Safety-Critical Control via Recurrent Tracking Functions

Jixian Liu and Enrique Mallada

***Abstract*— This paper addresses the challenge of synthesizing safety-critical controllers for high-order nonlinear systems, where constructing valid Control Barrier Functions (CBFs) remains computationally intractable. Leveraging layered control, we design CBFs in reduced-order models (RoMs) while regulating full-order models' (FoMs) dynamics at the same time. Traditional Lyapunov tracking functions are required to decrease monotonically, but systematic synthesis methods for such functions exist only for fully-actuated systems. To overcome this limitation, we introduce Recurrent Tracking Functions (RTFs), which replace the monotonic decay requirement with a weaker finite-time recurrence condition. This relaxation permits transient deviations of tracking errors while ensuring safety. By augmenting CBFs for RoMs with RTFs, we construct recurrent CBFs (RCBFs) whose zero-superlevel set is control τ-recurrent, and guarantee safety for all initial states in such a set when RTFs are satisfied. We establish theoretical safety guarantees and validate the approach through numerical experiments, demonstrating RTFs' effectiveness and the safety of FoMs.**

I. Introduction

Ensuring safety in autonomous systems requires controllers that can guarantee constraint satisfaction while achieving performance objectives. Control Barrier Functions (CBFs) provide a principled framework for certifying safety by rendering desired safe sets forward invariant through appropriate control actions. However, the practical deployment of CBF-based methods hinges on the ability to construct valid barrier functions for the system at hand—a challenge that grows increasingly difficult as system complexity and dimensionality increase [1].

Existing CBF synthesis methods typically leverage certain relative degree properties [2], [3], learning-based approaches [4], [5], sum-of-squares programs for polynomial dynamics [6], [7], or reachability-based constructions from Hamilton–Jacobi value functions [8], [9]. These approaches often rely on assumptions that fail in practice or suffer from the curse of dimensionality, limiting applicability to complex, high-dimensional systems; scalable CBF synthesis for higher-order dynamics in modern systems, such as aerial vehicles, legged robots, and complex power grids, therefore, remains open [10].

A promising alternative is layered control: a reduced-order model (RoM) captures safety-critical states (e.g., collision-relevant kinematics or center-of-mass motion) while a low-level controller regulates the full-order model (FoM) [11], [12]. Designing CBFs in the lower-dimensional RoM reduces computation, and safety transfers to the FoM when tracking errors admit exponential convergence certificates [10], [13]. However, beyond fully actuated systems, there is no systematic procedure for constructing such tracking functions to certify the Lyapunov condition for exponential convergence, which hinders practice. Robust RoM-based CBFs with predictive error margins [1] mitigate RoM–FoM mismatch, but the conservatism term δ discards useful system information.

In this paper, we address these issues by relaxing the Lyapunov condition to a recurrent one [14]–[17]: instead of enforcing monotone decay of the tracking function, we only require the decrease condition to recurrently hold on time instances that are separated by at most a uniform bound τ.

Concretely, we introduce the concept of **Recurrent Tracking Functions (RTFs)**, which leads to decreased conditions that can be verified using the norm function, thus yielding a more flexible RoM–FoM linkage for general nonlinear systems. Augmenting a CBF for RoM with an RTF produces a recurrent CBF h_V [17] whose zero-superlevel set S_V is control τ-recurrent (trajectories may leave for τ units of time), yet they never cross the unsafe region, thus ensuring safety for all admissible initial conditions. Our contributions are as follows:

1) **Introduce the RTF** as a generalized method to guarantee exponential regulation rather than strict invariance, where any norm of the tracking error can be a valid RTF if the error is exponentially stable for a Lipschitz continuous system [18];
2) **Theoretically guarantee safety** via the zero-superlevel set S_V of h_V: if the initial state lies in S_V, any controller satisfying the RTF condition guarantees safety at all times;
3) **Empirically validate** through numerical experiments demonstrating preserved agent safety and effectiveness on the FoM.

Organization. Section II reviews layered control, CBFs, and recurrence-based analysis. Section III presents the RTF-based method and safety proofs. Section IV reports numerical validations. Section V concludes and outlines future work.

II. Preliminaries and Problem Formulation

A. Layered Control System

We begin this paper by recalling the setting of a layered-control system, consisting of a **full-order model (FoM)** and a **reduced-order model (RoM)**, which interacts with the FoM through the projection of the full state onto the RoM [11].

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Definition 1 (Full-Order Model)**.** *The FoM represents the physical system, typically high-dimensional, nonlinear, and only partially known. Its dynamics are assumed to evolve as*

$$\dot{x} = F(x, u), \tag{1}$$

where $x \in \mathcal{X} \subseteq \mathbb{R}^N$ is the state and $u \in U \subseteq \mathbb{R}^M$ is the control input.

Given an interval $\mathcal{I} \subseteq \mathbb{R}_{\geq 0}$, et $\mathcal{U}^{\mathcal{I}} := \{u : \mathcal{I} \to U \mid u \text{ measurable}\}$. We use u for both instantaneous inputs $u \in U$ and signals $u \in \mathcal{U}^{\mathcal{I}}$, disambiguated by context, and $\mathcal{U} := \mathcal{U}^{\mathbb{R}_{\geq 0}}$. Given $x \in \mathbb{R}^N$ and $u \in \mathcal{U}^{(0,a]}$, let $\phi(t, x, u)$ be the trajectory of (1) at time $t \in (0, a]$.

Assumption 1 (Forward completeness)**.** *The control system* (1) *is forward complete: for every initial condition $x \in \mathcal{X} \subseteq \mathbb{R}^N$ and every admissible input signal $u \in \mathcal{U}$, the solution $\phi(t, x, u)$ of* (1) *exists for all $t \geq 0$.*

Assumption 2 (Uniform Local Lipschitz Continuity)**.** *The vector field $F : \mathcal{X} \times \mathrm{U} \to \mathbb{R}^N$ in* (1) *is continuous and locally Lipschitz in x, uniformly w.r.t. u, i.e., for every compact $S \subseteq \mathcal{X}$, there exists $L_{F_S} \geq 0$ such that*

$$\|F(y, u) - F(x, u)\| \leq L_{F_S} \|y - x\|, \forall x, y \in S, \forall u \in U.$$

Definition 2 (Reduced-order model)**.** *The RoM captures safety-relevant, lower-dimensional dynamics:*

$$\dot{z} = f(z, v), \tag{2}$$

with state $z \in \mathcal{Z} \subseteq \mathbb{R}^n$ $(n < N)$, and input $v \in \mathrm{V} \subseteq \mathbb{R}^m (m < M)$, where f is also locally Lipschitz continuous with constant L_{R_S} for some compact set $S \subseteq \mathcal{Z}$.

