable}\}$ . We use  $u$  for both instantaneous inputs  $u \in U$  and signals  $u \in \mathcal{U}^{\mathcal{I}}$ , disambiguated by context, and  $\mathcal{U} := \mathcal{U}^{\mathbb{R}_{\geq 0}}$ . Given initial condition  $x_0 \in \mathcal{X}$  and  $u \in \mathcal{U}^{(0, a]}$ , let  $\phi(t, x_0, u)$  be the trajectory of (1) at time  $t \in (0, a]$ .

**Assumption 1** (Forward Completeness). *The control system (1) is forward complete: for every initial condition  $x_0 \in \mathcal{X} \subseteq \mathbb{R}^N$  and every admissible input signal  $u \in \mathcal{U}$ , the solution  $\phi(t, x_0, u)$  of (1) exists for all  $t \geq 0$ .*

**Assumption 2** (Uniform Local Lipschitz Continuity). *The vector field  $F : \mathcal{X} \times U \rightarrow \mathbb{R}^N$  in (1) is continuous and locally Lipschitz in  $x$ , uniformly w.r.t.  $u$ , i.e., for every compact  $S \subseteq \mathcal{X}$ , there exists  $L_{F_S} \geq 0$  such that*

$$\|F(y, u) - F(x, u)\| \leq L_{F_S} \|y - x\|, \forall x, y \in S, \forall u \in U.$$

**Definition 2** (Reduced-order Model). *The RoM captures safety-relevant, lower-dimensional dynamics:*

$$\dot{z} = f(z, v), \quad (2)$$

with state  $z \in \mathcal{Z} \subseteq \mathbb{R}^n$  ( $n < N$ ), and input  $v \in V \subseteq \mathbb{R}^m$  ( $m < M$ ), where  $f$  is also locally Lipschitz continuous with constant  $L_{R_S}$  for every compact set  $S \subseteq \mathcal{Z}$ .

And a closed ball is defined as

**Definition 3** (Closed Ball). *For any  $y \in \mathbb{R}^d$  and  $r > 0$ , define the closed ball centered at  $y$  with radius  $r$  as*

$$\mathcal{B}_r(y) := \{w \in \mathbb{R}^d \mid \|w - y\| \leq r\}. \quad (3)$$

**RoM-FoM Interconnection.** The RoM and FoM are related through a projection map  $\Pi : \mathcal{X} \rightarrow \mathcal{Z}$  and a tracking interface  $K : \mathcal{X} \times V \rightarrow U$ . Let  $k : \mathcal{Z} \rightarrow V$  denote the feedback law of the RoM. Given a full-order state  $x \in \mathcal{X}$ , the projected state  $z = \Pi(x)$  is used to generate the reference input  $v = k(z)$ , which is then passed to the FoM controller  $u = K(x, v)$ . This induces the closed-loop FoM

$$\dot{x} = F_{\text{cl}}(x) := F(x, K(x, k(\Pi(x))))). \quad (4)$$

We denote by  $x_{\text{cl}}(t)$  the corresponding closed-loop state trajectory with initial condition  $x_0$ , i.e.,  $x_{\text{cl}}(0) = x_0$ .

### B. Problem Statement

To formalize the safety objective, let  $h : \mathcal{Z} \rightarrow \mathbb{R}$  be a smooth function that defines the safe set of the reduced-order model (RoM).

**Definition 4** (Safe State of the RoM). *A state  $z \in \mathcal{Z} \subseteq \mathbb{R}^n$  is safe if  $z \in \mathcal{S}_{\text{RoM}}$ , where*

$$\mathcal{S}_{\text{RoM}} := \{z \in \mathcal{Z} : h(z) \geq 0\} \quad (5)$$

is the RoM's safe set with boundary  $\partial \mathcal{S}_{\text{RoM}} = h^{-1}(0)$ .

**Definition 5** (Safe State of the FoM). *A state  $x \in \mathcal{X} \subseteq \mathbb{R}^N$  is safe if  $x \in \mathcal{S}_{\text{FoM}}$ , where*

$$\mathcal{S}_{\text{FoM}} := \{x \in \mathcal{X} : h(\Pi(x)) \geq 0\} \quad (6)$$

is the FoM's safe set with boundary  $\partial \mathcal{S}_{\text{FoM}} = \Pi^{-1}(h^{-1}(0))$ .

The objective is to guarantee the safety of the FoM by synthesizing a controller based on the safe reference generated by the RoM.

**Problem.** Given the RoM dynamics (2), the closed-loop FoM (4), and the safe set  $\mathcal{S}_{\text{FoM}}$ , design a RoM feedback law  $v = k(\Pi(x))$  and a compact set  $S \subseteq \mathcal{S}_{\text{FoM}}$  such that  $x \in S \Rightarrow x_{\text{cl}}(t) \in \mathcal{S}_{\text{FoM}}, \forall t \geq 0$ .

### C. Control Barrier Functions for the RoM

Control barrier functions (CBFs) provide a standard tool for certifying safety of the RoM (2). In particular, if a smooth function  $h$  satisfies a suitable differential inequality involving an extended class- $\mathcal{K}$  function, then its zero-superlevel set is forward invariant and therefore safe. We first recall the notion of an extended class- $\mathcal{K}$  function.

**Definition 6** (Extended Class- $\mathcal{K}$  Function). *A function  $\kappa : \mathbb{R} \rightarrow \mathbb{R}$  is an extended class  $\mathcal{K}$  function if it is continuous, strictly increasing, and satisfies  $\kappa(0) = 0$ .*

With this notion in place, we can formalize the CBF condition used throughout the paper.

