Stability Analysis and Data-driven Verification via Recurrent Lyapunov Functions

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***Abstract*—Lyapunov's direct method is an instrumental tool that provides a rigorous framework for stability analysis and control design for dynamical systems. A critical step that enables the application of the method is the availability of a Lyapunov function V—a function whose value monotonically decreases along the trajectories of the dynamical system. Unfortunately, finding a Lyapunov function is often tricky and requires ingenuity, domain knowledge, or significant computational power. At the core of this challenge is the fact that the method requires every sub-level set of V ($V_{\leq c}$) to be forward invariant, thus implicitly coupling the geometry of $V_{\leq c}$ and the trajectories of the system. In this paper, we seek to disentangle this dependence by developing a direct method that substitutes the concept of invariance with the more flexible notion of recurrence. A set is (τ-)recurrent if every trajectory that starts in the set returns to it (within τ seconds). We show that, under mild conditions, the recurrence of sub-level sets $V_{\leq c}$ is sufficient to guarantee stability and introduce the appropriate stronger notions to obtain asymptotic stability and exponential stability. Most notably, we provide norm-agnostic converse theorems showing that, under mild conditions, any norm satisfies our relaxed stability conditions, provided one is willing to certify a slightly weaker stability condition. We further develop GPU-based algorithms that can verify (practical) stability notions using purely trajectory data, and without the need of computing a Lyapunov function. Our analysis and methods further highlight an intrinsic trade-off between the sample/computational complexity and the certified performance that our algorithms navigate.**

I. Introduction

Lyapunov stability theory plays a central role in the study of dynamical systems. It provides a rigorous mathematical framework for qualitatively analyzing system solutions and has heavily influenced systems theory and engineering over the past century. A fundamental tool derived from this theory is the so-called Lyapunov direct method, a.k.a. Lyapunov's second method [1], which states mild conditions on a function $V(x)$ (non-increasing along trajectories and proper) that can certify stability of an equilibrium point. Since first proposed in 1892, Lyapunov's direct method has found ubiquitous applications across multiple branches of engineering, including aerospace, electrical, mechanical, and chemical, among others [2]–[5].

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A critical step in the application of Lyapunov's direct method is finding the function V that indeed satisfies all the conditions stated by the theory. Unfortunately, while such a function is known to exist via converse theorems, e.g., [6], manually finding a Lyapunov function is often tricky and relies on ingenuity and deep domain knowledge. To circumvent this step, a variety of computational methods have been proposed for finding Lyapunov functions [7], e.g., via the use of partial differential equation (PDE) solvers to solve Zubov's Equation [8], [9], linear programs (LPs) to find piece-wise linear Lyapunov functions [10], and semidefinite programs (SDPs) to solve linear matrix inequalities (LMIs) [11] or sum of square (SoSs) problems [12]. However, the computational complexity is known to exponentially increase with not only the dimension of the state space but also the parameterization of the Lyapunov function [7], [13].

This has led to multiple investigations into relaxing the conditions required for V, and in particular, its time derivative $\dot{V}$. Such relaxations can be broadly divided into three groups. The first group seeks LaSalle-Krasovskii type of conditions by relaxing the negative definiteness of $\dot{V}$, i.e., only requiring $\dot{V} \leq 0$; see [5], [14] and its generalization [15], [16]. The second group further relaxes the strict negative definite condition Lyapunov method by allowing $\dot{V} > 0$ on some regions of the state space. This is implicitly done by using generalizations of the comparison lemma [17] to impose conditions on higher order time derivatives of V that still ensure convergence of $V \to 0$ while allowing $\dot{V} > 0$ for some regions of the state space. The third group uses the so called discretization method, which considers a fixed parameter $T > 0$ and leverages the net decrement of V across any trajectory $x(t)$, i.e., $V(x(t+T)) - V(x(t))$, to reason about stability [18], [19]. Unfortunately, despite such efforts, the basic principle can still be traced back to the (indirect) construction of a Lyapunov function whose sub-level sets are invariant [20], [21], which still needs to be verified either analytically or via the solution of a convex program, rendering similar verification challenges as before.

The challenge of finding such functions lies in the fact that Lyapunov's direct method implicitly constrains its shape by requiring every sub-level set to be an invariant set. The goal of this paper is to relax this condition by replacing the invariance of sub-level sets with a weaker notion known as recurrence. We say that a set is (τ-)recurrent if every trajectory that starts in the set returns to it (within τ seconds). Such relaxation has been recently shown to provide a versatile mechanism for estimating regions of attractions of stable equilibrium points [22] as well as verifying the safety of a dynamical system [23]. Moreover, from an information theoretical standpoint, (control) recurrence can be achieved at lower data rates than invariance [24] and can often be enforced using a finite number of trajectory data [25].

In this paper, we seek to explore the role of recurrence in certifying different notions of stability of an equilibrium point. The contributions of our work are several:

- *Recurrent Lyapunov Functions:* We introduce the concept of Recurrent Lyapunov Functions (RLFs), which generalize classical Lyapunov functions by replacing the invariance condition on sub-level sets with a more flexible recurrence condition. This relaxation decouples the geometry of the trajectories from the geometry of the level sets.
- *Stability Guarantees:* We establish rigorous stability theorems demonstrating that τ-recurrence of a sequence of compact sub-level sets is sufficient to guarantee stability, asymptotic stability, exponential stability, and ultimate boundedness. These results provide an alternative framework for proving stability without requiring strict invariance conditions.
- *Norm-agnostic Converse Theorems:* We show that, under mild assumptions, any norm is guaranteed to satisfy our RLF conditions, provided one is willing to look for a slightly weaker stability property. This generality highlights the fundamental role of recurrence in stability analysis, and opens the door for the development of stability verification methods that do not require the computation of a Lyapunov function.
- *Data-Driven Verification:* We develop a computationally efficient, GPU-parallelized method to verify whether a given function, e.g., a norm, satisfies the proposed RLF conditions on a compact domain. Our approach enables scalable data-driven verification of dynamical systems, making stability analysis feasible even when an explicit Lyapunov function is difficult to construct.

A preliminary version of this paper has been presented in [26]. The present manuscript extends this work in multiple ways. First, we extend our stability analysis from local conditions to conditions over arbitrary sets containing the equilibrium. Second, beyond stability, we further provide conditions for ultimate boundedness. Third, we introduce novel converse theorems that show that any norm can satisfy our Recurrent Lyapunov Function conditions, and provide estimates on the sample complexity of verifying exponential stability on some bounded region of the state space. Finally, we provide a novel streamlined verification algorithm and thorough numerical validations that illustrate the merits of our framework.

Closely related work: The derived conditions are similar in spirit to the ones considered by Karafyllis in [21], which studies robust stability analogs (c.f. Proposition 2.3 and 2.5). Particularly, our asymptotic stability condition is closely related to [21, Prop. 2.5]. Our stability, exponential stability, and ultimate boundedness conditions are, however, new and not present in prior work. More importantly, the focus of our paper is on exploring the connection of such conditions with the recurrence of level sets of V and developing parallelizable algorithms that can be implemented on GPUs, whereas [21], focuses on robust stability and provides Matrosov-type conditions.

The rest of this paper is organized as follows. In Section II, we introduce preliminary definitions for dynamical systems and stability. Section III introduces the concept of recurrent sets and how they can be used to bound trajectories. Section IV characterizes properties of a function V that render its sub-level sets τ-recurrent, setting the stage for introduction of Recurrent Lyapunov Functions, and a novel proof of stability on an equilibrium. We further derive RLF conditions that guarantee asymptotic stability in Section V, and exponential stability and ultimate boundedness in Section VI. SectionVII is dedicated to demonstrating that the RLF can be widely satisfied using just norms, paving the way to stability verification algorithms that do not require the computation of a Lyapunov function VIII. Numerical experiements are presented in Section IX, and conclusions and future work are presented in Section X.

Notation: Throughout the text, we let $\|\cdot\|$ denote an arbitrary norm on $\mathbb{R}^n$, and define $B_r(x)$ as the closed ball of radius r centered at $x \in \mathbb{R}^n$. Given a set $S \subset \mathbb{R}^n$, the distance from a point $y \in \mathbb{R}^n$ to S is defined as

$$\mathrm{d}(y, S) := \inf_{x \in S} \|y - x\|.$$

The signed distance from a point $x \in \mathbb{R}^n$ to S is then given by

$$\mathrm{sd}(x, S) := \begin{cases} \mathrm{d}(x, \partial S), & \text{if } x \notin S, \\ -\mathrm{d}(x, \partial S), & \text{if } x \in S. \end{cases}$$

We also use $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} \mid x \geq 0\}$, $\mathbb{R}_{>0} := \{x \in \mathbb{R} \mid x > 0\}$, and $[n] := \{1, \ldots, n\}$.

II. PRELIMINARIES

We consider a continuous-time dynamical system

$$\dot{x} = f(x)\,, \tag{1}$$

where $x \in D \subset \mathbb{R}^n$ is the state, and the map $f : D \to \mathbb{R}^n$ is a continuous function defined over an *invariant and closed domain* D. Given an initial state x, we use $\phi(t, x)$ to denote the solution of (1). Throughout the paper, we make the following assumption about the vector field and its solutions.

Assumption 1. *The vector field $f(x)$ in* (1) *is locally Lipschitz. That is, for any compact set $S \subset D$, there exists a constant $\overline{L}_S \in \mathbb{R}_{\geq 0}$ such that*

$$\|f(y) - f(x)\| \leq \overline{L}_S \|y - x\|, \qquad \forall x, y \in S.$$

The local Lipchitz nature of the vector field implies that solutions must exist for some amount of time, which we will denote by the following:

Definition 1 (Interval of Existence)**.** *For $x \in D$, the **interval of existence** $I(x) \subset \mathbb{R}$ is the largest open interval such that $\phi(t, x)$ exists for all $t \in I(x)$.*

Whenever the initial condition is understood from the context, we will use $x(t) := \phi(t, x)$. Whenever the set S is understood from context, we will use $\overline{L}$ instead of $\overline{L}_S$.

For simplicity, we refer to the dynamical system (1) as the dynamical system f.

We next introduce the core building blocks of Lyapunov Stability Theory.

Definition 2 (Stability). *An equilibrium x^* is* ***stable*** *if for any $\varepsilon > 0$, $\exists \delta > 0$, such that if $\|x-x^*\| \leq \delta$ then $\|\phi(t,x)-x^*\| \leq \varepsilon$ $\forall t \geq 0$.*

Definition 3 (Attractivity). *An equilibrium x^* is* ***attractive on the set*** *S if every $x \in S$, $\|\phi(t,x) - x^*\| \to 0$ as $t \to \infty$.*

Definition 4 (Asymptotic Stability). *An equilibrium x^* is* ***asymptotically stable on the set*** *S, if it is stable, and attractive on S.*

Definition 5 (Exponential Stability). *An equilibrium x^* is* ***exponentially stable*** *on the set S if there exists constants $K > 0$, $\lambda > 0$ such that if $x \in S$, then*

$$\|\phi(t,x) - x^*\| \leq Ke^{-\lambda t}\|x - x^*\|, \quad \forall t \geq 0. \tag{2}$$

It will also be useful throughout our presentation to define sets that are of general use to characterize transient, as well as asymptotic behavior.

We start by characterizing the set of points that can be reached within a finite interval of time.

Definition 6 (Reachable Tube). *For the dynamical system f, a time $\tau > 0$, and a set $S \subset D$, we denote the τ-reachable tube from S within τ units of time by*

$$\mathcal{R}^\tau(S) = \bigcup_{x\in S, t\in[0,\tau]} \{\phi(t,x)\}.$$

Next, we formally define positive invariant sets.

Definition 7 (Positively Invariant Sets). *A set $S \subseteq \mathbb{R}^n$ is positively invariant w.r.t.* (1) *if and only if:*

$$x \in S \implies \phi(t,x) \in S, \quad \forall\, t \in \mathbb{R}_{\geq 0}.$$

Since we only consider here positive invariant sets, as opposed to negative invariant sets, we will often refer to them as plainly invariant sets. As mentioned before, the notion of positive invariance is a fundamental building block of Lyapunov Theory. By trapping trajectories on compact sub-level sets of a function one can guarantee boundedness of trajectories, stability, and asymptotic stability via a gradual reduction of the Lyapunov function value.