Coupling between RoM and FoM. These two models interact via a projection $\Pi : \mathcal{X} \to \mathcal{Z}$ and a tracking interface:

$$u := K(x, v), \quad v := k(z), \quad z := \Pi(x),$$

where K is the FoM's controller, k is the RoM's feedback controller, and Π the projection. This induces the **closed-loop** FoM:

$$\dot{x} = F_{\mathrm{cl}}(x) := F(\, x, K(x, k(\Pi(x)))\,). \tag{3}$$

For the system (3), we use $x(t) := \phi_{\mathrm{cl}}(t, x)$ to denote the closed-loop full state trajectory and $z(t)$ the corresponding projection $\Pi(x(t))$.

B. Problem Statement

To formalize the safety objective, consider constraints for the RoM defined via a smooth function $h : \mathcal{Z} \to \mathbb{R}$, which induces the RoM's safe set.

Definition 3 (Safe state of the RoM)**.** *A state $z \in \mathcal{Z} \subseteq \mathbb{R}^n$ is **safe** if $z \in \mathcal{S}_{\mathrm{RoM}}$, where*

$$\mathcal{S}_{\mathrm{RoM}} := \{z \in \mathcal{Z} : h(z) \geq 0\} \tag{4}$$

is the RoM's safe set with boundary $\partial\mathcal{S}_{\mathrm{RoM}} = h^{-1}(0)$.

Definition 4 (Safe state of the FoM)**.** *A state $x \in \mathcal{X} \subseteq \mathbb{R}^N$ is **safe** if $x \in \mathcal{S}_{\mathrm{FoM}}$, where*

$$\mathcal{S}_{\mathrm{FoM}} := \{x \in \mathcal{X} : h(\Pi(x)) \geq 0\} \tag{5}$$

is the FoM's safe set with boundary $\partial\mathcal{S}_{\mathrm{FoM}} = \Pi^{-1}(h^{-1}(0))$.

The objective is to guarantee the safety of the FoM by synthesizing a controller based on the safe reference generated by the RoM.

Problem. Given the RoM dynamics (2), the closed-loop FoM (3), and the safe set $\mathcal{S}_{\mathrm{FoM}}$, design a feedback controller $u = K(x, k(\Pi(x)))$ and a compact set $S \subseteq \mathcal{S}_{\mathrm{FoM}}$ such that $x \in S \Rightarrow x(t) \in \mathcal{S}_{\mathrm{FoM}}, \forall t \geq 0$.

C. Control Barrier Function for RoM

Control barrier functions (CBFs) provide a standard route to certify the safety for the RoM (2): by enforcing a differential inequality that lower–bounds $\dot{h}$ via an extended class-$\mathcal{K}$ function, the zero-superlevel set S of h becomes forward invariant and safe. We first recall the notion of extended class-$\mathcal{K}$ functions needed to state the CBF condition.

Definition 5 (Extended Class-$\mathcal{K}$ Function)**.** *A function $\kappa : \mathbb{R} \to \mathbb{R}$ is an extended class $\mathcal{K}$ function if it is continuous, strictly increasing, and satisfies $\kappa(0) = 0$.*

With this notion in place, we can formalize the CBF condition used throughout the paper.

Definition 6 (Control Barrier Function [19])**.** *A continuously differentiable function $h(z)$ is a CBF for the system* (2) *if there exists an extended class $\mathcal{K}$ function κ such that,*

$$\max_{v \in V} L_f h(z) + \kappa(h(z)) \geq 0, \tag{6}$$

for all $z \in \mathcal{Z}$, and where $L_f h(z) = \frac{\partial h}{\partial z}^\top f(z, v)$ denotes the Lie derivative.

Theorem 1 ([19])**.** *As a direct consequence of Definition 6, any Lipschitz-continuous controller $k(z)$ that satisfies*

$$k(z) \in \{u \in U \mid L_f h(z) + \kappa(h(z)) \geq 0\} \tag{7}$$

renders the set $h_{\geq 0} := \{z \mid h(z) \geq 0\}$ forward invariant. In particular, $h_{\geq 0}$ is control invariant.

In layered architectures, it is common to leverage RoM's safety together with tracking-based interface assumptions to ensure the safety on the FoM [1]. We adopt the following standard conditions to characterize such an interface.

Assumption 3 (Relative degree)**.** *There exists a projection $\Psi : \mathcal{X} \to \mathcal{V}$ such that*

$$\left.\frac{\partial \Pi}{\partial x}\right|_x F(x, u) = f(\Pi(x), \Psi(x)).$$

Assumption 4 (Boundedness)**.** *There exists $C_h > 0$ s.t.*

$$\|\nabla h(\Pi(x))\| \leq C_h, \quad \forall x \in \mathcal{S}_{\mathrm{FoM}}.$$

Assumption 3 ensures that u does not directly affect the time derivative of the CBF h for RoM, and Assumption 4 ensures the tracking error's effect is not unboundedly amplified.

D. Recurrent Lyapunov and Control Barrier Functions

Constructing a Lyapunov function for a general high-order system is often intractable. Reference [18] relaxes this requirement via Recurrent Lyapunov Functions (RLFs), which require the Lyapunov decrease condition to hold only recurrently rather than continuously. RLFs still guarantee exponential stability, and a converse theorem shows that any norm of the state satisfies the RLF conditions for exponentially stable systems. Since RLF conditions are defined for autonomous systems, we express the results in this section in terms of $x(t)$, which denotes solutions to (3).