**Definition 7** (Control Barrier Function [18]). *A continuously differentiable function  $h : \mathcal{Z} \rightarrow \mathbb{R}$  is a CBF for (2) if there exists an extended class- $\mathcal{K}$  function  $\kappa$  such that*

$$\max_{v \in V} \left( L_f h(z, v) + \kappa(h(z)) \right) \geq 0, \quad (7)$$

for all  $z \in \mathcal{Z}$ , where  $L_f h(z, v) := \frac{\partial h}{\partial z}(z)^\top f(z, v)$  denotes the Lie derivative of  $h$  along the RoM dynamics.

**Theorem 1** ([18]). *As a direct consequence of Definition 7, any Lipschitz-continuous controller  $k(z)$  that satisfies*

$$k(z) \in \{v \in V \mid L_f h(z) + \kappa(h(z)) \geq 0\} \quad (8)$$

renders the set  $h_{\geq 0} := \{z \mid h(z) \geq 0\}$  forward invariant. In particular,  $h_{\geq 0}$  is control invariant.

### D. Recurrent Lyapunov and Control Barrier Functions

Constructing a Lyapunov function for a general high-order system is often intractable. Reference [17] relaxes this requirement via Recurrent Lyapunov Functions (RLFs), which require the Lyapunov decrease condition to hold only recurrently rather than continuously. RLFs still guarantee exponential stability, and a converse theorem shows that any norm of the state satisfies the RLF conditions for exponentially stable systems. Since RLF conditions are defined for autonomous systems, we express the results in this section in terms of  $x_{\text{cl}}(t)$ , which denotes solutions to (4).

**Definition 8** (Exponential Stability). *Given  $S \subseteq \mathbb{R}^N$ , an equilibrium point  $x^*$  of (4) is exponentially stable on  $S$  if there exist constants  $M, \lambda > 0$  such that, for all  $x_0 \in S$ ,*

$$\|x_{\text{cl}}(t) - x^*\| \leq M e^{-\lambda t} \|x_0 - x^*\|, \quad \forall t \geq 0. \quad (9)$$

**Definition 9** (Reachable Tube). For system (4),  $\tau > 0$ , and  $S \subset \mathbb{R}^N$ , the  $\tau$ -reachable tube of  $S$  is

$$\mathcal{R}^\tau(S) = \bigcup_{\substack{x_0 \in S \\ t \in [0, \tau]}} \{x_{\text{cl}}(t)\}. \quad (10)$$

**Definition 10** (Containment Times). For  $S \subseteq \mathbb{R}^N$ ,  $x_0 \in \mathcal{X}$ , the containment time for (4) is defined as  $T_S(x_0) := \{t > 0 \mid x_{\text{cl}}(t) \in S\}$ . For constants  $a, b$ ,  $T_S(x_0; a, b) := T_S(x_0) \cap (a, a + b]$ , and  $T_S(x_0; b) := T_S(x_0; 0, b)$ .

**Definition 11** (Recurrent Lyapunov Function [17]). Given an equilibrium  $x^* \in \mathcal{X}$  of (4) and a compact  $S \subseteq \mathcal{X}$  with  $x^* \in \text{int}(S)$ , a continuous  $V : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$  is a *Recurrent Lyapunov Function (RLF)* over  $S$  if:

1) (**Positive definiteness**)  $\exists a_1, a_2 > 0$  such that

$$a_1 \|x - x^*\| \leq V(x) \leq a_2 \|x - x^*\|, \forall x \in S. \quad (11)$$

2) (**Exponential  $\tau$ -recurrence**)  $\exists \alpha, \tau > 0$  such that

$$\min_{t \in T_S(x_0; \tau)} e^{\alpha t} V(x_{\text{cl}}(t)) \leq V(x_0), \forall x_0 \in S. \quad (12)$$

Condition (12) enforces exponential convergence via recurrent returns, relaxing monotone decrease. Analogously, [16] defines a **Recurrent Control Barrier Function (RCBF)**  $h^r : \mathbb{R}^N \rightarrow \mathbb{R}$ , which replaces invariance with recurrence: trajectories may leave  $h^r \geq 0$  but must re-enter within time  $\tau$ , infinitely often; the resulting set is control  $\tau$ -recurrent.

**Remark 1.** The following requires extending the definition of containment times, i.e., Definition 10, for the FoM system (1). For  $S \subseteq \mathbb{R}^N$ ,  $x_0 \in \mathbb{R}^N$ , we use  $T_S(x_0, u) := \{t > 0 \mid \phi(t, x_0, u) \in S\}$  denoted as the containment time for (1) and  $T_S(x_0, u; \tau) := T_S(x_0, u) \cap (0, \tau]$ .

**Definition 12** (Recurrent Control Barrier Function). For (1) and a compact  $S \subseteq \mathbb{R}^N$ , a continuous  $h^r : \mathbb{R}^N \rightarrow \mathbb{R}$  is an *RCBF* over  $S$  if for every  $x \in S$  there exists  $u \in \mathcal{U}^{(0, \tau]}$  such that:

$$\max_{t \in T_S(x, u; \tau)} e^{\gamma(h^r(\phi(t, x, u)))t} h^r(\phi(t, x, u)) \geq h^r(x), \quad (13)$$

where the function  $\gamma : \mathbb{R} \rightarrow \mathbb{R}_{>0}$  is given.

Unlike invariance-based safety, which requires the invariant set to be disjoint from unsafe regions, RCBFs certify safety if the recurrent set avoids the backward  $\tau$ -reachable tube of the unsafe set [19].