III. Recurrence

To relax the notion of invariance, one must allow trajectories to temporarily leave a set. However, in order to still be able make statements about asymptotic behavior, our first condition requires trajectories to return infinitely often.

Definition 8 (Recurrent Set). *A set $S \subseteq \mathbb{R}^n$ is recurrent w.r.t.* (1)*, if for any $x \in S$, and $t \geq 0$,*

$$\exists\, t' > t, \quad \text{s.t.} \quad \phi(t',x) \in S.$$

Since trajectories are allowed to leave S, in our development, it will be useful to keep track of the time intervals where a trajectory $\phi(t,x)$ lies within a given set S for a given initial point $x \in D$.

Definition 9 (Containment Times). *Given a set $S \subset D$ and a point $x \in D$, we define the set of containment times, $T_S(x)$, as the set of times t for which the trajectory $\phi(t,x) \in S$, i.e.,*

$$T_S(x) := \{t \in \mathbb{R}_{>0} \mid \phi(t,x) \in S\}.$$

Given constants a, b, we write

$$T_S(x; a, b) := \; T_S(x) \cap (a, a+b].$$

For convenience we also write $T_S(x;b) := T_S(x;0,b)$. Finally, when the set S is clear from context, we may omit the subscript entirely.

The notion of recurrent sets introduced here is related to the classical notion of Poincare recurrence [27], and in particular, Poincare recurrent sets [28, Def. 2.4.1], which constitutes the union of Poincare recurrent points; a point x is Poincare recurrent if its backward and forward flows, i.e., $\{\phi(-t,x)\}_{t\geq 0}$ and $\{\phi(t,x)\}_{t\geq 0}$, get arbitrarily close to x, *infinitely often*. In fact, one can show that any open subset S of a Poincare Recurrent Set is a Recurrent Set according to Definition 8.

As we will soon see, Definition 8 will ensure that part of the ω-limit set of f must be contained within the recurrent set S. While this property suggests some notion of convergence (attractivity) to some set that intersects S, for stability analysis purposes, we further require control on how far the trajectory may depart from S. The following stronger notion of recurrence achieves this.

Definition 10 (τ-Recurrent Set). *A set $S \subseteq D$ is τ-recurrent w.r.t.* (1)*, if there exists a locally bounded function $\tau : D \to \mathbb{R}_{>0}$ s.t. for any $x \in S$,*

$$\exists\, t' > t, \quad \text{with} \quad t' - t \in (0, \tau(x)] \quad \text{and} \quad \phi(t',x) \in S.$$

We further say that S is strictly τ-recurrent, *if for any $x \in S$, and $t \geq 0$,*

$$\exists\, t' > t, \quad \text{with} \quad t' - t \in (0, \tau(x)] \quad \text{s.t.} \quad \phi(t',x) \in S \backslash \partial S.$$

One of the key properties of recurrent sets is that trajectories that start within the set S will visit it *infinitely often* (again and again), and *forever* (there is always a future time when it is visited again). This condition, implicitly assumes that trajectories that start in S are *forward complete*, i.e., $I(x) = [0,\infty)$. Moreover, while the recurrent property of definition 8 is sufficient for most of the development of our work, it is hard to verify or use. The next lemma shows that when the set S is compact, these complications become less relevant.

Lemma 1 (Characterization of τ-Recurrent Compact Sets). *Let $S \subset D$ be a compact set, and consider the system* (1) *under Assumption 1. Then, the following conditions are equivalent:*

(i) *S is τ-recurrent,*

(ii) *there exists locally bounded $\tau : D \to \mathbb{R}_{>0}$, such that for any $x \in S$, $\exists\, t' \in (0, \tau(x)] \subseteq I(x)$ with $\phi(t',x) \in S$,* *(iii)* *there exists $\tau > 0$ such that for any $x \in S$ there is a sequence $\{t_n\}_{n\in\mathbb{N}}$ satisfying,*

$$\lim_{n\to\infty} t_n = \infty\,, \quad \textit{with} \quad t_{n+1} - t_n \in (0, \tau]\,, \qquad (3)$$

and $\phi(t_n, x) \in S\ \forall n$. Thus, $I(x) = [0, \infty)\ \forall x \in S$.

Proof.
$(i) \Longrightarrow (ii)$: Follows from Definition 10 and choosing $t = 0$.
$(ii) \Longrightarrow (iii)$: Let $\tau := \sup_{x\in S} \tau(x)$, which, since S is compact and $\tau(\cdot)$ is locally bounded, is finite. Given $x \in S$, we build the sequence $\{t_n\}_{n\in\mathbb{N}}$ satisfying (3) and $\phi(t_n, x) \in S$ by induction. For the base case, let $t_0 = 0$ so that $\phi(t_0, x) = x$, and choose

$$t_1 = \max\{t \in (0, \tau] \mid \phi(t, x) \in S\};$$

note that the at least one time t with $\phi(t, x) \in S$ must exist, since property (ii) holds. By construction, $t_1 - t_0 \in (0, \tau]$, and since $\tau(x) \in I(x)$, we have $t_1 \in I(x)$. The inductive construction proceeds in a similar manner: given $t_1 < t_2 < \cdots t_n$, with $x_n := \phi(t_n, x) \in S$ and $t_n \in I(x)$, define

$$t_{n+1} = \max\{t \in (0, \tau] \mid \phi(t_n + t, x) \in S\}; \qquad (4)$$

Note that t_{n+1} always exists by (ii), and satisfies $t_{n+1} - t_n \in (0, \tau]$ as required. Further, since $x_n \in S$, $t_{n+1} - t_n \in I(\phi(t_n, x))$, which implies that $t_{n+1} \in I(x)$ It remains to show that $t_n \to \infty$, which we argue by contradiction. If, instead, the strictly increasing sequence of times was bounded, we would have $t_n \uparrow t^*$. Consider now the sequence $x_n := \phi(t_n, x)$, which satisfies $x_n \in S$ for all $n \in \mathbb{N}$. By continuity of $\phi(\cdot, x)$, it follows that $\lim_{n\to\infty} x_n = \phi(t^*, x)$. Moreover, since S is compact, we must have $\phi(t^*, x) \in S$. This implies, first, that the limit t^* must be achieved for finite n, say n^*. However, since S is τ-recurrent, we know that $\exists t' \in (0, \tau]$ such that $\phi(t^* + t', x) \in S$, implying that t^* cannot be an upper bound on the sequence $\{t_n\}_{n\in\mathbb{N}}$, since by definition of the sequence (4), $t_{n^*+1} \geq t^* + t' > t^*$. Thus, $t_n \to \infty$, as desired, and thus $I(x) = [0, \infty)$.
$(iii) \Longrightarrow (i)$: Finally, given $x \in S$ and $t \geq 0$, let n^* be the largest n s.t. $t_n \leq t$, then, it follows that $t_{n^*+1} - t \in (0, \tau]$, $\phi(t_{n^*+1}, x) \in S$. The result follows by defining $\tau(x) = \tau$, for all $x \in S$. □

As Lemma1 shows, for *compact* τ-recurrent set, excursions outside the set last at most $\tau = \sup_{x\in S} \tau(x) < +\infty$. We will use this property to bound the distance a trajectory can travel away from a τ-recurrent set. To that end, we recall here that the vector field (1) is assumed locally Lipschitz (Assumption 1). While such property suffices, it will prove convenient to obtain tighter bounds via locally one-sided Lipschitz constants.

Definition 11 (One-sided Lipschitz)**.** *We say f is locally one-sided Lipschitz if for any compact set $S \subset D$ there exists a constant $L_S \leq \overline{L} \in \mathbb{R}$ such that*

$$(y - x)^T (f(y) - f(x)) \leq L_S ||y - x||^2, \quad \forall x, y \in S$$

Finally, we will also need to estimate the maximum value of the norm of the vector field on a (compact) set Sr, i.e., $F_S := \max_S \|f(x)\|$. Using these definitions, we now introduce a Containment Lemma, which bounds how far trajectories can go from a compact set in which they start.

Lemma 2 (Containment Lemma)**.** *Let Assumption 1 hold. Consider a compact set $S \subset D$ and a constant $\tau > 0$. Then, for any $x \in S$ the following holds:*

$$\max_{t\in[0,\tau]} \mathrm{d}(\phi(t, x), S) \leq F_S\, h(\tau; L)$$

where $L := L_{\mathcal{R}^\tau(S)} < \infty$, and

$$h(\tau; L) := \begin{cases} \frac{e^{L\tau}-1}{L}, & L \neq 0, \\ \tau, & L = 0. \end{cases}$$

Proof. Since S is compact and Assumption 1 holds, it follows from Proposition 5.1 of [29] that the set $\mathcal{R}^\tau(S)$ is compact. Thus, L is finite. Let $x \in S$ and let $u(t) := \|\phi(t, x) - x\|$. Observe that since $x \in S$, $\mathrm{d}(\phi(t, x), S) \leq u(t)$. Thus, bounding $u(t)$ will be sufficient. Differentiating $u(t)^2$ w.r.t to time gives

$$\begin{aligned}
\frac{d}{dt}u(t)^2 = \frac{d}{dt}\|\phi(t, x) - x\|^2 &= \frac{d}{dt}\langle \phi(t, x) - x, \phi(t, x) - x\rangle \\
&= 2\langle \phi(t, x) - x, f(\phi(t, x))\rangle \\
&= 2\langle \phi(t, x) - x, f(\phi(t, x)) - f(x)\rangle \\
&\quad + 2\langle \phi(t, x) - x, f(x)\rangle \\
&\leq 2Lu(t)^2 + 2F_S u(t)
\end{aligned}$$

where the first step uses $\frac{d}{dt}\phi(t, x) = f(\phi(t, x))$, the second adds and subtracts $f(x)$, and the last inequality follows from the definition of L, F_S and $u(t)$.

It follows then that,

$$\frac{d}{dt}u(t)^2 = 2u(t)\dot{u}(t) \leq 2Lu(t)^2 + 2F_S u(t).$$

Since for $u(t) = 0$ our result holds trivially, we may assume $u(t) > 0$ for $t \in (0, \tau]$, which implies

$$\dot{u}(t) \leq Lu(t) + F_S.$$

Applying Grönwall's inequality (c.f [5], Lemma A.1) yields

$$u(t) \leq e^{Lt}u(0) + \frac{F_S}{L}(e^{Lt} - 1) \quad \text{and} \quad u(t) \leq u(0) + F_S t,$$

for $L \neq 0$ and $L = 0$, respectively. Finally, taking $u(0) = 0$ and maximizing the above values over the domain $t \in (0, \tau]$ yields precisely the claimed bounds. □

The Containment Lemma, which only provides containment guarantees for a finite time, can be combined with the recurrence property of Definition 8 and Lemma 1 to provide trajectory bounds for all positive times.

Corollary 1 (Boundedness of Trajectories)**.** *Let S be a compact τ-recurrent set. Then it follows that for any $x \in S$,*

$$\mathrm{d}(\phi(t, x), S) \leq F_S h(\tau; L), \qquad \forall t \geq 0,$$

where $\tau = \sup_{x\in S} \tau(x)$ and $L = L_{\mathcal{R}^\tau(S)}$. Moreover, the τ-reachable tube $\mathcal{R}^\tau(S)$ is invariant.

Proof. Applying Lemma 2, we have $\mathrm{d}(\phi(t, x), S) \leq F_S h(\tau; L)$ for $t \in (0, \tau]$. Now, construct a sequence as in

Lemma 1. Since $t_1 \in (0,\tau]$ with $\phi(t_1,x) \in S$, $\mathrm{d}(\phi(t,x),S) \leq F_S h(\tau;L)$ for $t \in (0, t_1+\tau]$. Continuing this inductively, we have this condition holding for $t \in \lim_{n\to\infty}(0,t_n]$, which is equivalent to $(0,\infty)$ since $t_n \to \infty$.