Definition 7 (Exponential Stability)**.** *Given $S \subseteq \mathbb{R}^N$, an equilibrium point x^* is* exponentially stable *on S if for all $x \in S$ there exists $u \in \mathcal{U}$ such that*

$$\|x(t) - x^*\| \leq Me^{-\lambda t}\|x - x^*\|, \quad \forall t \geq 0, \tag{8}$$

for some constants $M, \lambda > 0$.

Definition 8 (Reachable Tube)**.** *For system* (3)*, $\tau > 0$, and $S \subset \mathbb{R}^N$, the τ-reachable tube of S is*

$$\mathcal{R}^\tau(S) = \bigcup_{\substack{x \in S \\ t \in [0,\tau]}} \{x(t)\}. \tag{9}$$

Definition 9 (Containment Times)**.** *For $S \subseteq \mathbb{R}^N$, $x \in \mathbb{R}^N$, the containment time for* (3) *is defined as $T_S(x) := \{t > 0 \mid x(t) \in S\}$. For constants a, b, $T_S(x; a, b) := T_S(x) \cap (a, a + b]$, and $T_S(x; b) := T_S(x; 0, b)$.*

Definition 10 (Recurrent Lyapunov Function [18])**.** *Given an equilibrium $x^* \in \mathcal{X}$ of* (3) *and a compact $S \subseteq \mathcal{X}$ with $x^* \in \text{int}(S)$, a continuous $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ is an Recurrent Lyapunov Function (RLF) over S if:*

1) ***(Positive definiteness)*** *$\exists\, a_1, a_2 > 0$ such that*

$$a_1\|x - x^*\| \leq V(x) \leq a_2\|x - x^*\|, \forall x \in S. \tag{10}$$

2) ***(Exponential τ-recurrence)*** *$\exists \alpha, \tau > 0$ such that*

$$\min_{t \in T_S(x;\tau)} e^{\alpha t} V(x(t)) \leq V(x), \forall x \in S. \tag{11}$$

Condition (11) enforces exponential convergence via recurrent returns, relaxing monotone decrease.

Analogously, [17] defines a **Recurrent Control Barrier Function (RCBF)** $h : \mathbb{R}^N \to \mathbb{R}$, which replaces invariance with recurrence: trajectories may leave $h \geq 0$ but must re-enter within time τ, infinitely often; the resulting set is control τ-recurrent.

Remark 1. *The following requires extending the definition of containment times, i.e., Definition 9, for the FoM system* (1)*. For $S \subseteq \mathbb{R}^N, x \in \mathbb{R}^N$, we use $T_S(x, u) := \{t > 0 \mid \phi(x, u, t) \in S\}$ denoted as the containment time for* (1) *and $T_S(x, u; \tau) := T_S(x, u) \cap (0, \tau]$.*

Definition 11 (Recurrent Control Barrier Function)**.** *For* (1) *and a compact $S \subseteq \mathbb{R}^N$, a continuous $h : \mathbb{R}^N \to \mathbb{R}$ is an RCBF over S if for every $x \in S$ there exists $u \in \mathcal{U}^{(0,\tau]}$ such that:*

$$\max_{t \in T_S(x,u;\tau)} e^{\gamma(h(\phi(t,x,u)))} h(\phi(t, x, u)) \geq h(x), \tag{12}$$

where the function $\gamma : \mathbb{R} \to \mathbb{R}_{>0}$.

Unlike invariance-based safety, which requires the invariant set to be disjoint from unsafe regions, RCBFs certify safety if the recurrent set avoids the backward τ-reachable tube of the unsafe set [20].

A key advantage of the recurrence framework is that both RLFs and RCBFs can be constructed using simple norm and signed distance functions, respectively [15], [18], significantly simplifying synthesis compared to classical Lyapunov and barrier functions. This motivates replacing the exponential tracking certificates in [12] with (possibly norm-based) Recurrent Tracking Functions, yielding a systematic synthesis method aligned with (11)–(12).

III. Layered Safety-Critical Control via Recurrence

In this section, we extend the recurrence framework to guarantee safety in layered control architectures.

A. Recurrent Tracking Function

We start by formalizing the notion of **Recurrent Tracking Functions (RTFs)** for system (3) by defining a condition akin to (11) for tracking errors. To that end, we will consider trajectories $x(t)$ of the closed loop system (3), and $z(t)$ its projection.

Given a safe reference trajectory $z_s(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ for the RoM, the tracking error and its time derivative are

$$e(\cdot) := z(\cdot) - z_s(\cdot), \quad \dot{e}(\cdot) := \dot{z}(\cdot) - \dot{z}_s(\cdot), \tag{13}$$

with initial conditions $e := e(0)$ and $\dot{e} := \dot{e}(0)$. By Assumption 3, for any smooth function h,

$$\dot{h} = \nabla h(z)^\top \dot{z} = \nabla h(z)^\top (\dot{z}_s + \dot{e}). \tag{14}$$

If $h : \mathbb{R}^n \to \mathbb{R}$ is a CBF for the RoM (2) with linear class-$\mathcal{K}$ function $\kappa(h) = \alpha h$ for some $\alpha > 0$, the safe reference $\dot{z}_s$ satisfies

$$\nabla h(z)^\top \dot{z}_s \geq -\alpha h(z) \tag{15}$$

Definition 12 (Recurrent Tracking Function)**.** *Consider the system* (3) *and a compact set $S \subseteq \mathbb{R}^n \times \mathbb{R}^n$ with $0 \in \text{int}(S)$. A continuous function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a **Recurrent Tracking Function (RTF)** over S if:*

1) ***Positive definiteness with linear error bounds:*** *There exist $a_1, a_2 > 0$ such that*

$$a_1\|\dot{e}\| \leq V(z, \dot{e}) \leq a_2\|\dot{e}\|, \forall (z, \dot{e}) \in S. \tag{16}$$

2) ***β-exponential τ-recurrence:*** *There exist $\tau, \beta > 0$ such that for every $x \in \mathcal{X}$ s.t. $(z, \dot{e}) \in S$, the corresponding tracking error $(z(\cdot), \dot{e}(\cdot))$ satisfies*

$$\min_{t \in T_S(x;\tau)} e^{\beta t} V(z(t), \dot{e}(t)) \leq V(z, \dot{e}). \tag{17}$$

Like the RLF's Exponential Stability [18, Theorem 3], we can guarantee that the tracking error $\dot{e}(t)$ between the safe trajectory of the RoM, $\dot{z}_s(t)$ and the actual $\dot{z}(t) = \frac{d}{dt}\Pi(x(t))$ of the FoM exponentially converges to zero. We refer to this

condition as the system (3) having an exponentially stable tracking error $\|\dot{e}(t)\|$.