A key advantage of the recurrence framework is that both RLFs and RCBFs can be constructed using simple norm and signed distance functions, respectively [14], [17], significantly simplifying synthesis compared to classical Lyapunov and barrier functions. This motivates replacing the exponential tracking certificates in [11] with (possibly norm-based) Recurrent Tracking Certificates, yielding a systematic synthesis method aligned with (12)–(13).

### III. LAYERED SAFETY-CRITICAL CONTROL VIA RECURRENCE

In this section, we extend the recurrence framework to provide safety guarantees for layered control architectures. A standard approach in this setting is to combine RoM safety with tracking-based interface assumptions to certify FoM safety [1].

#### A. Ideal Tracking

Firstly, we consider trajectories  $x_{\text{cl}}(t)$  of the closed loop system (4), and  $z_{\text{cl}}(t)$  is its projection, i.e.,  $z_{\text{cl}}(t) = \Pi(x_{\text{cl}}(t))$ . Then consider a CBF  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  for the RoM (2) with linear extended class- $\mathcal{K}$  function  $\kappa(h) = \alpha h$  for some  $\alpha > 0$ , and the safe velocity reference  $\dot{z}_s(t)$  satisfies

$$\nabla h(z_{\text{cl}}(\cdot))^\top \dot{z}_s(\cdot) \geq -\alpha h(z_{\text{cl}}(\cdot)). \quad (14)$$

Moreover, along the projected closed-loop trajectory,

$$\begin{aligned} \dot{h}(z_{\text{cl}}(\cdot)) &= \nabla h(z_{\text{cl}}(\cdot))^\top \dot{z}_{\text{cl}}(\cdot) \\ &= \nabla h(z_{\text{cl}}(\cdot))^\top (\dot{z}_s(\cdot) + \dot{e}(\cdot)), \end{aligned} \quad (15)$$

where the tracking error between the true velocity  $\dot{z}_{\text{cl}}$  and  $\dot{z}_s$  is  $\dot{e}(\cdot) := \dot{z}_{\text{cl}}(\cdot) - \dot{z}_s(\cdot)$ , with initial condition  $\dot{e}_0 := \dot{e}(0)$ .

We impose the following regularity condition on the RoM safety function to bound its sensitivity with respect to the projected FoM state.

**Assumption 3** (Boundedness). There exists  $C_h > 0$  s.t.

$$\|\nabla h(\Pi(x))\| \leq C_h, \quad \forall x \in \mathcal{S}_{\text{FoM}}.$$

To capture the ideal tracking regime, we impose the following exponential tracking condition on the velocity error.

**Assumption 4** (Exponentially Stable Tracking). The velocity tracking error is assumed to decay exponentially, i.e., there exist positive constants  $M$  and  $\beta$  such that

$$\|\dot{e}(t)\| \leq M \|\dot{e}_0\| e^{-\beta t} \quad (16)$$

for all  $t \geq 0$ .

As shown later, this assumption provides a sufficient condition for constructing a valid recurrent tracking certificates.

#### B. Recurrent Tracking Function

We then formalize the notion of **Recurrent Tracking Functions (RTFs)** for system (4) by defining a condition akin to (12) for tracking errors. For a set  $S \subseteq \mathcal{Z} \times \mathbb{R}^n$  and initial condition  $x_0 \in \mathcal{X}$ , where  $(z_0, e_0) \in S$ , define  $T_S^e(x_0) := \{t > 0 \mid (z_{\text{cl}}(t), \dot{e}(t)) \in S\}$ , and  $T_S^e(x_0; \tau) := T_S^e(x_0) \cap (0, \tau]$ .

**Definition 13** (Recurrent Tracking Function). Consider the system (4) and a compact set  $S \subseteq \mathcal{Z} \times \mathbb{R}^n$  with  $0 \in \text{int}(S_e)$ , where  $S_e := \{\dot{e} \in \mathbb{R}^n : \exists z \in \mathbb{R}^n, (z, \dot{e}) \in S\}$ . A continuous function  $V : \mathcal{Z} \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is a **Recurrent Tracking Function (RTF)** over  $S$  if:

1) **Positive definiteness with linear error bounds:** There exist  $a_1, a_2 > 0$  such that

$$a_1 \|\dot{e}\| \leq V(z, \dot{e}) \leq a_2 \|\dot{e}\|, \forall (z, \dot{e}) \in S. \quad (17)$$

2)  **$\beta$ -exponential  $\tau$ -recurrence:** There exist  $\tau, \beta > 0$  such that for every  $x \in \mathcal{X}$  s.t.  $(z_0, \dot{e}_0) \in S$ , the corresponding tracking error  $(z_{cl}(\cdot), \dot{e}(\cdot))$  satisfies

$$\min_{t \in T_S^c(x_0; \tau)} e^{\beta t} V(z_{cl}(t), \dot{e}(t)) \leq V(z_0, \dot{e}_0). \quad (18)$$

In contrast to classical Lyapunov tracking functions, which must decrease monotonically along trajectories, RTF only require exponential  $\tau$ -recurrence. The exponential decay of the tracking error implies that  $\|\dot{e}\|$  serve as a valid RTF. This avoids the need to explicitly construct a monotone tracking Lyapunov function, a task that is generally difficult beyond simple systems.