For the second claim, let us assume for the sake of contradiction that $\mathcal{R}^\tau(S)$ is not invariant. Then, there must be an $x \in S$ and $t \geq 0$ such that $\phi(t,x) \notin \mathcal{R}^\tau(S)$. Observe that by construction of the sequence $\{t_n\}$, there must exist an n such that $t_n < t < t_{n+1}$ with $t_{n+1} - t_n < \tau$ wherein $\phi(t_n, x) \in S$. Therefore, $\phi(t - t_n, \phi(t_n,x)) \in \mathcal{R}^\tau(S)$, which yields contradiction to our assumption of non-invariance. Thus, $\mathcal{R}^\tau(S)$ is invariant, as desired

□

We finalize this section, noting that Corollary 1 imbues compact τ-recurrent sets with the same functional property of compact invariant sets, i.e., bounding trajectories. This provides the cornerstone to the development of a recurrence-based stability theory.

IV. Recurrent Lyapunov Functions

Having established the ability to bound trajectories using τ-recurrent sets, we now introduce the modified conditions on a function $V : D \to \mathbb{R}_{\geq 0}$, that relax the standard Lyapunov conditions for stability. In contrast to the classical counterpart, we do not require V to be monotonically non-increasing along trajectories. Rather, for any given initial $x \in D$, we allow $\tau(x)$ units of time to elapse before requiring the function to meet any requirements on its value. This leads to the proposed definition of Recurrent Lyapunov Functions.

Definition 12 (Recurrent Lyapunov Function (RLF))**.** *Given an equilibrium point $x^* \in D$ of* (1) *and a set $S \subseteq D$ satisfying $x^* \in \mathrm{int}(S)$. We say that a* continuous *function $V : D \to \mathbb{R}_{\geq 0}$ is a* ***Recurrent Lyapunov Function*** *over the set S if the following properties hold:*

(i) *V is* ***positive definite*** *around x^*, that is,*

$$V(x) > 0, \ \forall x \neq x^*, \ \text{and } V(x^*) = 0. \tag{5}$$

(ii) *V is* ***τ-recurrent*** *over S, that is, there is a locally bounded $\tau : S \to \mathbb{R}_{>0}$ such that*

$$\min_{s \in T_S(x;\,\tau(x))} V(\phi(s,x)) \leq V(x), \quad \forall x \in S. \tag{6}$$

We make the following remarks about Definition 12. First, the minimum in (6) is taken over the non-necessarily closed set $T_S(x;\tau(x)) = T_S(x) \cap (0,\tau(x)]$. As a result, for the $\min$ to be finite, one is required find $t \in (0,\tau(x)]$ with $\phi(t,x) \in S$. Second, we only require V to be continuous; while classical Lyapunov theory can be developed for non-differentiable functions, it usually requires increased complexity in the analysis. Our results can be readily stated for only continuous V. Finally, the τ-recurrent property (6) acts as a substitute to the standard differential inequality: $\dot{V} = \nabla V(x)^T f(x) \leq 0$. As we will see next, this condition allows us to substitute the standard invariance property with the more relaxed notion of recurrence.

Lemma 3. *Given any $c \geq 0$ and a compact set $S \subseteq D$. If $V : D \to \mathbb{R}_{\geq 0}$ is continuous and τ-recurrent over S (c.f. (ii) in Definition 12), then, the following holds:*

(i) The set S is τ-recurrent.

(ii) The set $V_{\leq c} \cap S = \{x \in S \mid V(x) \leq c\}$ is τ-recurrent.

Proof. We start by noting that since S is compact and contained in the domain of V, then there exists c large enough such that $S \cap V_{\leq c} = S$. As a result, property (i) follows directly from (ii). The proof of (ii) uses the property (ii) of Lemma 1. Precisely, by Definition 12, it follows from (6) that for any $x \in S \cap V_{\leq c}$, one can find $t' \in (0,\tau(x)]$ such that $\phi(t',x) \in S \cap V_{\leq c}$. Since $S \cap V_{\leq c}$ is compact and $\tau(\cdot)$ locally bounded, Lemma 1 (ii) holds for the set $S \cap V_{\leq c}$. Thus, by Lemma 1 (i), the set $S \cap V_{\leq c}$ is τ-recurrent. □

We are now ready to present the main result of this section, which states that the existence of an RLF is sufficient to guarantee the stability of the associated equilibrium point.

Theorem 1 (Stability)**.** *Let Assumption 1 hold. Consider an equilibrium point $x^* \in D$ of* (1) *and a set $S \subseteq D$ satisfying $x^* \in \mathrm{int}(S)$. Then, if $V : D \to \mathbb{R}_{\geq 0}$ is an RLF over S, the equilibrium x^* is stable.*

Proof. The proof is aligned with classical results. For any ε, we will aim to find a compact invariant set $I \subset B_\varepsilon(x^*)$ with $x^* \in \mathrm{int}(I)$. Precisely, let $\tau := \sup_{x\in S} \tau(x)$. Given any $\varepsilon > 0$, choose $0 < r \leq \varepsilon$ s.t. $B_r(x^*) \subset S$, let $L = L_{B_r(x^*)}$, and find $\varepsilon' > 0$ small enough such that

$$\varepsilon' + F_{\varepsilon'} h(\tau; L) < r \leq \varepsilon.$$

Now let $\alpha = \min_{\varepsilon' \leq \|x - x^*\| \leq r} V(x)$. Note that by construction $\alpha > 0$. Select β such that $0 < \beta < \alpha$ and introduce the *compact* set

$$\Omega_\beta := \{x \in B_{\varepsilon'}(x^*) : V(x) \leq \beta\}.$$

Now, consider any initial condition $x \in \Omega_\beta$. Since $\Omega_\beta \subset S$, it follows from τ-recurrence of V over S (Definition 12, (ii))), that there must exist a time $t \in (0,\tau]$ with $V(\phi(t,x)) \leq V(x) \leq \beta < \alpha$. Moreover, since by the Containment Lemma (Lemma 2) , $\mathcal{R}^\tau(\Omega_\beta) \subset B_r(x^*)$, it must be the case that $\phi(t,x) \in \Omega_\beta$ since otherwise one would necessarily have $V(\phi(t,x)) > \beta$ (contradiction). Thus, by the implication $(ii) \implies (i)$ of Lemma 1, Ω_β is τ-recurrent. It follows then from Corollary 1 (since Ω_β) is compact) $\mathcal{R}^\tau(\Omega_\beta)$ is a compact invariant satisfying

$$\mathcal{R}^\tau(\Omega_\beta) \subset B_r(x^*) \subseteq B_\varepsilon(x^*).$$

Finally, by choosing $\delta > 0$ small enough such that $B_\delta(x^*) \subset \Omega_\beta$, stability (Definition 2) follows. □

V. Asymptotic Stability

Now that we have proven stability with RLFs, we wish to expand the theory to encapsulate asymptotic stability. Similarly to Lyapunov's Direct Method, the extension essentially consists of strengthening the condition from a non-strict inequality to a strict one.

Definition 13 (Strict Recurrent Lyapunov Function (S-RLF))**.** *Given an equilibrium $x^* \in D$ and a set $S \subseteq D$ satisfying $x^* \in \mathrm{int}(S)$. We say that a continuous function $V : D \to \mathbb{R}_{\geq 0}$ is a **Strict Recurrent Lyapunov Function** over the set S, if the following holds:*

*(i) V is an **RLF**, i.e., Definition 12.*
*(ii) V is **strictly τ-recurrent** over S, that is, there is a locally bounded $\tau : S \to \mathbb{R}_{>0}$ such that*

$$\min_{s \in T_S(x;\tau(x))} V(\phi(s,x)) < V(x), \quad \forall x \in S\backslash\{x^*\}. \tag{7}$$

Since a Strict RLF is also an RLF, all the properties from our previous section hold for S-RLFs, particularly, Lemma 3 (S-restricted sub-level sets, $V_{\leq c} \cap S$, are τ-recurrent), which is the building block of Theorem 1. The key difference here is that the strict inequality in (7), acts as a substitute of the typical Lyapunov condition $\dot{V}(x) < 0$, $\forall x \in S\backslash\{x^*\}$, which leads to the strict decrease of $V(\phi(t,x))$ as a function of time. The next lemma shows that our new condition only requires decrements every so often.

Lemma 4. *Let $x^* \in D$ be an equilibrium and $S \subseteq D$ a compact set satisfying $x^* \in \mathrm{int}(S)$. Then, if $V : D \to \mathbb{R}_{\geq 0}$ is a Strict RLF, the following holds:*

(i) The solution $\phi(t,x)$ is bounded and forward complete for all $x \in S$.
(ii) Given $x \in S$, there exists a sequence of time instances $\{t_n\}_{n\in\mathbb{N}}$, such that for all $n \in \mathbb{N}$,

$$\lim_{n\to\infty} t_n = \infty \quad \textit{and} \quad t_{n+1} - t_n \in (0,\tau]\,, \tag{8}$$

with $\phi(t_n,x) \in S$, and whenever $x \in S\backslash\{x^*\}$,

$$V(\phi(t_{n+1},x)) < V(\phi(t_n,x)) < V(x). \tag{9}$$

Proof. Since S is compact and V is an S-RLF $\Rightarrow$ RLF, by Lemma 3, we conclude that for any $c \geq 0$ the set $V_{\leq c} \cap S$ is τ-recurrent. It follows then, by Corollary 1, that $\phi(t,x)$ is bounded and forward complete $\forall x \in S$, i.e., claim (i). Further, since $\phi(t,x^*) = x^* \in S$ $\forall t$, (ii) trivially holds. Thus, we assume from now on that $x \in S\backslash\{x^*\}$, and by uniqueness of solutions that $\phi(t,x) \neq x^*$ forall $t \geq 0$.

The inductive construction of the time sequence for the remaining cases is analogous to the proof of Lemma 1. Let $\tau = \sup_{x\in S} \tau(x)$, and consider $x \in S\backslash\{x^*\}$. For the base case, let $t_0 = 0$ so that $\phi(t_0,x) = x$ and $V(\phi(t_0,x)) = V(x)$, and choose

$$t_1 = \max\{\arg\min_{t\in T_S(x;\tau)} V(\phi(t,x))\};$$

note that the minimum exists by hypothesis (7); if there are multiple minimizing times, t_1 is defined as the largest. By construction then, $t_1 - t_0 \in T_S(x;\tau) \subseteq (0,\tau]$, $\phi(t_1,x) \in S$, and $V(\phi(t_1,x)) < V(x)$ as desired.

The inductive construction proceeds in a similar manner: given $t_1 < t_2 < \cdots t_n$, with $x_n := \phi(t_n,x) \in S$, define

$$t_{n+1} - t_n = \max\{\arg\min_{s\in T_S(x_n;\tau)} V(\phi(s,x_n))\}. \tag{10}$$

Note that $t_{n+1} - t_n \in T_S(x_n;\tau) \subseteq (0,\tau]$ as required. Also,

$$x_{n+1} := \phi(t_{n+1},x) = \phi(t_{n+1} - t_n, x_n) \in S\backslash\{x^*\}\,,$$

and satisfies $V(x_{n+1}) < V(x_n)$ by (7), so we verify (9).

It remains to show that $t_n \to \infty$, which we argue by contradiction. If, instead, the strictly increasing sequence of times was bounded, we would have $t_n \uparrow t^*$. Note that by continuity of $\phi(\cdot,x)$ and compactness of S, $x_n = \phi(t_n,x) \to \phi(t^*,x) \in S$. Further, since $\phi(t,x) \neq x^*$ for all $t \geq 0$, $\phi(t^*,x) \in S\backslash\{x^*\}$. Finally, continuity of V we have:

$$v_n := V(\phi(t_n,x)) \overset{n\to\infty}{\longrightarrow} V(\phi(t^*,x)) =: v^*.$$

Since $\{v_n\}$ is strictly decreasing we conclude that $v^* < v_n$ for all $n \in \mathbb{N}$. Now pick n such that $t_n \geq t^* - \tau$. Since $\phi(t^*,x) \in S$, it follows that $s^* := t^* - t_n \in (0,\tau]$ is in the feasible set for the minimization in (10), which by definition gives as minimum v_{n+1}, achieved at $t_{n+1} - t_n$. Now, since $v^* = V(\phi((s^*,x_n)) < v_{n+1}$, this means s^* achieves a smaller solution in (10) than $t_{n+1} - t_n$, which contradicts with the definition of $t_{n+1} - t_n$. Thus the sequence must be divergent, establishing claim (ii).