Theorem 2 (Exponential Stable Tracking Error)**.** *Suppose Assumption 2 holds, and let $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a Recurrent Tracking Function over the compact set S. Then, $\|\dot{e}(t)\|$ converges exponentially to zero with rate β on the set S. In particular, for every $(z, \dot{e}) \in S$ and every $t \geq 0$, it holds that*

$$\|\dot{e}(t)\| \leq Me^{-\beta t}\|\dot{e}(0)\|, \tag{18}$$

with $M := \frac{a_1}{a_2} e^{\beta\tau}(1 + L_{R_{\mathcal{K}^\tau(S)}} \tau e^{L\tau})$.

Proof. The proof follows closely similar results for [18, Theorem 3], and it is omitted due to space constraints. □

B. Safety Assessment

In this subsection, we combine an RTF for (3) with a CBF for (2) to characterize the set of initial conditions that can guarantee the safety of the closed loop system (3) when the trackin control K satisfies the RTF condition.

Theorem 3 (Recurrent CBF Construction)**.** *Consider system (3) and suppose $h : \mathbb{R}^n \to \mathbb{R}$ is a CBF for the RoM (2) with linear class-$\mathcal{K}$ function $\kappa(h) = \alpha h$ for some $\alpha > 0$. If $V(z, \dot{e})$ is a valid RTF over $S \subseteq \mathbb{R}^n \times \mathbb{R}^n$ with convergence rate $\beta > \alpha$ and recurrent time τ, then*

$$S_V := \{(z, \dot{e}) \in \mathbb{R}^n \times \mathbb{R}^n : h_V(z, \dot{e}) \geq 0\} \tag{19}$$

is a control τ-recurrent set, and

$$h_V(z, \dot{e}) = -V(z, \dot{e}) + \alpha_e h(z), \quad \alpha_e = \frac{a_1^2(\beta - \alpha)}{a_2 C_h M}, \tag{20}$$

is an RCBF on $S_V \subseteq S$.

In particular, for any $x \in \mathcal{S}_{\text{FoM}}$ s.t. $(\Pi(x), \dot{e}) \in S_V$, the induced control signal $u(\cdot) = K(x(\cdot), k(\Pi(x(\cdot))))$ satisfies the RCBF condition.

Proof. See Appendix A □

Remark 2. *Theorem 3 requires initial conditions to satisfy $(z, \dot{e}) \in S_V$, which means that $h_V(z, \dot{e}) = -V(z, \dot{e}) + \alpha_e h(z) \geq 0$, i.e. $h(z) \geq \frac{V(z,\dot{e})}{\alpha_e}$. According to the definition of $\mathcal{S}_{\text{FoM}}$, in fact, we know that Theorem 3 imposes a more conservative requirement on the problem than $h(z) \geq 0$.*

Theorem 4 (Safety Assessment)**.** *Let $V(z, \dot{e})$ be the RTF over $S \subseteq \mathbb{R}^n \times \mathbb{R}^n$ with rate β and consider an initial state $x \in \mathcal{S}_{\text{FoM}}$ of the FoM s.t. $(z, \dot{e}) \in S_V = \{(z, \dot{e}) \in \mathbb{R}^n \times \mathbb{R}^n : h_V(z, \dot{e}) \geq 0\}$. Then, the solution $x(t)$ of the closed system always remain in safe state set $\mathcal{S}_{\text{FoM}}$, i.e. $x(t) \in \mathcal{S}_{\text{FoM}}$ $\forall t \geq 0$.*

Proof. See Appendix B □

Remark 3 (Recurrence vs. Safety)**.** *Theorem 4 reveals a surprising property: although h_V is a recurrent CBF and trajectories may temporarily leave the recurrent set S_V, the system* never *leaves the safe set $\mathcal{S}_{\text{FoM}}$. This is because the tracking error converges at rate β while the RoM's CBF decays at rate α. The condition $\beta > \alpha$ ensures that the controller corrects tracking deviations faster than the system can approach the safety boundary, keeping $h(z(t)) \geq 0$ for all time despite the recurrent nature of h_V. Figure 1 illustrates this phenomenon: while V exhibits recurrent behavior, h remains strictly non-negative.*

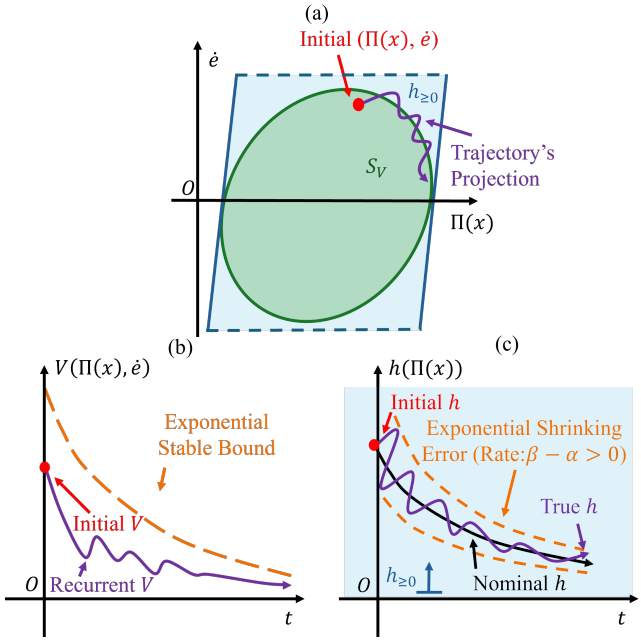

Fig. 1. (a) V, h, S_V and a safe trajectory's projection when $n = 1$. (b) Time evolution of V. (c) Time evolution of h.