**Theorem 2** (Norm is Valid RTF). Consider system (4), suppose Assumption 4 holds, given a compact set  $S \subseteq \mathcal{Z} \times \mathbb{R}^n$  with  $0 \in \text{int}(S_{\dot{e}})$ . then for any  $0 < \beta' < \beta$  and  $\tau \geq \frac{1}{\beta - \beta'} \ln(M \frac{b_2}{b_1})$ , where  $\mathcal{B}_{b_1}(0) \subseteq S_{\dot{e}} \subseteq \mathcal{B}_{b_2}(0)$ , the function  $V(z, \dot{e}) := \|\dot{e}\|$  is a valid RTF where

$$\min_{t \in T_S^c(x_0; \tau)} e^{\beta' t} V(z_{cl}(t), \dot{e}(t)) \leq V(z_0, \dot{e}_0). \quad (19)$$

*Proof.* The proof follows closely similar results for [17, Theorem 6], and it is omitted due to space constraints.  $\square$

### C. Safety Assessment

In this subsection, we combine a valid RTF for the closed-loop FoM (4) with a CBF for the RoM (2) to characterize the set of initial conditions from which safety of the closed-loop system can be guaranteed.

**Theorem 3** (Recurrent CBF Construction). Consider system (4) and suppose  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is a CBF for the RoM (2) with extended class- $\mathcal{K}$  function  $\kappa(h) = \alpha h$  for some  $\alpha > 0$ . If  $V(z, \dot{e})$  is a valid RTF given recurrent time  $\tau$  over  $S \subseteq \mathcal{Z} \times \mathbb{R}^n$  with convergence rate  $\beta > \alpha$ , then

$$h_V(z, \dot{e}) = -V(z, \dot{e}) + \alpha_e h(z) \quad (20)$$

is an RCBF, where  $\alpha_e = \frac{\alpha^2(\beta - \alpha)}{a_2 C_h M}$ , and its zero-superlevel set

$$S_V := \{(z, \dot{e}) \in S : h_V(z, \dot{e}) \geq 0\} \quad (21)$$

is a control  $\tau$ -recurrent set.

In particular, for any  $x_0 \in \mathcal{S}_{\text{FoM}}$  s.t.  $(\Pi(x_0), \dot{e}_0) \in S_V$ , the induced control signal  $u(\cdot) = K(x(\cdot), k(\Pi(x(\cdot))))$  satisfies the RCBF condition.

*Proof.* See Appendix A  $\square$

**Remark 2** (Conservativeness). Theorem 3 requires that  $(z_0, \dot{e}_0) \in S_V$ , equivalently  $h_V(z_0, \dot{e}_0) = -V(z_0, \dot{e}_0) + \alpha_e h(z_0) \geq 0$ , which implies  $h(z_0) \geq \frac{V(z_0, \dot{e}_0)}{\alpha_e}$ . Since the safe set  $\mathcal{S}_{\text{FoM}}$  is defined only by  $h(z_0) \geq 0$ , this condition is more conservative than merely requiring  $z_0 \in \mathcal{S}_{\text{FoM}}$ . We highlight, however, that a priori for underactuated systems it is not clear that the entire set  $\mathcal{S}_{\text{FoM}}$  is safe.

**Remark 3** (Feasibility). For systems with limited control authority, constructing a classical exponential tracking function may be difficult because monotone error decrease need not hold under admissible inputs. By contrast, our framework only requires a valid RTF and a nonempty induced set  $S_V$ . In particular, if the RoM safe set has nonempty interior and  $V(z, 0) = 0$ , then any  $z$  satisfying  $h(z) > 0$  also satisfies  $(z, 0) \in S_V$ , and thus  $S_V \neq \emptyset$ .

**Theorem 4** (Safety Assessment). Let  $V(z, \dot{e})$  be the RTF over  $S \subseteq \mathcal{Z} \times \mathbb{R}^n$  with rate  $\beta$  and consider an initial state  $x_0 \in \mathcal{S}_{\text{FoM}}$  of the FoM s.t.  $(z_0, \dot{e}_0) \in S_V = \{(z, \dot{e}) \in \mathcal{Z} \times \mathbb{R}^n : h_V(z, \dot{e}) \geq 0\}$ . Then, the solution  $x_{cl}(t)$  of the closed system always remain in safe state set  $\mathcal{S}_{\text{FoM}}$ , i.e.  $x_{cl}(t) \in \mathcal{S}_{\text{FoM}} \forall t \geq 0$ .

*Proof.* See Appendix B  $\square$

**Remark 4** (Recurrence vs. Safety). Theorem 4 reveals that although  $h_V$  is a recurrent CBF and trajectories may temporarily leave the recurrent set  $S_V$ , they never leave the safe set  $\mathcal{S}_{\text{FoM}}$ . The reason is that when  $\beta > \alpha$ , tracking deviations are corrected faster than the system can approach the safety boundary, ensuring  $h(z_{cl}(t)) \geq 0$  for all time despite the recurrent behavior of  $h_V$ . This is illustrated in Fig. 1:  $V$  is recurrent, while  $h$  remains strictly nonnegative.

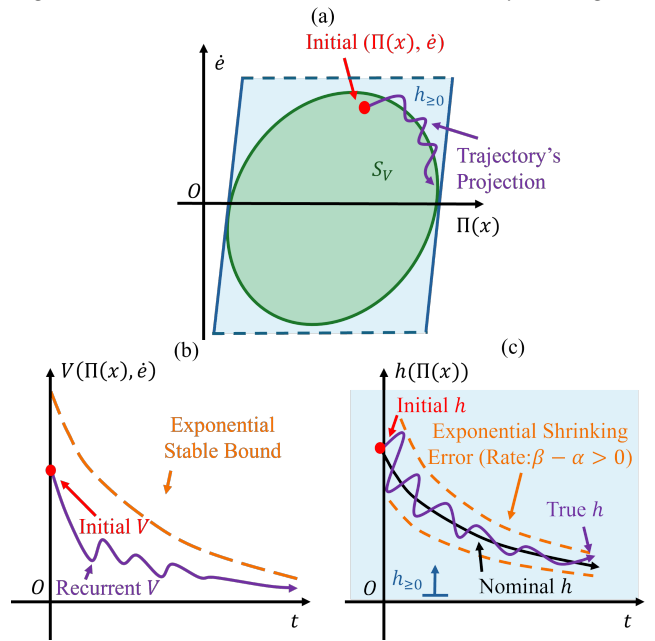


Fig. 1. (a)  $V, h, S_V$  and a safe trajectory's projection when  $n = 1$ . (b) Time evolution of  $V$ . (c) Time evolution of  $h$ .