□

By utilizing Lemma 4 and Theorem 1 we are able to demonstrate asymptotic stability.

Theorem 2 (Asymptotic Stability)**.** *Let Assumption 1 hold. Consider an equilibrium point $x^* \in D$ of* (1) *and a compact set $S \subseteq D$ satisfying $x^* \in \mathrm{int}(S)$. Then, if $V : D \to \mathbb{R}_{\geq 0}$ is an S-RLF over S, the equilibrium x^* is asymptotically stable on the set S.*

Proof. The stability requirement is already established by Theorem 1 and the fact that an S-RLF is also an RLF. Thus, we are only left to show the attractivity of x^* on the set S.

Following Lemma 4, for any $x \in S$, $\phi(t,x)$ is forward complete. Moreover, for any $x \in S\backslash\{x^*\}$, we can construct a sequence of times t_n and points $x_n = \phi(t_n,x) \in S\backslash\{x^*\}$ satisfying (8) and (9). Now, let $v_n := V(x_n)$ be the *strictly* decreasing sequence that follows from (9), and $\bar{v}$ be its limit, which exists since $v_n > 0$ for all n. It follows then that, $v_n > \bar{v} \geq 0$ for all n. Since $\{x_n\}$ is bounded in $\mathbb{R}^n$, we may take a convergent subsequence $x_{n_k} \overset{k\to\infty}{\longrightarrow} \bar{x}$. Since S is compact and $x^* \in \mathrm{int}(S)$, $\bar{x} \in S$. By continuity, $\bar{v} = V(\bar{x})$.

Suppose $\bar{v} > 0$, so $\bar{x} \neq x^*$. Let $\tau = \sup_{x\in S} \tau(x)$. Then, by the strict τ-recurrence of V (c.f. (7)), there exists $\bar{s} \in (0,\tau]$ satisfying $V(\phi(\bar{s},\bar{x})) < \bar{v}$ and $\phi(\bar{s},\bar{x}) \in S$. In fact, we must have $\phi(\bar{s},\bar{x}) \in S\backslash\{x^*\}$ by uniqueness and the assumption $\bar{x} \neq x^*$. Note that by continuity

$$V(\phi(\bar{s},x_{n_k})) \overset{k\to\infty}{\longrightarrow} V(\phi(\bar{s},\bar{x})) < \bar{v}. \tag{11}$$

However, by construction of the sequence according to Lemma 4, we have that

$$v_{n_k+1} = \min_{s\in T_S(x;0,\tau)} V(\phi(s,x_{n_k})) \leq V(\phi(\bar{s},x_{n_k})). \tag{12}$$

Combining (11) and (12) for k large enough we have

$$v_{n_k+1} \leq V(\phi(\bar{s},x_{n_k})) < \bar{v}$$

which leads to a contradiction since by construction of the sequence v_n we must have $v_{n_k+1} > \bar{v}$. Therefore we have shown that $v_n = V(\phi(t_n,x)) \overset{n\to\infty}{\longrightarrow} 0$.

An immediate consequence of (5) is that $x_n = \phi(t_n, x) \overset{n\to\infty}{\longrightarrow} x^*$. As such, $r_n := \|x_n - x^*\|$ a decreasing sequence with $\lim_{n\to\infty} r_n = 0$. To prove attractivity, let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} r_n = 0$, we can select an $N > 0$ be such that $r_N + F_{r_N} h(\tau; L) < \varepsilon$. For any $t > t_N$, by applying the Containment Lemma to $B_{r_N}(x^*)$, we have

$$\|\phi(t,x) - x^*\| \leq r_N + F_{r_N} h(\tau; L) < \varepsilon.$$

Since this argument holds for any $\varepsilon > 0$, we have that $\lim_{t\to\infty} \phi(t, x) = x^*$, proving attractivity.

□

Theorem 2 demonstrates, how by enforcing the required property (strict decrement the values of $V(\phi(t, x))$) on an infinite sequence of times with bounded difference (t_n, $t_{n+1} - t_n \in (0, \tau]$), it is possible to conclude properties of the entire trajectory. This principle seems to be rather general, and will be further exploited in the next section to prove exponential stability. Finally, we further point out that while we require S to be compact in Theorem 2 the results can easily extend to a global setting.

Corollary 2 (Global Asymptotic Stability). *Let Assumption 1 hold. Consider an equilibrium point $x^* \in D$ of* (1)*. Then, if $V : D \to \mathbb{R}_{\geq 0}$ is an S-RLF over D, and has **compact sub-level sets** $V_{\leq c} \subset D$, $\forall c \geq 0$, then the equilibrium x^* is globally asymptotically stable.*

Proof. Pick any $x \in D$ and let $c := V(x)$. Since V is S-RLF over D, property (7) implies there exists $t' \in (0, \tau(x)]$ satisfying

$$V(\phi(t', x)) < V(x) = c.$$

In fact, the same property holds for all $x \in V_{\leq c} \subset D$. It therefore follows that V is an S-RLF over $S := V_{\leq c}$, and since $V_{\leq c}$ is compact we can apply Theorem 2 to claim stability of $x^* \in V_{\leq c}$ and attractivity of $\phi(t, x)$. Finally, since x was chosen arbitrarily within D, results follows. □

VI. Exponential Stability

In the previous section, we showed that strictly τ-recurrent functions sequentially constrain trajectories, enforcing the desired condition (attractivity) at discrete times. We further showed that this was sufficient to ensure the same condition along the entire trajectory (for all subsequences). We now move towards exponential stability. As before, we seek conditions on the function V that enable us to enforce exponential convergence at discrete, recurrent times.

Typically, exponential stability in classical Lyapunov analysis is verified by ensuring an exponential decrease in the Lyapunov function along trajectories. More precisely, a common integral form condition is given by:

$$V(\phi(t, x)) \leq e^{-\alpha t} V(x), \quad \forall t \geq 0,$$

for some positive constant α. Such a condition tightly couples the geometry of V to trajectories, significantly complicating its verification and limiting practical applicability. To alleviate this, we now introduce a relaxed definition—*Exponential Recurrent Lyapunov Functions (E-RLFs)*—which relaxes exponential Lyapunov functions.

Definition 14 (Exponential Recurrent Lyapunov Function (E-RLF)). *Given an equilibrium point $x^* \in D$ of* (1) *and a set $S \subseteq D$ satisfying $x^* \in \mathrm{int}(S)$. We say that a* continuous *function $V : D \to \mathbb{R}_{\geq 0}$ is an **Exponential Recurrent Lyapunov Function** over the set S if the following properties hold:*

*(i) V is **positive definite and linearly contained** around x^*, that is, there exist constants $a_1, a_2 > 0$ such that*

$$a_1\|x - x^*\| \leq V(x) \leq a_2\|x - x^*\|, \ \forall x \in S. \quad (13)$$

*(ii) V is **α-exponentially τ-recurrent** over S, that is, there exist a locally bounded function $\tau : S \to \mathbb{R}_{>0}$ and a positive constant α such that*

$$\min_{s\in T_S(x;\,\tau(x))} e^{\alpha s} V(\phi(s, x)) \leq V(x), \ \forall x \in S. \quad (14)$$

Note that an α-exponentially τ-recurrent function is always strictly τ-recurrent, but not the other way around. We will use (14) to control (exponentially decreasing) upper and lower bounds of $V(\phi(t, x))$ as $t \to \infty$.

Theorem 3 (Exponential Stability). *Consider an equilibrium point $x^* \in D$ of* (1)*, and a compact set $S \subseteq D$ satisfying $x^* \in \mathrm{int}(S)$. Suppose Assumption 1 holds, and let $V : D \to \mathbb{R}_{\geq 0}$ be an Exponential Recurrent Lyapunov Function over the set S. Then, the equilibrium x^* is exponentially stable with rate α on the set S. In particular, for every $x \in S$ and every $t \geq 0$, it holds that*

$$\|\phi(t, x) - x^*\| \leq C\, e^{-\alpha t}\|x - x^*\|, \quad (15)$$

with $C := \frac{a_2}{a_1} e^{\alpha\tau}(1 + \bar{L}h(\tau; L))$, $\tau = \sup_{x\in S} \tau(x)$, $L := L_{R^\tau(S)}$, and $\bar{L} := \bar{L}_{R^\tau(S)}$.

Proof. Let τ, L, and $\bar{L}$ be as defined in the theorem statement, and pick any $x \in S$. Since E-RLF⇒S-RLF⇒RLF, it follows from Lemma 4, that $\phi(t, x)$ is bounded and forward complete. Moreover, a similar construction to that of Lemma 4 also defines a sequence $\{t_n\}_{n\in\mathbb{N}}$, with $t_0 = 0$, $\lim_{n\to\infty} t_n = \infty$ and $t_{n+1} - t_n \in (0, \tau]$, $\forall n$, such that

$$e^{\alpha t_{n+1}} V(\phi(t_{n+1}, x)) \leq e^{\alpha t_n} V(\phi(t_n, x)) \leq V(x), \ n \geq 1. \quad (16)$$

Using (13) and (16) we deduce that, for $n \geq 1$ we

$$\|\phi(t_n, x) - x^*\| \leq \frac{V(\phi(t_n, x))}{a_1} \leq \frac{e^{-\alpha t_n}}{a_1} V(x).$$

Let now $r_n := \frac{e^{-\alpha t_n}}{a_1} V(x)$ and $B_n := B_{r_n}(x^*) \cap S$. It follows from applying Lemma 2 on the compact set $B_n \subset S$ that

$$\|\phi(t, x) - x^*\| \leq r_n + F_{r_n} h(\tau; L), \ \forall t \in (t_n, t_{n+1}].$$

Furthermore, since by Assumption 1, f is $\bar{L}$-Lipschitz on $B_n \subset S$, and $f(x^*) = 0$ we have $F_{r_n} \leq \bar{L} r_n$, leading to

$$\begin{aligned}\|\phi(t, x) - x^*\| &\leq r_n(1 + \bar{L}h(\tau; L)) \\ &\leq \frac{e^{-\alpha t_n}}{a_1}(1 + \bar{L}h(\tau; L))V(x)\end{aligned}$$

for all $t \in (t_n, t_{n+1}]$ Note, further, that $t \leq t_{n+1} \leq t_n + \tau$, therefore $-t_n \leq \tau - t$ so $e^{-\alpha t_n} \leq e^{\alpha\tau} e^{-\alpha t}$, leading to

$$\|\phi(t,x) - x^*\| \leq e^{\alpha\tau} \frac{e^{-\alpha t}}{a_1}(1 + \bar{L}h(\tau; L))V(x).$$

Moreover, since the last bound is independent of n, and n was chosen arbitrarily, it must hold for all $t \geq 0$. Finally, applying the upper bound $V(x) \leq a_2\|x - x^*\|$ we establish (15) for any $x \in S$. □

The above theorem demonstrates the exponential stability of an equilibrium point x^* on a compact set S. Note though, that while the constant C is independent of x, as usually required, it does depend on the compact set S through τ, L and $\bar{L}$. This make the extension for global exponential stability slightly more involved. We provide here a restricted extension under additional assumptions.

Corollary 3 (Global Exponential Stability). *Consider the system* (1)*, and suppose the following hold:*

(i) The domain $D \subseteq \mathbb{R}^n$ is forward invariant, and the vector field f is globally Lipschitz on D with Lipschitz constant $\bar{L}$.
(ii) The function $V : D \to \mathbb{R}_{\geq 0}$ is an Exponential Recurrent Lyapunov Function over D with compact sublevel sets.
(iii) The recurrence time function $\tau : D \to \mathbb{R}_{>0}$ is globally bounded: $\sup_{x\in D} \tau(x) \leq \bar{\tau} < \infty$.

Then the equilibrium point x^* is globally exponentially stable. In particular, for all $x \in D$ and $t \geq 0$,

$$\|\phi(t,x) - x^*\| \leq C\, e^{-\alpha t}\|x - x^*\|,$$

where the constant $C := \frac{a_2}{a_1} e^{(\bar{L}+\alpha)\bar{\tau}} > 0$.