C. Effect of Disturbances

In practice, feedback controllers $K(x, k(\Pi(x)))$ cannot achieve ideal exponential tracking due to model uncertainties, actuation limits, and external disturbances. We model this mismatch via a bounded disturbance d, under which the tracking error becomes input-to-state stable (ISS):

$$\|\dot{e}(t)\| \leq M\|\dot{e}\|e^{-\beta t} + \mu(\|d_\infty\|) \tag{21}$$

for some class-$\mathcal{K}$ function $\mu(\cdot)$. The key insight is that while disturbances prevent the tracking error from vanishing completely, they introduce only a bounded steady-state offset $\mu(\|d_\infty\|)$.

To accommodate this, we modify the RTF condition (17) to account for the disturbance-induced offset:

$$\min_{t \in T_S(x;\tau)} e^{\beta t}(V(z(t), \dot{e}(t)) - \iota(\|d\|_\infty)) \leq V(z, \dot{e}) - \iota(\|d\|_\infty), \tag{22}$$

where $\iota(\|d\|_\infty) = \frac{a_2 e^{\beta\tau}\mu(\|d\|_\infty)}{M}$ quantifies the impact of disturbances on the RTF. This leads to a robust safety guarantee through an enlarged recurrent set:

$$S_{Vd} = \{(z, \dot{e}) \in S \times \mathbb{R}^n : h_{Vd}(z, \dot{e}) \geq 0\}, \tag{23}$$

$$h_{Vd}(z, \dot{e}) = h_V(z, \dot{e}) - \gamma(\|d\|_\infty), \tag{24}$$

where $\gamma(\|d\|_\infty) = \iota(\|d\|_\infty)/\alpha_e$ provides a safety margin that scales with the disturbance magnitude. The following corollary formalizes this robustness property.

Corollary 1 (Input-to-State Safety)**.** *For system* (3) *with* $\beta > \alpha$*, if an initial state* $x \in \mathcal{S}_{\text{FoM}}$ *satisfies* $(z, \dot{e}) \in S_{Vd}$ *and the ISS tracking condition* (22) *holds, then the system remains safe for all* $t \geq 0$*, where* α_e *is given in Theorem 3 and* $\gamma(\|d\|_\infty) = \iota(\|d\|_\infty)/\alpha_e$*.*

Proof. The proof follows by combining the arguments in Theorems 3 and 4 with the modified RTF condition (22), and is omitted due to space constraints. □

IV. Case Study

In this section, we validate the effectiveness of the proposed methodology using a 2D Double Integrator System. Consider the dynamics: $\ddot{z} = u$, where $z \in \mathbb{R}^2$ is the agent's position, the FoM's dimension is 4, the RoM is $\dot{z} = v$, and $u \in \mathbb{R}^2$. Our goal is to navigate the system from a start position z_0 to a goal z_g while avoiding obstacles. A simple solution is to realize the desired velocity $\dot{z}_d = -K_p(z - z_g)$ which is a proportional controller with gain $K_p \in \mathbb{R}_{>0}$.

In our setting, we consider a planar projection $\Pi(\mathcal{X}) \subseteq \mathbb{R}^2$ occupied with N circular (closed-disk) obstacles. The i-th obstacle is $\mathcal{O}_i = \{z \in \mathbb{R}^2 : \|z - o_i\| \leq r_i\}, i \leq N$, whose center is o_i and radius is r_i. Then the obstacle's configuration space is the union $\mathcal{O} = \cup_{i=1}^N \mathcal{O}_i$, and the collision-free configuration space is $\Pi(\mathcal{X})/\mathcal{O}$. The CBF we used for the RoM is $h(z) := \min_{i \in [1,N] \subset \mathbb{N}_+} \|z - o_i\| - r_i$, and its gradient $\nabla h(z) = \frac{(z-o_i)^\top}{\|z-o_i\|} = n_i^\top$ is an unit vector pointing from the nearest obstacle $\mathcal{O}_i$ to the agent. Then the safety velocity $\dot{z}_s$ w.r.t. $\mathcal{O}_i$ is given by the following quadratic program:

$$\arg\min_{\dot{z}_s \in \mathbb{R}^2} (\dot{z}_s - \dot{z}_d)^\top (\dot{z}_s - \dot{z}_d) \tag{25}$$

$$\text{s.t.} n_i^\top \dot{z}_s \geq -\alpha(\|z - o_i\| - r_i), \tag{26}$$

and its solution is $\dot{z}_s = \dot{z}_d + \max\{-n_i^\top \dot{z}_d - \alpha(\|z - o_i\| - r_i), 0\} n_i$ [12]. Then the safe velocity tracking controller can be defined as $u = -K_D(\dot{z} - \dot{z}_s)$ with $K_D > 0$.

We illustrate three different projected trajectories with three different choices of $\alpha = 0.5, 1, 5$ respectively, and $K_P = 1.8, K_D = 8$. Using standard linear control design and [12], we can show that when the CBF constraint is inactive, this choice of parameters leads to an exponentially convergent tracking error with $\beta = 2.45$ and $M = 3.24$. This allows us to characterize the set S_V of safe initial conditions. As shown in Fig 2 (a), the red dashed lines show the boundary of the obstacles $\{z \in \mathbb{R}^2 | \|z - [-0.1, 0.3]^\top\| \leq 0.5 \cup \|z - [1.3, -0.3]^\top\| \leq 0.5\}$, the union of blue dashed circles denotes the unsafe initial states when $\alpha = 0.5$, and the black ones denote the unsafe initial states when $\alpha = 1$. As Fig. 2 (a) and (b) show, when the CBF for RoM is invalid (α = 5) or the initial state is outside S_V (α = 1), the system's safety cannot be guaranteed, and the minimum of $h(t)$ is less than 0. Only when Theorem 3 and Theorem 4 are satisfied with a valid CBF in the RoM, can we ascertain that the trajectory is safe all the time ($\alpha = 0.5$). Besides,

