Generalized Barrier Functions: Integral Conditions & Recurrent Relaxations

Yue Shen, Hussein Sibai, and Enrique Mallada

***Abstract*— Barrier functions constitute an effective tool for assessing and enforcing safety-critical constraints on dynamical systems. To this end, one is required to find a function h that satisfies a Lyapunov-like differential condition, thereby ensuring the invariance of its zero super-level set $h_{\geq 0}$. This methodology, however, does not prescribe a general method for finding the function h that satisfies such differential conditions, which, in general, can be a daunting task. In this paper, we seek to overcome this limitation by developing a generalized barrier condition that makes the search for h easier. We do this in two steps. First, we develop integral barrier conditions that reveal equivalent asymptotic behavior to the differential ones, but without requiring differentiability of h. Subsequently, we further replace the stringent invariance requirement on $h \geq 0$ with a more flexible concept known as recurrence. A set is (τ-)recurrent if every trajectory that starts in the set returns to it (within τ seconds) infinitely often. We show that, under mild conditions, a simple sign distance function can satisfy our relaxed condition and that the (τ-)recurrence of the super-level set $h_{\geq 0}$ is sufficient to guarantee the system's safety.**

I. Introduction

The ability to enforce safety in a dynamical system is a fundamental requirement for many engineering systems. Air-traffic control [1], life support devices [2], robotics [3], and autonomous driving [4], are prominent examples of safety-critical applications where a failing event can be catastrophic. Informally, a system being safe implies that no "bad" event will ever happen [5]. From a formal standpoint, verifying a safety property usually involves an invariance agrument [6], and thus, it is not surprising that guaranteeing the safety of a dynamical system usually involves finding invariant sets, as it is the case for backward reachable sets from reachability theory [1] and barrier functions [7]–[9].

Barrier functions, particularly, have become a prominent option. First proposed in [7], barrier functions build on Nagumo's invariance theorem [10] to establish the invariance of a set $\mathcal{C}$. Under this framework, one is requiblack to find both the target set $\mathcal{C}$ and a function h, with $\mathcal{C} = h_{\geq 0} := \{x : h(x) \geq 0\}$, that satisfies a differential condition (c.f., e.g., (3)) which, among other things, renders $\mathcal{C}$ invariant. The success of this approach is ingrained in the flexibility its framework, which allows extensions to account for the role of stochasticity [8], incorporate robustness guarantees [9], [11], and design safety controllers for systems with inputs [12]. Leading to the so-called Control Barrier Function (CLF)–a generalization of Sontag's Control Lyapunov Function [13]–that can be combined with performance requirements such as stabilization [14], [15]. We refer the reader to [16] for a good recount of the history, theory, and applications of CLFs. Unfortunately, despite the many benefits of barrier functions, some limitations prevent their widespread use. First, there is no a priori computationally efficient method to find either the set $\mathcal{C}$ or the function h. Though Sum-of-Squares (SoS) methods have been proposed in the literature [17]–[19], such methods do not scale beyond a few dimensions. Second, when the system to be controlled is underactuated, finding the set $\mathcal{C}$ is highly non-trivial and often renders very conservative solutions [20], [21].

The goal of this paper is to relax the conditions that characterize BFs, intending to make the search for them an easier undertaking. Our approach stems from the insight that though invariance is a safety property, not every safety property is an invariance one. In fact, despite the many benefits that invariance has brought to control theory, e.g., Lyapunov Theory, we argue that it is the invariance conditions on $\mathcal{C}$ and the super-level sets of h that makes the search for barrier functions intricate. From a theoretical standpoint, invariance imposes stringent conditions on the geometry of the set $\mathcal{C}$ that can be hard to meet for many parameterizations of h. Moreover, from a practical standpoint, even when $\mathcal{C}$ and h are found, enforcing invariance (strictly speaking) would require solving a quadratic program instantaneously at all times [15].

To overcome these challenges, we seek to relax the invariance condition on h and substitute it with the more flexible notion of recurrence. A set is (τ-)recurrent if every trajectory that starts in the set returns to it (within τ seconds) infinitely often. τ-recurrent sets allow trajectories to leave the set and thus constitute a strict relaxation of invariance—every invariant set is τ-recurrent, but not the other way around. Recent work has shown recurrence to be a powerful mechanism for analyzing dynamical systems, including estimating regions of attractions of stable equilibrium points [22] and certifying stability via generalized Lyapunov conditions [23]. Note that, from an information theoretical viewpoint, making a set τ-recurrent requires less information than to make it invariant [24], thus making it a more beneficial search target. We further refer the reader to [25], [26], where a similar definition of recurrence (p-invariance) was introduced in the context of constrained model pblackictive control.