Proof. Let $x \in D$ be arbitrary, and define the compact sublevel set $S := V_{\leq V(x)} \subset D$. Since V is an Exponential Recurrent Lyapunov Function over D, it satisfies the ERLF conditions over its sublevel set S as well.

By Theorem 3, the exponential stability bound holds on S with:

$$\|\phi(t,x) - x^*\| \leq C_S e^{-\alpha t}\|x - x^*\|,$$

where the constants $C_S := \frac{a_2}{a_1} e^{\alpha\tau_S}(1 + \bar{L}_S h(\tau_S; L_S))$ and $\tau_S := \sup_{x\in S} \tau(x) \leq \sup_{x\in D} \tau(x) \leq \bar{\tau}$.

By assumption, since f is globally Lipschitz on D with constant $\bar{L}$, both the standard and one-sided Lipschitz constants over $R^{\tau_S}(S) \subseteq D$ satisfy $L_S \leq \bar{L}$ and $\bar{L}_S \leq \bar{L}$.

Moreover, the function $h(\tau, L) := \frac{e^{L\tau}-1}{L}$ is strictly increasing in both τ and L for $\tau > 0$. Therefore, we conclude that

$$C_S \leq \frac{a_2}{a_1} e^{\alpha\bar{\tau}}(1 + \bar{L}h(\bar{\tau}; \bar{L})) = C.$$

Since this bound is independent of the particular choice of $x \in D$, we obtain:

$$\|\phi(t,x) - x^*\| \leq Ce^{-\alpha t}\|x - x^*\|, \quad \forall x \in D,\ \forall t \geq 0.$$

□

We conclude this section by relaxing the requirement in Theorem 3 that $x^* \in \text{int}(S)$, which leads to an ultimate boundedness condition that we now formally define.

Definition 15 (Exponential Ultimate Boundedness). *The solutions of system* (1) *are said to be* exponentially ultimately bounded *over a set $S \subset D$ with rate $\alpha > 0$ and bound $\varepsilon' > 0$ if there exists $C > 0$ such that for every $x \in S$ and all $t \geq 0$,*

$$\|\phi(t,x) - x^*\| \leq Ce^{-\alpha t}\|x - x^*\| + \varepsilon'. \tag{17}$$

To show this new result we are required to define a slight variation for the exponential τ-recurrent condition in (14).

Definition 16 (α-Exponential τ-Recurrence over S relative to S'). *Let $S, S' \subseteq D$, with $S \subseteq S'$. We say that a function $V : D \to \mathbb{R}_{\geq 0}$ is* α-exponentially τ-recurrent over S relative to S' *if there exists a locally bounded function $\tau : D \to \mathbb{R}_{>0}$ and a constant $\alpha > 0$ such that:*

$$\min_{s\in T_{S'}(x;\tau(x))} e^{\alpha s} V(\phi(s,x)) \leq V(x), \quad \forall x \in S.$$

Definition 17 (Relative Exponential Recurrent Lyapunov Function). *Let $S \subseteq S' \subseteq D$. We say that a continuous function $V : D \to \mathbb{R}_{\geq 0}$ is an* Exponential Recurrent Lyapunov Function over S relative to S' *if:*

(i) V is linearly contained, i.e., (13)*, for all $x \in S'$,*
(ii) V is α-exponentially τ-recurrent over S relative to S'.

Theorem 4 (Ultimate Boundedness). *Consider an equilibrium point $x^* \in D$ of* (1)*. Suppose Assumption 1 holds, and let $V : D \to \mathbb{R}_{\geq 0}$ be an E-RLF over a compact set $S \subseteq D$ relative to $S \cup B_\varepsilon(x^*)$, for some $\varepsilon > 0$ such that*

$$x^* \in \text{int}(S \cup B_\varepsilon(x^*)), \quad \textit{and} \quad \partial B_\varepsilon(x^*) \subseteq S.$$

Then, the solutions of (1) *are exponentially ultimately bounded over $S \cup B_\varepsilon(x^*)$, i.e.,* (17)*, with rate α and bound*

$$\varepsilon' := \varepsilon + F_\varepsilon h(\bar{\tau}; L), \tag{18}$$

where $\bar{\tau} := \frac{1}{\alpha}\log(C)$, $C := \frac{a_2}{a_1} e^{\alpha\tau}(1 + \bar{L}h(\tau; L))$, $\tau = \sup_{x\in S} \tau(x)$, $L := L_{R^\tau(S)}$, and $\bar{L} := \bar{L}_{R^\tau(S)}$.

Proof. Let $x \in S \cup B_\varepsilon(x^*)$. Assume first $x \in S$. Since V is an E-RLF over S relative to $S \cup B_\varepsilon(x^*)$, it follows from Theorem 3 that the trajectory satisfies

$$\|\phi(t,x) - x^*\| \leq Ce^{-\alpha t}\|x - x^*\|, \quad \forall t \in [0, t'] \tag{19}$$

where t' is the first time instance when $\phi(t', x) \in B_\varepsilon(x^*)$.

Now, given any $x' \in B_\varepsilon(x^*)$ since $\partial B_\varepsilon(x^*) \subset S$, if $\phi(t, x')$ leaves $B_\varepsilon(x^*)$, it must come back to it. In fact, using again (19), at most, it will take $\bar{\tau} = \frac{1}{\alpha}\log(C)$ to get back to $B_\varepsilon(x^*)$. It follows then from Lemma 1 that $B_\varepsilon(x^*)$ is $\bar{\tau}$-recurrent. By Lemma 2, any excursion from $B_\varepsilon(x^*)$ remains within distance ε' of x^*, where ε' is given in (18).

Thus, any trajectory starting from S either decays exponentially while in S, or remains within the uniform bound ε' once it enters $B_\varepsilon(x^*)$. Hence, for all $t \geq 0$,

$$\|\phi(t,x) - x^*\| \leq Ce^{-\alpha t}\|x - x^*\| + \varepsilon',$$

establishing exponential ultimate boundedness over $S \cup B_\varepsilon(x^*)$. □

VII. Norm-Based Converse Theorems

We are now ready to show that the recurrence conditions developed in the previous sections are naturally satisfied by standard norms under classical stability assumptions. In particular, we demonstrate that if a system is asymptotically or exponentially stable, then any norm satisfies the corresponding recurrence inequality over compact subsets of the domain. We begin by showing that any norm satisfies the τ-recurrence condition over compact subsets of the domain of attraction when the system is asymptotically stable.

Theorem 5 (Asymptotic Stability Implies Norm is S-RLF)**.** *Given system* (1)*. Let $x^* \in D$ be an asymptotically stable equilibrium on a compact set $S \subseteq D$ (Definition 4) satisfying $x^* \in \text{int}(S)$. Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. Then, the function $V(x) := \|x - x^*\|$ is a Strict Recurrent Lyapunov Function (S-RLF) over S.*

Proof. The proof follows closely the results in Remark 2/4 of [21]. Let $V(x) := \|x-x^*\|$. Since x^* is asymptotically stable on S, standard converse Lyapunov arguments (e.g., Lemma 4.5 in [5]) guarantee the existence of a class $\mathcal{KL}$ function β such that for all $x \in S$:

$$V(\phi(t,x)) \leq \beta(V(x), t), \quad \forall t \geq 0.$$

Now pick an arbitrary constant $\mu \in (0,1)$ and $x \in S$. Since $x^* \in \text{int}(S)$ and is asymptotically stable on S, there exist $\delta > 0$ satisfying $B_\delta(x^*) \subseteq S$, and a finite time $t_0(x) > 0$, such that

$$\phi(t,x) \in B_\delta(x^*), \quad \forall t \geq t_0(x).$$

Now, for each $x \in S \setminus \{x^*\}$, define explicitly:

$$\tau(x) := \min\{t \geq t_0(x) : \beta(V(x), t) \leq \mu V(x)\}.$$

Since $t_0(x) > 0$, the function $\beta(s,t)$ is continuous, strictly decreasing in t, and satisfies $\lim_{t\to\infty} \beta(s,t) = 0$, it follows that this minimum exists, is unique, and strictly positive. Furthermore, continuity of β combined with compactness of S ensures $\tau(x)$ is locally bounded on S.

Clearly, by construction, we have for all $x \in S \setminus \{x^*\}$:

$$V(\phi(\tau(x), x)) \leq \beta(V(x), \tau(x)) \leq \mu V(x) < V(x),$$

thus satisfying the strictly decreasing recurrence condition (7). Therefore, the function $V(x) = ||x - x^*||$ is strictly τ-recurrent over the compact set S. Finally, since by definition a norm is always proper, it follows from Definition 13 that $V(x) = ||x - x^*||$ is an S-RLF. ☐

Having established that asymptotic stability implies strict τ-recurrence of standard norms, we now turn our attention to exponential stability. As mention, exponential stability will lead to norms satisfying an E-RLF condition, providing one gives some slack on the exponential rate that can be verified.

Theorem 6 (Exponential Stability Implies Norm is E-RLF)**.** *Consider system* (1)*, and let $x^* \in D$ be a λ-exponentially stable equilibrium on a compact set S, i.e., Definition 5, satisfying $x^* \in \text{int}(S)$. Then for any $0 < \alpha < \lambda$, the function $V(x) := \|x - x^*\|$ is an Exponential Recurrent Lyapunov Function (E-RLF) on S for any l.b. function $\tau(x)$ satisfying*

$$\tau(x) \geq \tau := \frac{1}{\lambda - \alpha} \ln\left(K \frac{a_2}{a_1}\right), \quad \forall x \in S, \tag{20}$$

where K, λ are given in (2) *and a_1, a_2 are positive constants satisfying: $B_{a_1}(x^*) \subseteq S \subseteq B_{a_2}(x^*)$.*

Proof. Let $V(x) := \|x - x^*\|$, where $\|\cdot\|$ is the norm of Definition 5. By hypothesis, the equilibrium x^* is λ-exponentially stable over the compact set S, implying that for all $x \in S$ and all $t \geq 0$, $\|\phi(t,x) - x^*\| \leq Ke^{-\lambda t}\|x - x^*\|$.

Therefore it follows that for all $t \geq \tau$

$$\begin{aligned}&\|\phi(t,x) - x^*\| \leq Ke^{-\lambda t}\|x - x^*\| \\ &= Ke^{-\lambda \frac{1}{\lambda-\alpha}\ln\left(K\frac{a_2}{a_1}\right)}||x - x^*|| \\ &\leq Ke^{-\lambda \frac{1}{\lambda}\ln\left(K\frac{a_2}{a_1}\right)}||x - x^*|| \leq \frac{a_1}{a_2}||x - x^*|| \leq a_1\end{aligned}$$

implying that $\phi(\tau, x) \in B_{a_1}(x^*) \subset S$.

Analogously, for all $x \in S$ and τ as in (20) we have:

$$\begin{aligned}e^{\alpha\tau}\|\phi(\tau,x) - x^*\| &\leq Ke^{(\alpha-\lambda)\tau}\|x - x^*\| \\ &\leq \frac{a_1}{a_2}\|x - x^*\| \leq \|x - x^*\|\end{aligned}$$

It therefore follows that for any locally bounded function $\tau(x)$ satisfying (20) we must have

$$\begin{aligned}\min_{t\in T_S(x;\tau(x))} e^{\alpha t}\|\phi(t,x) - x^*\| &\leq e^{\alpha\tau}\|\phi(\tau,x) - x^*\| \\ &\leq \|x - x^*\|,\end{aligned}$$

which means that $V(x) = \|x - x^*\|$ is an E-RLF over S, as desired. ☐

VIII. Verification of Exponential RLFs

So far, we have defined E-RLFs and provided theoretical guarantees of exponential stability, under the assumption that the exponential τ-recurrence condition (14) is satisfied. Furthermore, we established that standard norms themselves are E-RLFs. We now leverage these insights to provide a practical mechanism to verify condition (14) explicitly using trajectory data. Our development in this section is closely related to the work on topological entropy of dynamical systems [] as well as its extensions to control [].

In particular, in order to certify a specific behavior (α-exponential τ-recurrence) over a set S, we will first focus on using a trajectory $\phi(t,x)$ of fixed duration $[0,\tau]$ to certify such behavior over a neighborhood $B_\varepsilon(x)$ (Section VIII-A). We will then extend our method to verify said behavior on a set S. For reasons that will become clear later, our focus will be on sets S satisfying the E-RLF conditions for exponential ultimate boundedness of trajectories around an equilibrium x^*.