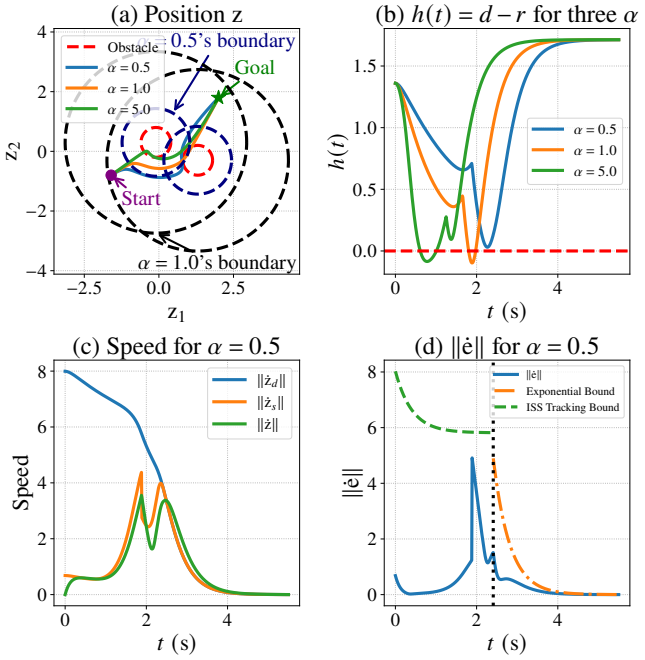

Fig. 2. (a) 2D path with circular obstacles. (b) Barrier value h vs. time for the three α. (c) Speeds for $\alpha = 0.5$ of $\|\dot{z}_d\|$, $\|\dot{z}_s\|$, and $\|\dot{z}\|$. (d) Tracking-error speed $\|\dot{e}\|$ for $\alpha = 0.5$.

Fig. 2 (c) and (d) verified that the tracking error shrink to 0 exponentially fast when (17) is satisfied, and the tracking error is bounded by the ISS tracking bound computed by the $\|d\|_\infty = \|\ddot{q}_s\|_\infty$ when the safety filter is activated.

V. Conclusion and Future Work

We developed a recurrence-based framework for layered safety-critical control of high-order nonlinear systems. By introducing Recurrent Tracking Functions (RTFs), we relax the strict Lyapunov tracking requirement to finite-time recurrence, accommodating transient deviations while preserving safety. Critically, any norm of an exponentially stable tracking error can serve as an RTF, providing a certificate that is applicable to general nonlinear systems. We prove that augmenting RoM-based CBFs with RTFs yields recurrent CBFs of the form $h_V(z, \dot{e}) = -V(z, \dot{e}) + \alpha_e h(z)$, which guarantee safety for all initial conditions in $S_V = \{h_V \geq 0\}$. The condition $\beta > \alpha$ ensures that tracking errors converge faster than the system approaches safety boundaries, maintaining $h(z(t)) \geq 0$ despite the recurrent nature of h_V. Numerical simulations validate our theoretical results. Future work will leverage recent advances in data-driven verification of recurrent sets [18], [21] to develop learning-based methods for RTF construction directly from trajectory data, eliminating the need for explicit construction of tracking functions. Additionally, experimental validation on physical robotic platforms remains an important next step.

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REFERENCES

[1] W. D. Compton, M. H. Cohen, and A. D. Ames, "Learning for layered safety-critical control with predictive control barrier functions," *arXiv preprint arXiv:2412.04658*, 2024.

[2] Q. Nguyen and K. Sreenath, "Exponential control barrier functions for enforcing high relative-degree safety-critical constraints," in *2016 American Control Conference (ACC)*, IEEE, 2016, pp. 322–328.

[3] C. Wang, Y. Meng, Y. Li, S. L. Smith, and J. Liu, "Learning control barrier functions with high relative degree for safety-critical control," in *2021 European Control Conference (ECC)*, IEEE, 2021, pp. 1459–1464.

[4] A. Robey, L. Lindemann, S. Tu, and N. Matni, "Learning robust hybrid control barrier functions for uncertain systems," *IFAC-PapersOnLine*, vol. 54, no. 5, pp. 1–6, 2021.

[5] M. Srinivasan, A. Dabholkar, S. Coogan, and P. A. Vela, "Synthesis of control barrier functions using a supervised machine learning approach," in *2020 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*, Ieee, 2020, pp. 7139–7145.

[6] A. Clark, "Verification and synthesis of control barrier functions," in *2021 60th IEEE Conference on Decision and Control (CDC)*, IEEE, 2021, pp. 6105–6112.

[7] H. Dai and F. Permenter, "Convex synthesis and verification of control-lyapunov and barrier functions with input constraints," *arXiv preprint arXiv:2210.00629*, 2022.

[8] J. J. Choi, D. Lee, K. Sreenath, C. J. Tomlin, and S. L. Herbert, "Robust control barrier–value functions for safety-critical control," in *2021 60th IEEE Conference on Decision and Control (CDC)*, IEEE, 2021, pp. 6814–6821.

[9] S. Tonkens and S. Herbert, "Refining control barrier functions through hamilton-jacobi reachability," in *2022 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*, IEEE, 2022, pp. 13 355–13 362.

[10] M. H. Cohen, N. Csomay-Shanklin, W. D. Compton, T. G. Molnar, and A. D. Ames, "Safety-critical controller synthesis with reduced-order models," *arXiv preprint arXiv:2411.16479*, 2024.

[11] N. Matni, A. D. Ames, and J. C. Doyle, "A quantitative framework for layered multirate control: Toward a theory of control architecture," *IEEE Control Systems*, vol. 44, no. 3, 2024.

[12] T. G. Molnar, R. K. Cosner, A. W. Singletary, W. Ubellacker, and A. D. Ames, "Model-free safety-critical control for robotic systems," *IEEE robotics and automation letters*, vol. 7, no. 2, pp. 944–951, 2021.

[13] T. G. Molnar and A. D. Ames, "Safety-critical control with bounded inputs via reduced order models," *arXiv preprint arXiv:2303.03247*, 2023.

[14] R. Siegelmann, Y. Shen, F. Paganini, and E. Mallada, "A recurrence-based direct method for stability analysis and gpu-based verification of non-monotonic lyapunov functions," in *62nd IEEE Conference on Decision and Control (CDC)*, IEEE, Dec. 2023, pp. 6665–6672.