Building on this literature, in this paper, we relax the conditions defining classic barrier functions toward conditions that render $\mathcal{C}$ τ-recurrent. We achieve this goal in two steps. First, we introduce integral barrier conditions that eschew the requirement for differentiability while providing the same guarantees as classical barrier functions. Subsequently, we replace the stringent invariance condition, a cornerstone of

Yue Shen and Enrique Mallada are with the Department of Electrical and Computer Engineering, Johns Hopkins University, 3400 N Charles St., Baltimore, MD 21218, USA. Email:{yshen50,mallada}@jhu.edu.

Hussein Sibai is with the Department of Computer Science and Engineering, Washington University in St. Louis, One Brookings Drive, St. Louis, MO 63130, USA. Email:sibai@wustl.edu

This work was supported by NSF through grants CAREER 1752362, CPS 2136324 and Global Centers 2330450.

classical and integral barrier functions, with the more flexible concept of recurrence. Notably, we show that under mild conditions, the existence of an exponential barrier function [19] is sufficient to guarantee that virtually any set that contains the super-level set of that function, equipped with a signed distance, satisfies our recurrent barrier conditions and that despite allowing trajectories to leave, recurrent barrier functions can be used to certified safety.

The remainder of this paper is structublack as follows: Section II introduces preliminary definitions pertinent to dynamical systems and safety and revisits classical barrier conditions that will be utilized throughout this work. In Section III, we detail the development of integral-based barrier functions. This discussion is expanded in Section IV, where we introduce and elaborate on the recurrence conditions. Under these conditions, one can certify that a super-level set of h is τ-recurrent, leading to the boundedness of trajectories. In Section V, we demonstrate the generality of the recurrence conditions, specifically showing that almost every set would satisfy our recurrent barrier conditions with a bounded time horizon τ. In Section VI, we argue that recurrent sets are functionally equivalent to invariant sets and, thus, can be indicative of safety. Finally, we conclude in Section VII.

Notation: Given a set S and an arbitrary norm (denoted by $\|\cdot\|$), we use $\mathrm{sd}(x, S)$ to denote the signed distance between a point x and S, i.e.,

$$\mathrm{sd}(x,S) := \begin{cases} \inf_{y\in\partial S} \|y-x\| & \text{if } x \notin S \\ -\inf_{y\in\partial S} \|y-x\| & \text{if } x \in S. \end{cases}$$

We further use $\mathrm{P}_S(x)$ to denote the set of projections of a point x on a non-empty closed set S, i.e.,

$$\mathrm{P}_S(x) := \arg\min_{y\in S} \|y-x\|.$$

We respectively use $[x]_+$ and $[x]_-$ to denote the projection of a number $x \in \mathbb{R}$ on the set of all non-negative real numbers and the set of all non-positive real numbers, respectively. For a continuous function h, we use $h_{<0}$ ($h_{\leq 0}$), $h_{>0}$ ($h_{\geq 0}$), and $h_{=0}$ to denote the open (closed) sub-level set, open (closed) super-level set and closed level set, respectively.

II. Preliminary Results

Throughout the paper, we consider a continuous-time dynamical system described using the following ordinary differential equation:

$$\dot{x} = f(x)\,, \tag{1}$$

where $x \in D \subset \mathbb{R}^n$ is the state, and the map $f : D \to \mathbb{R}^n$ is a continuous and locally Lipschitz function defined over a domain D. Given an initial state x_0, we use $\phi(t, x_0)$ to denote the solution of (1). We assume system 1 is forward complete, which is specified in the following assumption.

Assumption 1 (Forward Completeness)**.** *For any $x \in D$, the trajectory $\phi(\cdot, x)$ is defined for all $t \in [0, \infty)$.*

In the following, we formally define the notions of safety and invariance.

Definition 1 (Safety)**.** *Let $\mathcal{X}_u \subseteq D$ be a set of unsafe states, a trajectory $\phi(t, x_0)$ of* (1) *is unsafe if there exists a time $t \geq 0$ such that $\phi(t, x_0) \cap \mathcal{X}_u \neq \emptyset$.*

We say that a set $\mathcal{X}_s \subseteq D$ is a safe state space region if there are no unsafe trajectories starting from $\mathcal{X}_s$.