A. Trajectory-based Verification of a Neighborhood

We start by deriving conditions to verify condition (14) around a neighborhood of a trajectory.

Theorem 7 (Trajectory-based Verification of E-RLF)**.** *Consider the system* (1)*, an equilibrium point $x^* \in D$, a compact* *set $S' \subseteq D$, and constants $\varepsilon > 0$, $\alpha > 0$, and $\tau > 0$. Assume that $L = L_{\mathcal{R}^\tau(S')}$ and that for some $x \in S'$, with $B_\varepsilon(x) \subseteq S'$, there exists $t \in (0, \tau]$ satisfying simultaneously:*

$$e^{\alpha t}\big(\|\phi(t,x) - x^*\| + \varepsilon e^{Lt}\big) \le \|x - x^*\| - \varepsilon, \tag{21a}$$

$$\mathrm{sd}(\phi(t,x), S') + \varepsilon e^{Lt} \le 0, \tag{21b}$$

where $\mathrm{sd}(\cdot, S')$ is the signed distance w.r.t S' based on the same norm $\|\cdot\|$. Then, the function $V(x) = \|x - x^*\|$ is an E-RLF over $B_\varepsilon(x)$ relative to S'.

Proof. Let $x \in S'$ be such that $B_\varepsilon(x) \subseteq S'$, and define $V(\cdot) := \|\cdot - x^*\|$. Let $\tau > 0$ and $L := L_{\mathcal{R}^\tau(S')}$, as in the statement.

Since system (1) is L-one-sided-Lipschitz on $\mathcal{R}^\tau(S')$, for all $y \in B_\varepsilon(x)$ and all $t \in [0, \tau]$, we have:

$$\|\phi(t,x) - \phi(t,y)\| \le e^{Lt}\|x - y\| \le \varepsilon e^{Lt}$$
$$\implies \|\phi(t,y) - x^*\| \le \|\phi(t,x) - x^*\| + \varepsilon e^{Lt}.$$

Let $t^* \in (0, \tau]$ be a time that satisfies the verification condition (21a)–(21b) at point x. In particular,

$$e^{\alpha t^*}\left(\|\phi(t^*,x) - x^*\| + \varepsilon e^{Lt^*}\right) \le \|x - x^*\| - \varepsilon.$$

Now, fix any $y \in B_\varepsilon(x)$. Then:

$$\begin{aligned} e^{\alpha t^*}\|\phi(t^*,y) - x^*\| &\le e^{\alpha t^*}\left(\|\phi(t^*,x) - x^*\| + \varepsilon e^{Lt^*}\right) \\ &\le \|x - x^*\| - \varepsilon \le \|y - x^*\|, \end{aligned}$$

where the last inequality holds since $y \in B_\varepsilon(x)$. This shows that $||\cdot - x^*||$ would satisfies the exponential decrease condition (14) at every point $y \in B_\varepsilon(x)$ provided that $\phi(t^*, y) \in S'$.

To complete the argument, assume that $\phi(t^*, y) \notin S'$ for some $y \in B_\varepsilon(x)$. By the assumptions of the theorem, $\mathrm{sd}(\phi(t^*,x), S') \le 0$ which implies $\phi(t^*,x) \in S'$. Now let $y_p \in \partial S'$ be such that $y_p = \lambda\phi(t^*,x) + (1-\lambda)\phi(t^*,y)$ for $\lambda \in [0,1]$. It follows then by Grönwall's inequality (c.f [5], Lemma A.1), that

$$\begin{aligned} \varepsilon e^{Lt} &\ge \|\phi(t^*,y) - \phi(t^*,x)\| \\ &= ||\phi(t^*,y) - y_p|| + ||\phi(t^*,x) - y_p|| \\ &\ge \inf_{z \in \partial S'} ||\phi(t^*,y) - z|| + \inf_{z \in \partial S'} ||\phi(t^*,x) - z|| \\ &= \mathrm{sd}(\phi(t^*,y), S') - \mathrm{sd}(\phi(t^*,x), S') \end{aligned}$$

Therefore we conclude that

$$\mathrm{sd}(\phi(t^*,y), S') \le \mathrm{sd}(\phi(t^*,x), S') + \varepsilon e^{Lt^*} \le 0,$$

contradicting the assumption that $\phi(t^*,y) \notin S'$, since by compactness of S' implies $\mathrm{sd}(\phi(t^*,y), S') > 0$.

Thus, $\phi(t^*,y) \in S'$ for all $y \in B_\varepsilon(x)$, and therefore $V(y) := \|y - x^*\|$ is an Exponential Recurrent Lyapunov Function (E-RLF) over $B_\varepsilon(x)$ relative to S'. □

Theorem 7 provides a mechanism to verify that every point $y \in B_\varepsilon(x)$ will satisfy condition (14). We aim to leverage this conditions to verify the Relative E-RLF property of a norm $\|\cdot\|$ over S relative to S', $S \subseteq S'$. It is therefore natural to consider an ε-cover[1] of such set. However, this approach poses two main challenges. First, condition (21a) cannot be satisfied for any point within $B_\varepsilon(x^*)$, which requires ε to be small as one gets closer to x^* and is aligned with he well known fact that exponential stability cannot be verify using a finite number of trajectories [30, Example 2.1]. Second, any ε-cover of S will necessarily require $O((R/\varepsilon)^d)$ number of trajectories, making its verification a computationally difficult problem.

[1] An ε-cover of a set S is a collections of balls $B_\varepsilon(x_i)$ s.t. $S \subseteq \cup_i B_\varepsilon(x_i)$

B. Trajectory-based Verification of a Region

In this section we overcome the above mentioned problems by focusing on the verification of sets of the form $S = \mathrm{cl}(B_R(x^*) \backslash B_\varepsilon(x^*))$ and $S' = B_R(x^*)$, which satisfy the conditions of Theorem 4 for exponential ultimate boundedness. This allows us to limit the radius ε of the balls needed to cover S. Moreover, we will allow the radius of such cover to gradually increase as the center of the balls move away from x^*. The result is a strategy that allows to verify the region $S = \mathrm{cl}(B_R(x^*) \backslash B_\varepsilon(x^*))$ using significantly less number of trajectories.

Theorem 8 (Sample Complexity for Verifying E-RLF)**.** *Consider system* (1)*, an equilibrium point $x^* \in D \subseteq \mathbb{R}^d$, and constants $R > 0$, $\varepsilon \in (0, R)$, and $\tau > 0$. Let $L := L_{\mathcal{R}^\tau(B_R(x^*))}$ denote the one-sided Lipschitz constant of* (1) *over the τ-reachable set from $B_R(x^*)$. Suppose that x^* is λ-exponentially stable over $B_R(x^*)$ with constant $K \ge 1$, choose any rate $\alpha \in (0, \lambda)$, and define $\beta := \frac{\lambda - \alpha}{\lambda + L}$, $K_\beta := K^{\frac{1-\beta}{\beta}}$, and assume $\beta \in (0,1)$.*

Then, the function $V(x) := \|x - x^*\|$ can be verified to be an Exponential Recurrent Lyapunov Function (E-RLF) over the set $S = B_R(x^*) \setminus B_\varepsilon(x^*)$, relative to $S' = B_R(x^*)$, using at most

$$N(\varepsilon) \;=\; d \ln\left(\frac{R}{\varepsilon}\right) \frac{e}{\beta}\Big(2\,(2 + K_\beta)\Big)^d = O\left(\log\left(\frac{R}{\varepsilon}\right)\right).$$

trajectory evaluations of duration

$$\tau = \frac{1}{\lambda - \alpha} \ln\left(K(2 + K_\beta)e^{\beta/d}\right).$$

Proof. We construct a layered covering of the region $B_R(x^*) \setminus B_\varepsilon(x^*)$ using concentric annuli A_i centered at x^*, i.e., $A_i := \{x \in B_R(x^*) : R_i \ge \|x - x^*\| \ge R_{i+1}\}$, with $i \in \{0, \dots, N-1\}$, with $R_0 = R$, $R_i = \rho R_{i-1} = \rho^i R$, and $\rho, \mu \in (0,1)$. We aim to cover each layer with verification balls of radius also given by $\varepsilon_i := \mu R_i$. The key idea is to progressively chose different radii ε_i so as to limit the total number of balls, while still guaranteeing that $V(x) := \|x - x^*\|$ satisfies the exponential recurrence condition.

We aim to find values of ρ and μ such that for any given $i \in \{0, \dots, n\}$, the function V is an E-RLF over $B_{\varepsilon_i}(x)$, for all $x \in A_i$, relative to $B_R(x^*)$. By Theorem 7, it is sufficient to show that:

$$\min_{t \in T_{B_R(x^*)}(x;\tau)} e^{\alpha t}\big(\|\phi(t,x) - x^*\| + \varepsilon_i e^{Lt}\big) \le \|x - x^*\| - \varepsilon_i. \tag{23}$$

Since x^* is λ-exponentially stable over $B_R(x^*)$, the left hand side of (23) can be upper-bounded for all $x \in A_i$ by:

$$Ke^{(\alpha-\lambda)\tau}\|x-x^*\|+\mu R_i e^{(\alpha+L)\tau} \le R_i(Ke^{(\alpha-\lambda)\tau}+\mu e^{(\alpha+L)\tau}).$$

Now, using the fact that $\|x - x^*\| \ge R_{i+1} = \rho R_i$ and $\varepsilon_i = \mu R_i$, it follows that inequality (23) is satisfied whenever:

$$f(\tau, \mu) := Ke^{(\alpha-\lambda)\tau} + \mu e^{(\alpha+L)\tau} \le \rho - \mu. \tag{24}$$

Choosing τ large enough, i.e.,

$$\tau(\mu) = \frac{1}{\lambda - \alpha} \ln\left(\frac{K}{\mu^\beta}\right) > \frac{1}{\lambda - \alpha} \ln(K), \tag{25}$$

ensures the existence of some μ small enough s.t. (24) hold. In particular, since

$$f(\tau(\mu), \mu) = \mu^\beta + \mu\left(\frac{K}{\mu^\beta}\right)^{\frac{1-\beta}{\beta}} = (1 + K_\beta)\mu^\beta$$

and $\beta \in (0, 1)$, it is sufficient

$$\mu(\rho) := \frac{\rho^{1/\beta}}{\sqrt[\beta]{2 + K_\beta}} \tag{26}$$

for (24) to hold, thus ensuring that (23) is satisfied for all i.