### D. Effect of Disturbances

In practice, feedback controllers  $K(x, k(\Pi(x)))$  and  $k(\Pi(x))$  often cannot achieve ideal exponential tracking due to model uncertainties, actuation limits, and external disturbances. We model this mismatch via a bounded disturbance  $d$ , under which the tracking error becomes Input-to-State Stable (ISS):

$$\|\dot{e}(t)\| \leq M \|\dot{e}_0\| e^{-\beta t} + \mu(\|d\|_{\infty}) \quad (22)$$

for some class- $\mathcal{K}$  function  $\mu(\cdot)$ . The key insight is that while disturbances prevent the tracking error from vanishing completely, they introduce only a bounded steady-state offset  $\mu(\|d_\infty\|)$ .

To accommodate this, we modify the RTF condition (18) to account for the disturbance-induced offset:

$$\min_{t \in T_s(x; \tau)} e^{\beta t} (V(z_{cl}(t), \dot{e}(t)) - \iota(\|d\|_\infty)) \leq V(z_0, \dot{e}_0) - \iota(\|d\|_\infty), \quad (23)$$

where  $\iota(\|d\|_\infty) = \frac{a_2 e^{\beta \tau} \mu(\|d\|_\infty)}{M}$  [20, Theorem 1] quantifies the impact of disturbances on the RTF.

Besides, considering the tracking disturbance, rather than enforcing safety, we require Input-to-State Safety (ISSf), namely, the forward invariance of an enlarged set  $S_d \supseteq h_{\geq 0}$  defined by

$$S_d = \{z \in \mathcal{Z} : h_d(z) \geq 0\}, \quad (24)$$

$$h_d(z) = h(z) + \gamma(\|d\|_\infty)/\alpha_e, \quad (25)$$

where  $\gamma$  is a class- $\mathcal{K}$  function to be specified. This leads to a robust safety guarantee through an enlarged recurrent set:

$$S_{Vd} = \{(z, \dot{e}) \in \mathcal{Z} \times \mathbb{R}^n : h_{Vd}(z, \dot{e}) \geq 0\}, \quad (26)$$

$$h_{Vd}(z, \dot{e}) = h_V(z, \dot{e}) + \gamma(\|d\|_\infty), \quad (27)$$

where  $\gamma(\|d\|_\infty) = \frac{(2\beta - \alpha)\iota(\|d\|_\infty)}{\alpha}$  provides a safety margin that scales with the disturbance magnitude. The following corollary formalizes this ISSf property.

**Corollary 1 (Input-to-State Safety).** *For system (4) with  $\beta > \alpha$ , if an initial state  $x_0 \in \mathcal{S}_{\text{FoM}}$  satisfies  $(z_0, \dot{e}_0) \in S_{Vd}$  and the ISS tracking condition (23) holds, then the ISSf for all  $t \geq 0$ , where  $\gamma(\|d\|_\infty) = (2\beta - \alpha)\iota(\|d\|_\infty)/\alpha$  and  $\alpha_e = \frac{a_1^2(\beta - \alpha)}{a_2 C_h M}$  is guaranteed.*

*Proof.* The proof follows by combining the arguments in Theorems 3 and 4 with the modified RTF condition (23), and is omitted due to space constraints.  $\square$

#### IV. CASE STUDY

In this section, we use a 2D double-integrator example as an illustrative proof-of-concept to visualize the role of RTFs and the induced recurrent safe set  $S_V$ . Consider the dynamics:  $\ddot{z} = u$ , where  $z \in \mathbb{R}^2$  is the agent's position, the FoM's dimension is 4, the RoM is  $\dot{z} = v$ , and  $u \in \mathbb{R}^2$ . Our goal is to navigate the system from a start position  $z_0$  to a goal  $z_g$  while avoiding obstacles. A simple solution is to realize the desired velocity  $\dot{z}_d = -K_p(z - z_g)$  which is a proportional controller with gain  $K_p \in \mathbb{R}_{>0}$ .

In our setting, we consider a planar projection  $\Pi(\mathcal{X}) \subseteq \mathbb{R}^2$  occupied with  $N$  circular (closed-disk) obstacles. The  $i$ -th obstacle is  $\mathcal{O}_i = \{z \in \mathbb{R}^2 : \|z - o_i\| \leq r_i\}$ ,  $i \leq N$ , whose center is  $o_i$  and radius is  $r_i$ . Then the obstacle's configuration space is the union  $\mathcal{O} = \bigcup_{i=1}^N \mathcal{O}_i$ , and the collision-free configuration space is  $\Pi(\mathcal{X})/\mathcal{O}$ . The CBF we used for the RoM is  $h(z) := \min_{i \in [1, N]} \min_{j \in \mathbb{N}_+} \|z - o_i\| - r_i$ , and its gradient  $\nabla h(z) = \frac{(z - o_i)^\top}{\|z - o_i\|} = n_i^\top$  is a unit vector pointing from the

nearest obstacle  $\mathcal{O}_i$  to the agent. Then the safety velocity  $\dot{z}_s$  w.r.t.  $\mathcal{O}_i$  is given by the following quadratic program:

$$\arg \min_{\dot{z}_s \in \mathbb{R}^2} (\dot{z}_s - \dot{z}_d)^\top (\dot{z}_s - \dot{z}_d) \quad (28)$$

$$\text{s.t. } n_i^\top \dot{z}_s \geq -\alpha(\|z - o_i\| - r_i), \quad (29)$$

and its solution is  $\dot{z}_s = \dot{z}_d + \max\{-n_i^\top \dot{z}_d - \alpha(\|z - o_i\| - r_i), 0\} \cdot n_i$  [11]. Then the safe velocity tracking controller can be defined as  $u = -K_D(\dot{z} - \dot{z}_s)$  with  $K_D > 0$ .