[15] Y. Shen, H. Sibai, and E. Mallada, "Generalized barrier functions: Integral conditions & recurrent relaxations," in *60th Allerton Conference on Communication, Control, and Computing*, Sep. 2024, pp. 1–8, presented.

[16] H. Sibai and E. Mallada, "Recurrence of nonlinear control systems: Entropy and bit rates," in *Proceedings of the 27th ACM International Conference on Hybrid Systems: Computation and Control (HSCC)*, ser. HSCC '24, New York, NY, USA: Association for Computing Machinery, May 2024, pp. 1–9.

[17] J. Liu and E. Mallada, "Recurrent control barrier functions: A path towards nonparametric safety verification," *2025 IEEE 64th Conference on Decision and Control (CDC)*, 2025.

[18] R. Siegelmann, Y. Shen, F. Paganini, and E. Mallada, "Stability Analysis and Data-driven Verification via Recurrent Lyapunov Functions," *IEEE Transactions on Automatic Control*, Jul. 2025, submitted.

[19] A. D. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, "Control barrier functions: Theory and applications," in *2019 18th European control conference (ECC)*, IEEE, 2019, pp. 3420–3431.

[20] S. Bansal, M. Chen, S. Herbert, and C. J. Tomlin, "Hamilton-jacobi reachability: A brief overview and recent advances," in *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, IEEE, 2017, pp. 2242–2253.

[21] Y. Shen, M. Bichuch, and E. Mallada, "Model-free learning of regions of attraction via recurrent sets," in *61st IEEE Conference on Decision and Control (CDC)*, Dec. 2022, pp. 4714–4719.

APPENDIX

A. Proof of Theorem 3

Proof. For simplicity let T denote $T_S((z,\dot{e});\tau)$. To prove the Theorem 3, for any $x \in \mathcal{S}_{\text{FoM}}$ s.t. $(\Pi(x),\dot{e}) \in S_V$, our goal is to find a feedback control signal $u(\cdot) = K(x(\cdot), k(\Pi(x(\cdot)))) \in \mathcal{U}^{(0,\infty]}$ and $\gamma : \mathbb{R} \to \mathbb{R}_{>0}$ such that:

$$\max_{t\in T}\ e^{\gamma(h_V(z(t),\dot{e}(t)))} h_V(z(t),\dot{e}(t)) \geq h(z,\dot{e}). \tag{27}$$

Then we can prove, when $u(\cdot)$ is the feedback controller that satisfies (17) and $\gamma(h_V(z(t))) = \alpha t$, (27) is satisfied and S_V is the corresponding control τ-recurrent set.

For the trajectory $x(\cdot)$ generated by the feedback controller that satisfies (17), we have

$$\max_{t\in T} e^{\alpha t} h_V(z(t),\dot{e}(t)) - h_V(z,\dot{e}) \tag{28}$$

$$= \max_{t\in T} \int_0^t \frac{d(e^{\alpha s} h_V(z(s),\dot{e}(s)))}{ds} ds \tag{29}$$

$$= \max_{t\in T} \int_0^t e^{\alpha s}(-\dot{V}(z(s),\dot{e}(s)) \dots + \alpha_e \nabla h^\top (\dot{z}_s + \dot{e}) + \alpha h_V(z(s),\dot{e}(s)))ds \tag{30}$$

$$\geq \max_{t\in T} \int_0^t e^{\alpha s}(-\dot{V}(z(s),\dot{e}(s)) - \alpha_e \alpha h(z(s)) \dots - \alpha_e \|\nabla h\| \|\dot{e}\| + \alpha h_V(z(s),\dot{e}(s)))ds \tag{31}$$

$$\geq \max_{t\in T} \int_0^t e^{\alpha s}(-\dot{V}(z(s),\dot{e}(s)) - (\alpha + \frac{\alpha_e C_h}{a_1}) \dots \cdot V(z(s),\dot{e}(s)))ds \tag{32}$$

$$= \max_{t\in T} \int_0^t -e^{-\frac{\alpha_e C_h}{a_1} s} \cdot \frac{d(e^{(\alpha + \frac{\alpha_e C_h}{a_1})s} V(z(s),\dot{e}(s)))}{ds} ds \tag{33}$$

$$= \max_{t\in T} V(z,\dot{e}) - e^{\alpha t} V(z(t),\dot{e}(t)) \dots + \int_0^t \frac{d(e^{-\frac{\alpha_e C_h}{a_1} s})}{ds} e^{(\alpha + \frac{\alpha_e C_h}{a_1})s} V(z(s),\dot{e}(s))\, ds \tag{34}$$

$$= \max_{t\in T} V(z,\dot{e}) - e^{\alpha t} V(z(t),\dot{e}(t)) \dots - \frac{\alpha_e C_h}{a_1} \int_0^t e^{\alpha s} V(z(s),\dot{e}(s)) ds \tag{35}$$

$$\geq \max_{t\in T} V(z,\dot{e}) - e^{\alpha t}V(z(t),\dot{e}(t)) \dots$$
$$- \frac{a_2\alpha_e C_h}{a_1}\int_0^t M e^{-(\beta-\alpha)s}\|\dot{e}\|\, ds \tag{36}$$
$$\geq \max_{t\in T}[1 - \frac{a_2\alpha_e C_h M}{a_1^2(\beta-\alpha)}(1-e^{-(\beta-\alpha)t})]V(z,\dot{e}) \dots$$
$$- e^{\alpha t}V(z(t),\dot{e}(t)) \tag{37}$$
$$= \max_{t\in T}[1 - \frac{a_2\alpha_e C_h M}{a_1^2(\beta-\alpha)}(1-e^{-(\beta-\alpha)t})] \dots$$
$$\cdot [V(z,\dot{e}) - e^{\beta t}V(z(t),\dot{e}(t))], \tag{38}$$

where the equality (29) holds from the Fundamental Theorem of Calculus, the equality (30) follows from the principle of integration by parts, the equality (33) is equavalent to (20), and the equality (38) holds since $\alpha_e = \frac{a_1^2(\beta-\alpha)}{a_2 C_h M}$. The inequality (31) can be obtained from Cauchy Inequality and $\nabla h(z)^\top \dot{z}_s \geq -\alpha h(z)$, the inequality (32) and the inequality (37) can be obtained from the linear lower bound of the tracking error $a_1\|\dot{e}\| \leq V(\dot{e})$, and the inequality (36) is equvalent to the upper bound of the tracking error $V(\dot{e}) \leq a_2\|\dot{e}\|$.