Definition 2 (Invariant Set)**.** *A set $S \subseteq D$ is invariant w.r.t.* (1) *if and only if:*

$$x_0 \in S \implies \phi(t, x_0) \in S, \quad \forall\, t \in \mathbb{R}_{\geq 0}.$$

The notion of invariance is closely related to the barrier function methods that characterize safe state space regions. By trapping trajectories on level sets of a function, one can ensure the system's safety whenever its initial state belongs to an invariant set S that does not intersect with $\mathcal{X}_u$.

Next, we review some classic formulations of barrier functions that certify the invariance of their super-level sets. These formulations require the barrier functions to be differentiable, which we will relax in the following sections. We start with the most basic formulation: Nagumo's. It only requires the time derivative of the barrier function h, which is also its Lie derivative along f, to be non-negative at any state of its zero level set $h_{=0}$. That implies that whenever a trajectory reaches the boundary of that set $h_{\geq 0}$ from its interior, h must not decrease. Thus, the trajectory must remain in the set, ensuring the latter's invariance.

Theorem 1 (Nagumo's Barrier Functions)**.** *[27, Th 3.1] Consider a dynamical system* (1) *and a differentiable function $h : D \subset \mathbb{R}^n \to \mathbb{R}$, then h is a Nagumo's Barrier Function (NBF) satisfying:*

$$L_f h(x) := \lim_{t\to 0} \frac{h(\phi(t,x)) - h(x)}{t} \geq 0, \quad \forall x \in h_{=0}, \tag{2}$$

if and only if the super-level set $h_{\geq 0}$ is invariant.

The following theorem adds an additional constraint to Nagumo's definition that further lower-bounds the rates at which h *can* at most decrease along the trajectories starting from the interior of its super-level set and the least at which it *should* increase along the trajectories starting from the exterior of that set. That ensures safe trajectories approach the boundary slowly, if at all, and possibly unsafe ones converge to the safe set fast enough, ensuring its stability. The latter is not guaranteed by Nagumo's version.

Definition 3 (Extended class $\mathcal{K}$ function)**.** *[15, Def 2] A continuous function $\zeta := (-b, a) \to (-\infty, \infty)$ is said to belong to extended class $\mathcal{K}$ for some $a, b > 0$ if it is strictly increasing and $\zeta(0) = 0$.*

Theorem 2 (Zeroing Barrier Functions)**.** *[15, Prop 1] Consider a differentiable function $h : D \to \mathbb{R}$ and an extended class $\mathcal{K}$ function ζ. Assume there exists a super-level set $D_0 := h_{\geq -c} \subseteq D$ for some $c \geq 0$ such that:*

$$L_f h(x) \geq -\zeta(h(x)), \quad \forall x \in D_0, \tag{3}$$

then:

(i) h is called a Zeroing Barrier Functions (ZBF), and the super-level set $h_{\geq 0}$ is invariant.

(ii) *whenever $x \in h_{<0} \cap D_0$, then as long as $h(\phi(t,x)) < 0$, $h(\phi(t,x))$ must monotonically increase to zero, at least, with a positive rate of $-\zeta(h(\phi(t,x)))$, and*

(iii) *whenever $h(\phi(t,x)) > 0$, then $h(\phi(t,x))$ may decrease to zero, at most, with a negative rate of $-\zeta(h(\phi(t,x)))$.*

The bounds on the boundary-approaching rates of Zeroing barrier functions in Theorem 2 are non-uniform and state-dependent. In the following remark and the rest of the section, we describe the special case when ζ is a (piece-wise) linear function of $h(x)$, which results in uniform exponential bounds on the evolution of $h(\phi(t,x))$.

Remark 1. *[15, Remark 6] A special case of* (3) *is:*

$$L_f h(x) \geq -\alpha h(x),\ \forall x \in D_0, \tag{4}$$

for some $\alpha > 0$. Since $\zeta(s) = \alpha s$ is an extended class $\mathcal{K}$ function, the super-level set $h_{\geq 0}$ is invariant as stated in Theorem 2. This formulation is commonly used since it leads to a convex problem that can be efficiently solved using techniques like SoS programming [17], [19].

The following lemma provides the formal statement on the exponential bounds on the convergence rates of the function value $h(\phi(t,x))$ under condition (4).The proof is analogous to [19, Thm 1], with an extension to consider all states $x \in h_{\leq -c}$ instead of just $h_{\leq 0}$.