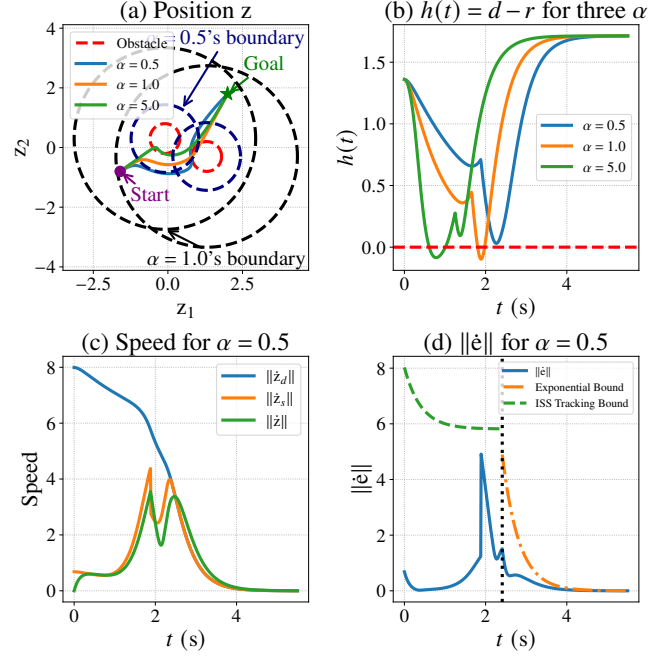


Fig. 2. (a) 2D path with circular obstacles. (b) Barrier value  $h$  vs. time for the three  $\alpha$ . (c) Speeds for  $\alpha = 0.5$  of  $\|\dot{z}_d\|$ ,  $\|\dot{z}_s\|$ , and  $\|\dot{z}\|$ . (d) Tracking-error speed  $\|\dot{e}\|$  for  $\alpha = 0.5$ .

We illustrate three different projected trajectories with three different choices of  $\alpha = 0.5, 1, 5$  respectively, and  $K_p = 1.8, K_D = 8$ . Using standard linear control design and [11], we can show that when the CBF constraint is inactive, this choice of parameters leads to an exponentially convergent tracking error with  $\beta = 2.45$  and  $M = 3.24$ . This allows us to characterize the set  $S_V$  of safe initial conditions. As shown in Fig 2 (a), the red dashed lines show the boundary of the obstacles  $\{z \in \mathbb{R}^2 : \|z - [-0.1, 0.3]^\top\| \leq 0.5 \cup \|z - [1.3, -0.3]^\top\| \leq 0.5\}$ , the union of blue dashed circles denotes the unsafe initial states when  $\alpha = 0.5$ , and the black ones denote the unsafe initial states when  $\alpha = 1$ . As Fig. 2 (a) and (b) show, when the CBF for RoM is invalid ( $\alpha = 5$ ) or the initial state is outside  $S_V$  ( $\alpha = 1$ ), the system's safety cannot be guaranteed, and the minimum of  $h(t)$  is less than 0. Only when Theorem 3 and Theorem 4 are satisfied with a valid CBF in the RoM, can we ascertain that the trajectory is safe all the time ( $\alpha = 0.5$ ). Besides, Fig. 2 (c) and (d) verified that the tracking error shrink to 0 exponentially fast when (18) is satisfied, and the tracking error is bounded by the ISS tracking bound computed by the  $\|d\|_\infty = \|\dot{z}_s\|_\infty$  when the safety filter is activated.

## V. CONCLUSION AND FUTURE WORK

We developed a recurrence-based framework for layered safety-critical control of high-order nonlinear systems. By introducing Recurrent Tracking Functions (RTFs), we relax the classical Lyapunov-based tracking requirement to a finite-time recurrence condition, thereby allowing transient deviations while still preserving safety. In particular, any norm of an exponentially stable tracking error can serve as an RTF, yielding a certificate applicable to general nonlinear systems. We showed that augmenting RoM-based CBFs with RTFs leads to recurrent CBFs of the form  $h_V(z, \dot{e}) = -V(z, \dot{e}) + \alpha_e h(z)$ , which guarantee safety for all initial conditions in  $S_V = \{h_V \geq 0\}$ . The condition  $\beta > \alpha$  ensures that the tracking error converges faster than the system approaches the safety boundary, thereby maintaining  $h(z_{cl}(t)) \geq 0, \forall t \geq 0$ , despite the recurrent nature of  $h_V$ . Numerical simulations further validate the theoretical results.

Future work will build on recent advances in data-driven verification of recurrent sets [20], [21] to develop learning-based approaches for constructing RTFs directly from trajectory data, eliminating the need for explicit tracking function design. Experimental validation on physical robotic platforms also remains an important direction for future research.