Next, we compute the number of annuli required to cover the region $B_R(x^*) \setminus B_\varepsilon(x^*)$. Since the radii follow a geometric progression $R_i = \rho^i R$, we stop when $R_n \le \varepsilon$, which implies

$$\rho^n R \le \varepsilon \quad \Rightarrow \quad n = \ln\left(\frac{R}{\varepsilon}\right) \frac{1}{\ln(\rho^{-1})}. \tag{27}$$

To estimate the number of balls of radius $\varepsilon_i = \mu R_i$ required to cover the annular region A_i, we leverage properties of covering and packing numbers of a set K, i.e., $\mathcal{N}(K, \varepsilon)$ and $\mathcal{P}(K, \varepsilon)$, respectively. In particular, for a given set, it is easy to show [31]:

$$\mathcal{N}(K, 2\varepsilon) \le \mathcal{P}(K, \varepsilon) \le \mathcal{N}(K, \varepsilon) \le \mathcal{P}(K, \tfrac{\varepsilon}{2})$$

as well as

$$\mathcal{P}(K, \varepsilon) \le \frac{\mathrm{vol}(K)}{\mathrm{vol}(B_\varepsilon)} \le \mathcal{N}(K, \varepsilon)$$

It follows then that

$$\begin{aligned} &\mathcal{N}(A_i, \varepsilon_i) \le \mathcal{N}(B_{R_i}, \varepsilon_i) - \mathcal{P}(B_{R_{i+1}}, \varepsilon_i) \\ &\le \mathcal{P}(B_{R_i}, \tfrac{\varepsilon_i}{2}) - \mathcal{N}(B_{R_{i+1}}, 2\varepsilon_i) \le \frac{\mathrm{vol}(B_{R_i})}{\mathrm{vol}(B_{\frac{\varepsilon_i}{2}})} - \frac{\mathrm{vol}(B_{R_{i+1}})}{\mathrm{vol}(B_{2\varepsilon_i})} \\ &= \frac{4^d \mathrm{vol}(B_{R_i}) - \rho^d \mathrm{vol}(B_{R_i})}{2^d \mathrm{vol}(B_{\varepsilon_i})} \le \frac{2^d}{\mu(\rho)^d}, \end{aligned} \tag{28}$$

where in the last step we kept track of constants that depend on d. Summing over all annuli, the total number of trajectories required is

$$n \cdot \mathcal{N}(A_i, \varepsilon_i) \le \frac{2^d}{\mu(\rho)^d} \frac{\ln\left(\frac{R}{\varepsilon}\right)}{\ln(\rho^{-1})} = \frac{\ln\left(\frac{R}{\varepsilon}\right) 2^d (2 + K_\beta)^d}{\rho^{\frac{d}{\beta}} \ln(\rho^{-1})}. \tag{29}$$

where the first step follows from (27) and (28), and the last from (26). Optimizing for the maximum $\rho \in (0, 1)$ in $g(\rho) = \rho^{\frac{d}{\beta}} \ln(\rho^{-1})$ leads to $\rho^* = e^{-\frac{\beta}{d}}$ and $g(\rho^*) = \frac{\beta}{d}\frac{1}{e}$. Substituting ρ^* in (29) and $\mu(\rho^*) = (\frac{e^{-\frac{1}{d}}}{C_\beta + 1})^{\frac{1}{\beta}}$ in (25) finishes the proof. □

Remark 1 (Performance vs Complexity Trade-off)**.** *Theorem 8 highlight the intrinsic trade-off between the performance gap $\lambda - \alpha$ and the sample complexity of the verification process. Notably, when $\lambda - \alpha \to 0^+$, $\beta \to 0^+$, and $K_\beta \to \infty$. However, a constant gap enables us to get sample complexity that is exponentially better thant he naive $O\left(\left(\frac{R}{\varepsilon}\right)^d\right)$.*

Remark 2 (Highly Performing Systems)**.** *When $\lambda \to \infty$, $\beta, K_\beta \to 1$, thus leading to*

$$N(\varepsilon) = O\left(d \ln\left(\frac{R}{\varepsilon}\right) 6^d\right).$$

In this setting, since the trajectories converge very, very fast, one can use arbitrarily small $\tau \to 0^+$.

IX. Numerical Methods

In this section, we build on Theorem 4 to develop parallelizable algorithms for certifying exponential ultimate boundedness around an equilibrium point. Rather than verifying exponential recurrence over a dense set, we use the neighborhood-based condition from Theorem 7 to efficiently certify local balls around individual sample points.

We present two complementary algorithms. The first (Section IX-B) takes a target region $S = B_R(x^*) \setminus B_\varepsilon(x^*)$ and computes the largest value of α for which $V(x) = \|x - x^*\|$ is an ERLF over S relative to $S' = B_R(x^*)$. The second (Section IX-C) takes a fixed decay rate α and incrementally constructs a certified region S where $V(x) := \|x - x^*\|$ is an ERLF relative to $S \cup B_\varepsilon(x^*)$. We validate both algorithms in Sections IX-D and IX-E, respectively. In all of our algorithms below, L, τ, are precomputed ahead of time and globally shared. We further assume here that $x^* = 0$, and $V(x) = \|x\| := \max_{i\in[n]} |x_i|$, i.e., the ℓ_∞-norm.

A. Supporting Methods and Algorithms

In this section we introduce a set of supporting routines that are used in our methods.

1) Verifying a Ball at a Fixed Point: Given a grid point $x \in \mathbb{R}^d$ and a radius $r > 0$, we first build a routine that determines the largest rate α for which the ball $B_r(x)$ satisfies the exponential recurrence condition (21). We formulate this as a simple one-dimensional optimization over α, based on Theorem 7. Note that checking (21b) is not necessary when S' is a sub-level set of $V(x)$. Also, failure to certify is implied by $\alpha_{\max}(x, r; S') < 0$, and in particular, when (21b) fails $\alpha_{\max}(x, r; S') = -\infty$. For simplicity, we use $\alpha_{\max}(x, r) := \alpha_{\max}(x, r; B_R(x^*))$.

Algorithm 1: $\alpha_{\max}(x, r; S')$ — Maximum α certifying exponential recurrence for $B_r(x)$

Input: Center $x \in \mathbb{R}^d$, radius $r > 0$
Output: Maximum α such that $B_r(x)$ satisfies (21)
Find $\alpha^* = \max\{\alpha \mid \exists t \in (0, \tau]$ s.t. (21a) $\wedge$ (21b);
return α^*;

2) Splitting Failed Points: In cases where Algorithm 1 fails to certify a ball $B_r(x)$ (i.e., $\alpha_{\max}(x,r) < 0$), the failure may result from evaluating the recurrence condition over a region that is too large to satisfy the assumptions of Theorem 7. To improve local resolution, we subdivide $B_r(x)$ into 3^d smaller balls of radius $r/3$, each centered on a uniformly spaced grid. This allows us to recursively refine problematic regions while maintaining full parallelizability. The subdivision procedure is described in Algorithm 2, and an illustration is provided in Figure 1 for the 2D case.

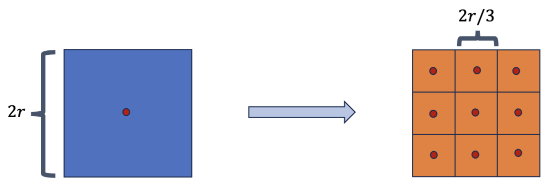

Fig. 1: Splitting a Ball according to Algorithm 2

Algorithm 2: $\text{Split}(x, r)$ — Splitting a ball $B_r(x)$ into 3^d sub-balls

---

Input: Center $x = (x_i) \in \mathbb{R}^d$, radius $r > 0$
Output: List of sub-balls of radius $r/3$
Let $S \leftarrow \{-\frac{2}{3}r, 0, \frac{2}{3}r\}$;
$Y \leftarrow \{(y_i) \in \mathbb{R}^d \mid y_i \in x_i + S, i \in [d]\}$;
$\texttt{splits} \leftarrow \{(y, r/3) \mid y \in Y\}$;
return $\texttt{splits}$;

---

3) Initial Grid Setup: Given a center x^*, inner radius ε, and outer radius R, our goal is to verify exponential recurrence over the region $S := B_R(x^*) \setminus B_\varepsilon(x^*)$. To do this efficiently, we construct a layered grid of candidate points. Specifically, we divide the region into m layers, where the radius of the center point in the ℓ-th layer is given by $r_\ell := 3^{\ell-1}\varepsilon$, for $\ell \in \{1, \dots, m\}$. Each layer contributes $3^d - 1$ points (excluding the origin), so the total number of initial grid points is $\mathcal{O}(3^d m)$.

To ensure that the full annular region S is covered, we choose the number of layers m such that:

$$R \le \varepsilon + \sum_{\ell=1}^{m} 2r_\ell = \varepsilon \left(1 + 2\sum_{\ell=1}^{m} 3^{\ell-1}\right) = 3^m \varepsilon.$$

This construction is exponentially more efficient than a uniform ε-grid, which would require $\mathcal{O}((R/\varepsilon)^d)$ points. A two-layer example is shown in Figure 2.

4) Estimation of L and τ: To apply Theorem 7, we require a one-sided Lipschitz constant L valid over a set S' that contains all trajectories starting from the region $S = B_R(x^*) \setminus B_\varepsilon(x^*)$. To this end, we seek a conservative outer set of the form $S' := B_{R'}(x^*)$ for some $R' > R$. We begin by constructing a uniform grid $G \subset \partial B_R(x^*)$ with spacing at most $\ell > 0$. From each point $x \in G$, we simulate the trajectory $\phi(t, x)$ over the interval $t \in [0, \tau]$, and verify that it reenters $B_R(x^*)$ within time τ. If any trajectory fails to return, τ must be increased.

Next, we compute the maximum excursion of all trajectories: $R_{\max} := \max_{x \in G,\, t \in [0,\tau]} \|\phi(t,x) - x^*\|$, and set $R' := R_{\max} + \delta$, where $\delta > 0$ is a small buffer. Finally, we estimate the one-sided Lipschitz constant $L := \max_{z \in B_{R'}(x^*)} L_z$ using a fine grid. To ensure robustness with respect to perturbations around grid points, we verify the inequality:

$$\max_{t \in (0,\tau]} \max_{x \in G} \left(R' - \|\phi(t,x)\|\right) e^{-tL} \ge \ell.$$

This condition guarantees that all trajectories starting within an ℓ-neighborhood of the grid remain safely inside $B_{R'}(x^*)$ for the entire interval $[0, \tau]$. If this fails, we halve the grid resolution ℓ and repeat the process until the condition is met.

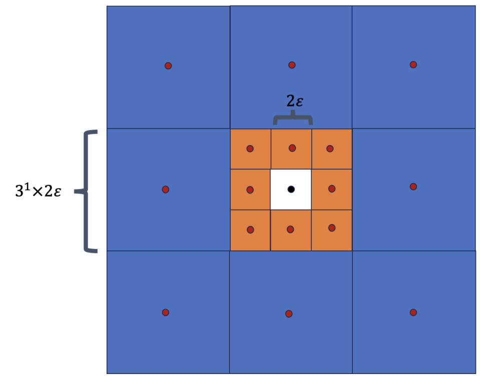

Fig. 2: Illustration of initial grid setup for $R = 3^m\varepsilon$, with $m = 2$ layers. The red dots are the grid points, while the central black dot is x^*

B. Verification of a Region

We now integrate the routines developed in Section IX-A into a complete algorithm for certifying exponential recurrence over the annular region $S := B_R(x^*) \setminus B_\varepsilon(x^*)$. We begin by computing a valid recurrence time τ and a one-sided Lipschitz constant L over a reachable set $S' \supseteq S$, as described in Section IX-A4 (*Estimation of L and τ*). Using these constants, we construct an initial layered grid over S as explained in Section IX-A3 (*Initial Grid Setup*). Each grid point x is associated with a ball $B_r(x)$, where the radius r scales with the distance to the equilibrium.

To evaluate recurrence at each point, we apply the routine $\alpha_{\max}(x, r)$ from Algorithm 1, which returns the largest value of α for which the ball satisfies condition (21). For each point x, we compute both:

- a **lower bound** $\underline{\alpha}(x) := \alpha_{\max}(x, r)$, which certifies condition (21) over the full ball,
- an **upper bound** $\overline{\alpha}(x) := \alpha_{\max}(x, 0)$, which certifies condition (21) only at the center.

If the worst-case relative $(\overline{\alpha}(x) - \underline{\alpha}(x))/\underline{\alpha}(x)$ the lowest $\underline{\alpha}(x)$ across the grid exceeds a specified threshold θ, we iteratively refine the k lowest-scoring points using the SPLIT routine (Algorithm 2). This process continues for at most m refinement steps.

The full verification procedure is summarized in Algorithm 3. If successful, it returns a uniform lower bound α such that $V(x) = \|x\|_\infty$ is an ERLF over S, as guaranteed by Theorem 4.