Lemma 1. *Consider a continuous function $h : D \to \mathbb{R}$, an $\alpha > 0$, and a super-level set $D_0 := h_{\geq -c}$ for some $c \geq 0$, then condition* (4) *implies the following exponential convergence result:*

$$h(\phi(t,x)) \geq e^{-\alpha t} h(x), \quad \forall t \geq 0,\, x \in D_0.$$

Proof. Rearranging condition (4) gives:

$$g(x) := L_f h(x) - (-\alpha) h(x) \geq 0,\ \forall x \in D_0$$

We then have the following differential equation:

$$\begin{aligned} \frac{dh(\phi(t,x))}{dt} &= L_f h(\phi(t,x)) \\ &= g(\phi(t,x)) + (-\alpha) h(\phi(t,x)) \end{aligned} \tag{5}$$

Solving (5) gives:

$$h(\phi(t,x)) = e^{-\alpha t}\left(\int_0^t g(\phi(s,x)) e^{\alpha s} ds + h(x)\right). \tag{6}$$

Note that condition (4) ensures $h_{\geq 0}$ is invariant and the function value $h(\phi(t,x))$ is strictly increasing along the trajectory starting from $x \in h_{<0} \cap D_0$. Therefore, D_0 is also an invariant set, and thus

$$\int_0^t g(\phi(s,x)) ds \geq 0,\ \ \forall t \geq 0, x \in D_0. \tag{7}$$

By applying (7) to (6), one can conclude that:

$$h(\phi(t,x)) \geq e^{-\alpha t} h(x), \quad \forall t \geq 0,\, x \in D_0,$$

and the result follows. □

We end this section by generalizing the case in Remark 1 to one where the bounds on the different sides of the boundary differ, calling the resulting functions *exponential barrier functions (EBFs)*. The need for this generalization will become clear in Section V, where we show the generality of our newly developed conditions.

Theorem 3 (Exponential Barrier Functions)**.** *Consider a differentiable function $h : D \subset \mathbb{R}^n \to \mathbb{R}$, and parameters $\alpha, \beta > 0$. Assume there exists a super-level set $D_0 := h_{\geq -c}$ for some $c \geq 0$ such that:*

$$L_f h(x) \geq -(\beta [h(x)]_- + \alpha [h(x)]_+), \quad \forall x \in D_0, \tag{8}$$

then:

(i) *we call h an Exponential Barrier Function (EBF), the super-level set $h_{\geq 0}$ is positively invariant,*

(ii) *whenever $x \in h_{<0} \cap D_0$, then as long as $h(\phi(t,x)) < 0$, $h(\phi(t,x))$ must monotonically increase to zero, at least, with a positive linear rate of $-\beta h(\phi(t,x))$, which implies,*

$$h(\phi(t,x)) \geq e^{-\beta t} h(x),\ \forall t \geq 0,\, x \in h_{<0} \cap D_0, \tag{9}$$

and

(iii) *whenever $x \in h_{>0}$, then $h(\phi(t,x))$ may decrease to zero, at most, with a negative linear rate of $-\alpha h(\phi(t,x))$, which implies,*

$$h(\phi(t,x)) \geq e^{-\alpha t} h(x), \quad \forall t \geq 0,\, x \in h_{\geq 0}. \tag{10}$$

Proof. (i): Starting from any initial state $x \in h_{=0}$, condition (8) requires $L_f h(x) \geq 0$, i.e., condition (2). According to Theorem 1, this guarantees the invariance of the super-level set $h_{\geq 0}$.

(ii): As demonstrated in the proof of Lemma 1, the following two conditions hold for all $x \in D_0 \cap h_{<0}$:

$$\begin{aligned} g(x) &= L_f h(x) - (-\beta) h(x) \geq 0, \\ h(\phi(t,x)) &= e^{-\beta t}\left(\int_0^t g(\phi(s,x)) e^{\beta s} ds + h(x)\right). \end{aligned} \tag{11}$$

Note that the super-level set $h_{\geq 0}$ is invariant, and the function value $h(\phi(t,x))$ increases strictly along the trajectory starting from any $x \in D_0 \cap h_{<0}$. Hence, D_0 is an invariant set.

Now, starting from $x \in h_{<0} \cap D_0$, if there is a time t' with $h(\phi(t',x)) \geq 0$, then condition (9) is automatically satisfied for any time $t \geq t'$. Thus, w.l.o.g. we concentrate on the scenario where $h(\phi(t,x)) < 0$ for all $t \geq 0$. In this case, $\phi(t,x)$ remains within $D_0 \cap h_{<0}$, and consequently:

$$\int_0^t g(\phi(s,x)) ds \geq 0,\ \ \forall t \geq 0. \tag{12}$$

By applying (12) to (11), we successfully verify (10).