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## APPENDIX

### A. Proof of Theorem 3

*Proof.* For simplicity let  $T$  denote  $T_S((z, \dot{e}); \tau)$ . To prove the Theorem 3, for any  $x_0 \in \mathcal{S}_{\text{FoM}}$  s.t.  $(\Pi(x_0), \dot{e}_0) \in S_V$ , our goal is to find a feedback control signal  $u(\cdot) = K(x_{cl}(\cdot), k(\Pi(x_{cl}(\cdot)))) \in \mathcal{U}^{(0, \infty]}$  and  $\gamma : \mathbb{R} \rightarrow \mathbb{R}_{>0}$  such that:

$$\max_{t \in T} e^{\gamma(h_V(z_{cl}(t), \dot{e}(t)))} h_V(z_{cl}(t), \dot{e}(t)) \geq h(z_0, \dot{e}_0). \quad (30)$$

Then we can prove, when  $u(\cdot)$  is the feedback controller that satisfies (18) and  $\gamma(h_V(z_{cl}(t))) = \alpha t$ , (30) is satisfied and  $S_V$  is the corresponding control  $\tau$ -recurrent set.

For the trajectory  $x_{cl}(\cdot)$  generated by the feedback controller that satisfies (18), we have

$$\max_{t \in T} e^{\alpha t} h_V(z_{cl}(t), \dot{e}(t)) - h_V(z_0, \dot{e}_0) \quad (31)$$

$$= \max_{t \in T} \int_0^t \frac{d(e^{\alpha s} h_V(z_{cl}(s), \dot{s}(s)))}{ds} ds \quad (32)$$

$$= \max_{t \in T} \int_0^t e^{\alpha s} (-\dot{V}(z_{cl}(s), \dot{s}(s)) \dots + \alpha_e \nabla h^\top (\dot{z}_s + \dot{s}) + \alpha h_V(z_{cl}(s), \dot{s}(s))) ds \quad (33)$$

$$\geq \max_{t \in T} \int_0^t e^{\alpha s} (-\dot{V}(z_{cl}(s), \dot{s}(s)) - \alpha_e \alpha h(z_{cl}(s)) \dots - \alpha_e \|\nabla h\| \|\dot{s}\| + \alpha h_V(z_{cl}(s), \dot{s}(s))) ds \quad (34)$$

$$\geq \max_{t \in T} \int_0^t e^{\alpha s} (-\dot{V}(z_{cl}(s), \dot{s}(s)) - (\alpha + \frac{\alpha_e C_h}{a_1}) \dots \cdot V(z_{cl}(s), \dot{s}(s))) ds \quad (35)$$

$$= \max_{t \in T} \int_0^t -e^{-\frac{\alpha_e C_h}{a_1} s} \cdot \frac{d(e^{(\alpha + \frac{\alpha_e C_h}{a_1}) s} V(z_{cl}(s), \dot{s}(s)))}{ds} ds \quad (36)$$

$$= \max_{t \in T} V(z_0, \dot{s}_0) - e^{\alpha t} V(z_{cl}(t), \dot{s}(t)) \dots + \int_0^t \frac{d(e^{-\frac{\alpha_e C_h}{a_1} s})}{ds} e^{(\alpha + \frac{\alpha_e C_h}{a_1}) s} V(z_{cl}(s), \dot{s}(s)) ds \quad (37)$$

$$= \max_{t \in T} V(z_0, \dot{s}_0) - e^{\alpha t} V(z_{cl}(t), \dot{s}(t)) \dots - \frac{\alpha_e C_h}{a_1} \int_0^t e^{\alpha s} V(z_{cl}(s), \dot{s}(s)) ds \quad (38)$$

$$\geq \max_{t \in T} V(z_0, \dot{s}_0) - e^{\alpha t} V(z_{cl}(t), \dot{s}(t)) \dots - \frac{a_2 \alpha_e C_h}{a_1} \int_0^t M e^{-(\beta - \alpha) s} \|\dot{s}_0\| ds \quad (39)$$

$$\geq \max_{t \in T} [1 - \frac{a_2 \alpha_e C_h M}{a_1^2 (\beta - \alpha)} (1 - e^{-(\beta - \alpha) t})] V(z_0, \dot{s}_0) \dots - e^{\alpha t} V(z_{cl}(t), \dot{s}(t)) \quad (40)$$

$$= \max_{t \in T} [1 - \frac{a_2 \alpha_e C_h M}{a_1^2 (\beta - \alpha)} (1 - e^{-(\beta - \alpha) t})] \dots \cdot [V(z_0, \dot{s}_0) - e^{\beta t} V(z_{cl}(t), \dot{s}(t))], \quad (41)$$

where the equality (32) holds from the Fundamental Theorem of Calculus, the equality (33) follows from the principle of integration by parts, the equality (36) is equivalent to (20), and the equality (41) holds since  $\alpha_e = \frac{a_1^2 (\beta - \alpha)}{a_2 C_h M}$ . The inequality (34) can be obtained from Cauchy Inequality and  $\nabla h(z_{cl})^\top \dot{z}_s \geq -\alpha h(z_{cl})$ , the inequality (35) and the inequality (40) can be obtained from the linear lower bound of the tracking error  $a_1 \|\dot{s}(\cdot)\| \leq V(\dot{s}(\cdot))$ , and the inequality (39) is equivalent to the upper bound of the tracking error  $V(\dot{s}(\cdot)) \leq a_2 \|\dot{s}(\cdot)\|$ .