Algorithm 3: Find-$\alpha_{\min}(R, \varepsilon, \theta)$ — Find best rate α for region $B_R(0) \setminus B_\varepsilon(0)$ via parallel ball certification

```
Input: Outer radius R > 0, inner radius ε ∈ (0, R), threshold θ ∈ (0, 1), max number of refinements m
Output: Lower bound on the certified value of α
Construct initial grid G ← {(x_i, r_i)} covering B_R(x*) \ B_ε(x*); set counter ← 0;
while counter ≤ m − 1 do
    For all (x_i, r_i) ∈ G, compute α̲_i ← α_max(x_i, r_i) and ᾱ_i ← α_max(x_i, 0) ;   // upper and lower bounds
    Let i* ← arg min_{i∈[|G|]} α̲_i;                                                   // lowest estimate of α in region
    if (ᾱ_{i*} − α̲_{i*})/α̲_{i*} > θ then
        Replace the k lowest–α̲_i balls (x_i, r_i) in G with SPLIT((x_i, r_i)) ;
        counter ← counter + 1;
    else
        break;
return α̲_{i*}
```

Algorithm 4: Find-α-RoA$(R, \varepsilon, \alpha, G_0, \texttt{Trim})$ — Find certified region for fixed exponential rate α

```
Input: Target α > 0, range parameters R > 0, ε ∈ (0, R), initial grid G_0 = {(x_i, r_i)}, boolean Trim, max splits m
Output: Subset of G_0 certifying ‖x‖_∞ as an ERLF with rate at least α
Set G ← G_0, Positives ← ∅, counter ← 0;
while counter ≤ m − 1 and G ≠ ∅ do
    Define Region ← ⋃_{(x,r)∈Positives∪G} B_r(x) ;                       // Used if trimming is enabled
    foreach (x_i, r_i) ∈ G do
        α_i ← Trim α_max(x_i, r_i; Region) : α_max(x_i, r_i) ;            // Evaluate recurrence
        if α_i ≥ α then
            Positives ← Positives ∪ {(x_i, r_i)} ;                          // Add to certified set
    G ← SPLIT(G \ Positives) ;                                              // Refine remaining uncertified balls
    counter ← counter + 1;
return Positives ;                                                          // Region certified for recurrence at rate α
```

C. Exponential Region of Attraction Mining

We now consider the complementary task to Section IX-B: given a fixed decay rate $\alpha > 0$, identify a maximal subset of the state space over which $V(x) = \|x\|_\infty$ can be certified as an Exponential Recurrent Lyapunov Function (ERLF) with rate at least α.

To that end, we define an initial covering grid $G_0 := \{(x_i, r_i)\}$ over the annular region $B_R(x^*) \setminus B_\varepsilon(x^*)$, where each pair (x_i, r_i) represents a candidate ball $B_{r_i}(x_i)$. The union of all such balls forms a discrete approximation to the candidate region of attraction. Using this grid, we iteratively apply the ball certification routine $\alpha_{\max}(x_i, r_i)$ (Algorithm 1). Balls that pass the condition $\alpha_{\max}(x_i, r_i) \geq \alpha$ are added to the certified set. Those that fail are recursively subdivided using the SPLIT routine, up to a maximum number of refinements. This growth procedure is encapsulated in Algorithm 4. To improve efficiency, Algorithm 4 supports a Boolean flag `Trim`. When set to `False`, each ball is verified using only local information. Once the region has been expanded, the same algorithm is rerun with `Trim` set to `True`, ensuring that all recurrence trajectories remain entirely within the certified region (as required by Theorem 4).

The result is a data-driven, self-consistent inner approximation of the α-region of attraction.

D. Numerical Validation of Algorithm 3

We continue by providing a preliminary validation of the proposed Algorithm 3. To investigate the efficiency of our proposed method, we consider the following systems:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B_1 \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}; \tag{30}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0.5 & -1 & 0 \\ 0.5 & 0.5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + B_2 \begin{bmatrix} x_1^2 \\ ... \\ x_3^2 \end{bmatrix}, \tag{31}$$

where $B_1 \in \mathbb{R}^{2\times 3}$ and $B_2 \in \mathbb{R}^{3\times 9}$ are drawn independently from a Gaussian distribution, i.e., $[B_1]_{ij}, [B_2]_{ij} \sim \mathcal{N}(0, \sigma)$. We will increase the standard deviation σ as a means to increase the complexity of the dynamics. In our experiments, we choose the ℓ_∞ norm as our choice of $V(x)$ and as the norm used to measure distances between trajectories. Thus, $\alpha_1 = \alpha_2 = 1$. Sample trajectories for the system (30) with $\sigma = 0.3$ are shown in Figure 3, where we also illustrate the ball of radius R (blue) selected, the computed ball of radius R' (red), and the small region around the origin (x^*) not certified (black). We also show in Figure 4 the verified region and a coloring scheme illustrating the different ball sizes used at different parts of $B_R(x^*) \setminus B_\varepsilon(x^*)$.

In these experiments we use $R = 0.7$ and $\varepsilon = 0.01$. Table I and Table II summarize the results obtained by running

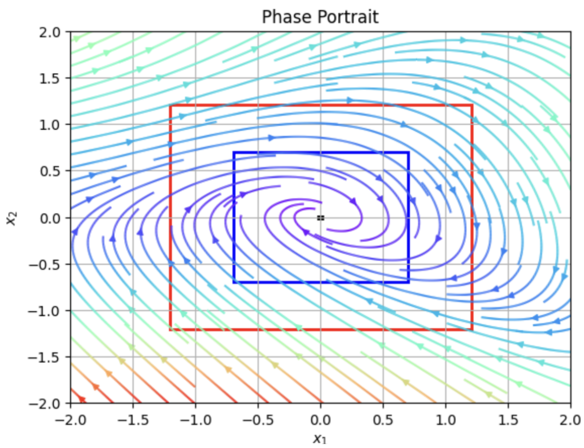

Fig. 3: Phase portrait of system (30), wherein the black box surrounds the region which we do not verify ($B_\varepsilon(x^*)$), the blue box represents surrounds the region which we verify in Algorithm 3 ($B_R(x^*)$), and the red box surrounds the region which trajectories that begin in the blue box do not leave ($B_{R'}(x^*)$).

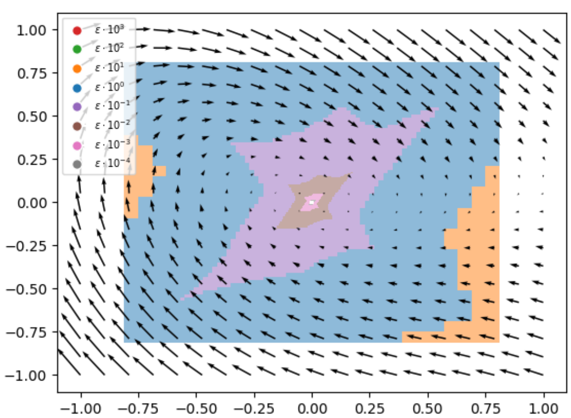

Fig. 4: Sizes of blocks resulting from applying Algorithm 3 to system (30).

Algorithm 3 together with a comparison with SOSTOOLS. When running our algorithm, we use the Torchode toolbox [32] to compute system trajectories in parallel. We also show the solving times of our algorithm and SOSTOOLS [33]. It can be seen that as σ grows - i.e., the system becomes more nonlinear - and the dimension grows, our algorithm begins to significantly outperform SOSTOOLS.

2D sys. (30)	0.3,	0.6	1
Alg 3 α:	**0.470**	**0.414**	**0.349**
SoS α:	0.360	0.247	0.223
Alg 3 Time:	10.95s	15.01s	15.30s
SoS Time:	**0.97s**	**1.06s**	**0.94s**

TABLE I: Parameter values and performance comparison between our algorithm and the SOSTOOLS for system (30)

3D sys. (31)	$\sigma = 0.1$,	0.3	0.4	0.5
Alg 3 α:	**0.635**	**0.449**	**0.370**	**0.359**
SoS α:	0.309	0.341	0.256	0.213
Alg 3 Time:	140.85s	**144.33s**	**150.07s**	**154.92s**
SoS Time:	**54.89s**	276.10s	299.77s	632.55s

TABLE II: Parameter values and performance comparison between our algorithm and the SOSTOOLS for system (31)

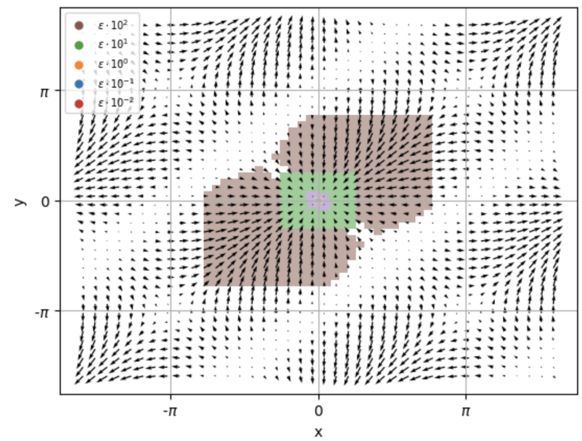

Fig. 5: Phase portrait of system (32). The the blue region composed of distinct cells is the verified region resulting from Algorithm 4 with a maximum split count of 1.

E. Numerical Validation of Algorithm 4

We end this section by providing preliminary experiments for Algorithm 4. To do so, we consider the Kuramoto Oscillator with uniform coupling constants, i.e., for an n-dimensional system and some constant k, oscillator θ_i is defined by:

$$\dot{\theta}_i = \frac{k}{n} \sum_{j=1}^{n} \sin(\theta_j - \theta_i) \tag{32}$$

To eliminate the rotational symmetry of this system, we consider the change of variables $\varphi_i = \theta_i - \theta_n$, which effectively reduces the dimensions by one.

We next consider the use of Algorithm 4 to find the α-RoA. First we consider the two-oscillator system with $\alpha = 1$ and $k = 10$. Figures 5 and 6 are the phase portraits along with the 1-RoA formed of the verified balls, when usnig maximum split counts of 1 and 6, respectively.

Finally, we investigate how the size of the certified region and the computational cost of verification scale with the ambient dimension. Specifically, we fix the system dynamics and all algorithmic parameters, and vary only the state dimension. Figure 7 reports the volume of the region certified by Algorithm 4 and the corresponding computation time, for dimensions ranging from 2 to 6 and maximum split counts from 0 to 6 (represented by the dots).

X. Conclusions

In this paper, we seek to relax the notion of set invariance, a fundamental tool in the analysis of dynamical systems. We thus propose and use the notion of set recurrence and

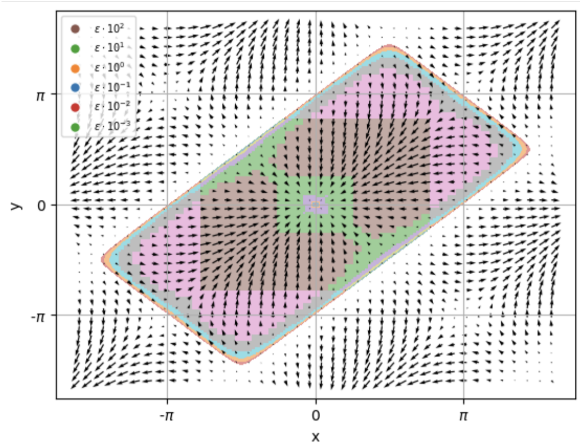

Fig. 6: Phase portrait of system (32). The the blue region composed of distinct cells is the verified region resulting from Algorithm 4 with a maximum split count of 6.

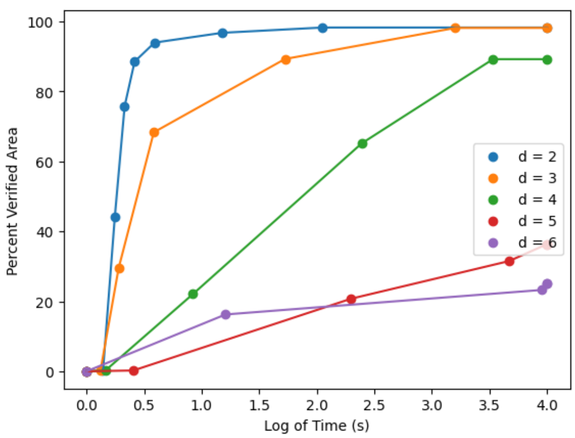

Fig. 7: Area of region verified by Algorithm 4 when applied to system 32 and associated computational time requirements for split counts ranging from 0 to 6, with each split count corresponding to a dot and dimensions 2 through 6. Time is provided in log-scale.

show that under mild conditions, recurrence can be used to guarantee stability, asymptotic stability, and exponential stability of an equilibrium point. On the back of this theory, we have constructed an algorithm that lets us verify that a set, other than a ball arbitrarily close to the equilibrium, is κ-exponentially τ-decreasing. This algorithm is entirely deterministic and can be run in parallel on GPUs, resulting in time or accuracy improvements over state-of-the-art methods based using Sum of Squares.

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