(iii): This result is trivially obtained by applying Lemma 1 with $D_0 = h_{\geq 0}$. □

We finalize by reminding that all the results of this section require the differentiability of h. In the next section, we relax this requirement.

III. Integral Barrier Functions

This section examines integral-based versions of the classic barrier function formulations previously discussed. We will explore how these integral forms relate to earlier formulations and their implications for the invariance of a set. Specifically, in Theorems 4, 5, and 6, we present the integral-based versions of the barrier function formulations of Theorems 1, 2, and 3, respectively. Additionally, we discuss in Remarks 2, 3, and 4 the equivalence between the integral formulations and classical ones when the barrier function is differentiable.

Theorem 4 (Integral Nagumo's Barrier Functions)**.** *Consider a continuous function $h : D \to \mathbb{R}$. The super-level set $h_{\geq 0}$ is invariant if and only if h satisfies:*

$$h(\phi(t, x)) \geq 0, \forall x \in h_{=0}, t \geq 0, \tag{13}$$

in which case we call h an Integral Nagumo's Barrier Function (INBF).

Proof. ($\Longrightarrow$): This direction follows directly from the definition of the invariant set. Precisely, the super-level set $h_{\geq 0}$ being invariant implies:

$$\phi(t, x) \in h_{\geq 0} \implies h(\phi(t, x)) \geq 0,$$

for all $t \geq 0$ and $x \in h_{\geq 0}$.

($\Longleftarrow$): Suppose $h_{\geq 0}$ is not invariant. Then there must exist an initial state $x \in h_{\geq 0}$ and a time instant $t^* \geq 0$, such that $\phi(t^*, x) \notin h_{\geq 0}$. We then use t' to denote the last time the trajectory $\phi(t, x)$ stays within the closed set $h_{\geq 0}$ before t^*, i.e.,

$$x' := \phi(t', x) \in h_{=0} \quad \text{and} \quad h(\phi(t, x)) < 0, \ \forall t \in (t', t^*].$$

This contradicts condition (13), which requires $h(\phi(t, x'))$ to always remain non-negative when starting from $h(x') = h(\phi(t', x)) = 0$. Therefore, the result follows. ☐

Remark 2. *Theorem 4 only requires h to be continuous. If h is also differentiable, then* (13) *is equivalent to* (2)*. Specifically, starting from an arbitrary point $x \in h_{=0}$, condition* (13) *implies that $h(\phi(t, x)) \geq 0$ for all $t \geq 0$. Therefore, $h(\phi(t, x)) - h(x) \geq -h(x) = 0, \ \forall x \in h_{=0}, t \geq 0$. Then, by evaluating the Lie derivative:*

$$L_f h(x) = \lim_{t \to 0} \frac{h(\phi(t, x)) - h(x)}{t} \geq 0,$$

condition (2) *follows.*

For the other direction, note that condition (2) *implies $h_{\geq 0}$ is invariant, which is equivalent to condition* (13)*.*

In the following theorem, we relax the differentiability conditions on zeroing barrier functions recalled in Theorem 2 and define the *integral zeroing barrier* functions.

Theorem 5 (Integral Zeroing Barrier Functions)**.** *Consider a continuous function $h : D \subset \mathbb{R}^n \to \mathbb{R}$, and an extended class $\mathcal{K}$ function ζ. Assume there exists a super-level set $D_0 := h_{\geq -c}$ for some $c \geq 0$ such that:*

$$h(\phi(t, x)) + \int_0^t \zeta(h(\phi(s, x)))ds \geq h(x), \tag{14}$$

for all $t \geq 0$ and $x \in D_0$, then:

(i) we call h an Integral Zeroing Barrier Function (IZBF), the super-level set $h_{\geq 0}$ is invariant, and

(ii) the conditions (ii)-(iii) stated in Theorem 2 are satisfied.

Proof. (i): Suppose $h_{\geq 0}$ is not invariant. As stated in the proof of Theorem 4, there exists a $t \geq 0$ and $x \in h_{=0}$ such that:

$$h(\phi(s, x)) < 0, \ \forall s \in (0, t].$$

However, condition (14) requires that $h(\phi(t, x)) \geq h(x) - \int_0^t \zeta(h(\phi(s, x)))ds$, and the right-hand side would be non-negative in this case. This is a contradiction.