Note that  $e^{-(\beta - \alpha)t} - e^{-(\beta - \alpha)\tau} \geq 0, \forall t \in T$ , since  $\beta > \alpha$  and  $\min_{t \in T} e^{\beta t} V(z_{cl}(t), \dot{s}(t)) - V(z_0, \dot{s}_0) \leq 0$ , i.e.  $\max_{t \in T} V(z_0, \dot{s}_0) - e^{\beta t} V(z_{cl}(t), \dot{s}(t)) \geq 0$ , then we have

$$\max_{t \in T} [e^{-(\beta - \alpha)t} - e^{-(\beta - \alpha)\tau}] \dots \quad (42)$$

$$\cdot [V(z_0, \dot{s}_0) - e^{\beta t} V(z_{cl}(t), \dot{s}(t))] \quad (43)$$

$$\geq [e^{-(\beta - \alpha)t} - e^{-(\beta - \alpha)\tau}] \dots \quad (44)$$

$$\cdot \max_{t \in T} [V(z_0, \dot{s}_0) - e^{\beta t} V(z_{cl}(t), \dot{s}(t))] \geq 0, \quad (45)$$

where (45) holds naturally from the property of the maximum,

$$\text{i.e., (41)} \geq e^{-(\beta - \alpha)\tau} \max_{t \in T} [V(z_0, \dot{s}_0) - e^{\beta t} V(z_{cl}(t), \dot{s}(t))] \geq 0, \quad (46)$$

thus,  $\max_{t \in T} e^{\alpha t} h_V(z_{cl}(t), \dot{s}(t)) \geq h_V(z_0, \dot{s}_0)$ , and  $S_V$  is the corresponding control  $\tau$ -recurrent set [16, Theorem 2], which completes the proof.  $\square$

#### B. Proof of Theorem 4

*Proof.* Firstly, we know that  $\nabla h(z_{cl}(\cdot))^\top \dot{z}_s(\cdot) \geq -\alpha h(z_{cl}(\cdot))$  and  $\|\nabla h(\cdot)\| \leq C_h$ . Along the FoM trajectory  $x_{cl}(\cdot)$ , we have

$$e^{\alpha t} h(z_{cl}(t)) - h(z_0) = \int_0^t \frac{d}{ds} (e^{\alpha s} h(z_{cl}(s))) ds \quad (47)$$

$$= \int_0^t e^{\alpha s} (\dot{h}(z_{cl}(s)) + \alpha h(z_{cl}(s))) ds \quad (48)$$

$$= \int_0^t e^{\alpha s} (\nabla h(z_{cl}(s))^\top (\dot{z}_s + \dot{s}) + \alpha h(z_{cl}(s))) ds \quad (49)$$

$$\geq \int_0^t e^{\alpha s} (-\alpha h(z_{cl}(s)) + \nabla h(z_{cl}(s))^\top \dot{s} + \alpha h(z_{cl}(s))) ds \quad (50)$$

$$= \int_0^t e^{\alpha s} \nabla h(z_{cl}(s))^\top \dot{s} ds \geq -C_h \int_0^t e^{\alpha s} \|\dot{s}\| ds, \quad (51)$$

where the equality (47) follows from the Fundamental Theorem of Calculus, and the equality (48) holds naturally from Leibniz rule. Inequalities (50) and (51) follow from  $\nabla h(z_{cl})^\top \dot{z}_s \geq -\alpha h(z_{cl})$  and Cauchy Inequality, respectively. Then  $\forall t$ , we have

$$h(z_{cl}(t)) \geq e^{-\alpha t} h(z_{cl}) - C_h \int_0^t e^{-\alpha(t-s)} \|\dot{s}\| ds \quad (52)$$

$$\geq e^{-\alpha t} h(z_0) - C_h M e^{-\alpha t} \int_0^t e^{-(\beta - \alpha)s} \|\dot{s}_0\| ds \quad (53)$$

$$= e^{-\alpha t} h(z_0) - \frac{C_h M}{\beta - \alpha} e^{-\alpha t} (1 - e^{-(\beta - \alpha)t}) \|\dot{s}_0\| \quad (54)$$

$$\geq e^{-\alpha t} \frac{V(z_0, \dot{s}_0)}{\alpha_e} - \frac{C_h M}{\beta - \alpha} e^{-\alpha t} (1 - e^{-(\beta - \alpha)t}) \|\dot{s}_0\| \quad (55)$$

$$\geq e^{-\alpha t} V(z_0, \dot{s}_0) \left[ \frac{1}{\alpha_e} - \frac{C_h M}{a_1 (\beta - \alpha)} (1 - e^{-(\beta - \alpha)t}) \right] \quad (56)$$

$$\geq e^{-\alpha t} V(z_0, \dot{s}_0) \left[ \frac{1}{\alpha_e} - \frac{C_h M}{a_1 (\beta - \alpha)} \right] \quad (57)$$

$$= e^{-\alpha t} V(z_0, \dot{s}_0) \frac{C_h M}{a_1 (\beta - \alpha)} \left( \frac{a_2}{a_1} - 1 \right) \geq 0, \quad (58)$$

where the inequality (53) holds from Theorem 2, i.e., the exponential stability of tracking error under recurrence condition, the inequality (55) holds since  $(z_0, \dot{s}_0) \in S_V$ , the inequality (56) is obtained from the linear lower bound of the tracking error  $a_1 \|\dot{s}_0\| \leq V(z_0, \dot{s}_0)$ , and the inequality (58) follows from the fact that  $a_1 \leq a_2$  and  $\beta > \alpha$ . Since  $h(\Pi(x_{cl}(t))) \geq 0, \forall t \geq 0$ , then we have  $x_{cl}(t) \in \mathcal{S}_{\text{FoM}}, \forall t \geq 0$ , i.e., the trajectory is always in the safe region, which completes the proof.  $\square$