Note that $e^{-(\beta-\alpha)t} - e^{-(\beta-\alpha)\tau} \geq 0, \forall t \in T$, since $\beta > \alpha$ and $\min_{t\in T} e^{\beta t}V(z(t),\dot{e}(t)) - V(z,\dot{e}) \leq 0$, i.e. $\max_{t\in T} V(z,\dot{e}) - e^{\beta t}V(z(t),\dot{e}(t)) \geq 0$,then we have

$$\max_{t\in T}[e^{-(\beta-\alpha)t} - e^{-(\beta-\alpha)\tau}] \dots \tag{39}$$
$$\cdot [V(z,\dot{e}) - e^{\beta t}V(z(t),\dot{e}(t))] \tag{40}$$
$$\geq [e^{-(\beta-\alpha)t} - e^{-(\beta-\alpha)\tau}] \dots \tag{41}$$
$$\cdot \max_{t\in T}[V(z,\dot{e}) - e^{\beta t}V(z(t),\dot{e}(t))] \tag{42}$$
$$\geq 0, \tag{43}$$
$$\tag{44}$$

where (42) holds naturally from the property of the maximum,

i.e., (38)
$$\geq e^{-(\beta-\alpha)\tau}\max_{t\in T}[V(z,\dot{e}) - e^{\beta t}V(z(t),\dot{e}(t))] \geq 0, \tag{45}$$

thus,

$$\max_{t\in T} e^{\alpha t}h_V(z(t),\dot{e}(t)) \geq h_V(z,\dot{e}), \tag{46}$$

and S_V is the corresponding control τ-recurrent set [17, Theorem 2], which completes the proof. □

B. Proof of Theorem 4

Proof. To prove the Theorem 4, note that states in S_V impose the pointwise budget inequality $\frac{V(z,\dot{e})}{\alpha_e} \leq h(z)$. We know that $\nabla h(z)^\top \dot{z}_s \geq -\alpha h(z)$ and $\|\nabla h(z)\| \leq C_h$. Along the FoM trajectory $x(\cdot)$, we have

$$e^{\alpha t}h(z(t)) - h(z) \tag{47}$$
$$= \int_0^t \frac{d}{ds}(e^{\alpha s}h(z(s)))ds \tag{48}$$
$$= \int_0^t e^{\alpha s}(\dot{h}(z(s)) + \alpha h(z(s)))ds \tag{49}$$
$$= \int_0^t e^{\alpha s}(\nabla h(z(s))^\top(\dot{z}_s + \dot{e}(s)) + \alpha h(z(s)))ds \tag{50}$$
$$\geq \int_0^t e^{\alpha s}(-\alpha h(z(s)) + \nabla h(z(s))^\top \dot{e}(s) + \alpha h(z(s)))ds \tag{51}$$
$$= \int_0^t e^{\alpha s}\nabla h(z(s))^\top \dot{e}(s)ds \tag{52}$$
$$\geq -C_h\int_0^t e^{\alpha s}\|\dot{e}(s)\|ds, \tag{53}$$

where the equality (48) follows from the Fundamental Theorem of Calculus, and the equality (49) holds naturally from Leibniz rule. Inequalities (51) and (53) follow from $\nabla h(z)^\top \dot{z}_s \geq -\alpha h(z)$ and Cauchy Inequality, respectively. Then $\forall t$, we have

$$h(z(t)) \geq e^{-\alpha t}h(z) - C_h\int_0^t e^{-\alpha(t-s)}\|\dot{e}(s)\|ds. \tag{54}$$

To be further, $\forall t \geq 0$, we have

$$h(z(t))$$
$$\geq e^{-\alpha t}h(z) - C_h\int_0^t e^{-\alpha(t-s)}\|\dot{e}(s)\|ds \tag{55}$$
$$\geq e^{-\alpha t}h(z) - C_h M e^{-\alpha t}\int_0^t e^{-(\beta-\alpha)s}\|\dot{e}\|ds \tag{56}$$
$$= e^{-\alpha t}h(z) - \frac{C_h M}{\beta-\alpha}e^{-\alpha t}(1-e^{-(\beta-\alpha)t})\|\dot{e}\| \tag{57}$$
$$\geq e^{-\alpha t}\frac{V(z,\dot{e})}{\alpha_e} - \frac{C_h M}{\beta-\alpha}e^{-\alpha t}(1-e^{-(\beta-\alpha)t})\|\dot{e}\| \tag{58}$$
$$\geq e^{-\alpha t}V(z,\dot{e})[\frac{1}{\alpha_e} - \frac{C_h M}{a_1(\beta-\alpha)}(1-e^{-(\beta-\alpha)t})] \tag{59}$$
$$\geq e^{-\alpha t}V(z,\dot{e})[\frac{1}{\alpha_e} - \frac{C_h M}{a_1(\beta-\alpha)}] \tag{60}$$
$$= e^{-\alpha t}V(z,\dot{e})\frac{C_h M}{a_1(\beta-\alpha)}(\frac{a_2}{a_1} - 1) \tag{61}$$
$$\geq 0, \tag{62}$$

where the inequality (56) holds from Theorem 2, i.e., the exponential stability of tracking error under recurrence condition, the inequality (58) holds since $(z,\dot{e}) \in S_V$, the inequality (59) is obtained from the linear lower bound of the tracking error $a_1\|\dot{e}\| \leq V(z,\dot{e})$, and the inequality (62) follows from the fact that $a_1 \leq a_2$ and $\beta > \alpha$.

Since $h(\Pi(x(t))) \geq 0, \forall t \geq 0$, then we have $x(t) \in \mathcal{S}_{\mathrm{FoM}}, \forall t \geq 0$, i.e., the trajectory is always in the safe region, which completes the proof. □