(ii): We first evaluate the following Taylor expansion for t around 0:

$$\int_0^t \zeta(h(\phi(s, x)))ds = 0 + \zeta(h(\phi(0, x)))t + o(t).$$

This, together with condition (14), implies the following bound on the convergence rate expressed in the form of the lower-right Dini derivative [28]:

$$\begin{aligned} D_+ h(x) :=& \liminf_{t \to 0} \frac{h(\phi(t, x)) - h(x)}{t} \\ \geq& \liminf_{t \to 0} \frac{-\int_0^t \zeta(h(\phi(s, x)))ds}{t} \\ =& \liminf_{t \to 0} \frac{0 - \zeta(h(\phi(0, x)))t - o(t)}{t} \\ =& -\zeta(h(x)), \quad \forall x \in D_0. \end{aligned}$$

☐

Remark 3. *Theorem 5 only requires h to be continuous. If h is also differentiable,* (14) *is equivalent to* (3)*, and thus the asymptotic convergence results (ii)-(iii) stated in Theorem 2 are satisfied.*

Precisely, condition (3) *follows by evaluating the following Lie derivative everywhere under condition* (14)*:*

$$\begin{aligned} L_f h(x) =& \lim_{t \to 0} \frac{h(\phi(t, x)) - h(x)}{t} \geq \lim_{t \to 0} \frac{-\int_0^t \zeta(h(\phi(s, x)))ds}{t} \\ =& \lim_{t \to 0} \frac{0 - \zeta(h(\phi(0, x)))t - o(t)}{t} = -\zeta(h(x)). \end{aligned}$$

For the other direction, condition (14) *follows directly from* (3) *by applying the 'fundamental theorem of calculus II' [29].*

In the last theorem of this section, we relax the differentiability conditions on exponential barrier functions introduced in Theorem 3 and define the *integral exponential barrier* functions.

Theorem 6 (Integral Exponential Barrier Functions)**.** *Consider a continuous function $h : D \subset \mathbb{R}^n \to \mathbb{R}$ and parameters $\alpha, \beta > 0$. Assume there exists a super-level set $D_0 := h_{\geq -c}$ for some $c \geq 0$ such that:*

$$h(\phi(t, x)) \geq e^{-\beta t}[h(x)]_- + e^{-\alpha t}[h(x)]_+, \tag{15}$$

for all $x \in D_0$ and $t \geq 0$, then:

(i) we call h an Integral Exponential Barrier Function (IEBF), the super-level set $h_{\geq 0}$ is invariant, and

(ii) the conditions (ii)-(iii) stated in Theorem 3 are satisfied.

Proof. (i): Suppose $h_{\geq 0}$ is not invariant. As stated in the proof of Theorem 4, there exists a $t \geq 0$ and $x \in h_{=0}$ such that:

$$h(\phi(s,x)) < 0,\ \ \forall s \in (0,t].$$

This contradicts condition (15), which requires the function value $h(\phi(t,x)) \geq h(x)e^{-\alpha t} = 0$ for all $t \geq 0$. Therefore, the result follows.

(ii): This part follows trivially given condition (15). □

Remark 4. *Theorem 6 only requires h to be continuous. If h is also differentiable, then* (15) *is equivalent to* (8). *In particular, condition* (8) *follows by evaluating the following Lie derivative everywhere under condition* (15)*, i.e.,*

Whenever $x \in h_{\geq 0}$:

$$L_f h(x) = \lim_{t\to 0} \frac{h(\phi(t,x)) - h(x)}{t} \geq \lim_{t\to 0} \frac{h(x)e^{-\alpha t} - h(x)}{t}$$
$$= h(x) \lim_{t\to 0} \frac{e^{-\alpha t} - 1}{t} \qquad = -\alpha h(x).$$

Similarily, whenever $x \in D_0 \cap h_{<0}$:

$$L_f h(x) = \lim_{t\to 0} \frac{h(\phi(t,x)) - h(x)}{t} \geq -\beta h(x).$$

For the other direction, we have shown condition (8) *implies* (15) *in Theorem 3 part (ii-iii).*

IV. Recurrent Barrier Functions

We are now ready to provide a relaxation to the barrier conditions that lead to the invariance of the super-level set $h_{\geq 0}$. To relax the notion of invariance, one must allow trajectories to temporarily leave a set. Our recurrent condition, first proposed in [22, Def 4], requires trajectories to return to the set within a finite time, infinitely many times.

To be able to provide safety guarantees under recurrence, we additionally require the trajectories to return back to the set within a predefined finite duration every time they leave it. This is formalized in a stronger notion of recurrence known as τ-recurrence [22, Def 5].

Definition 4 (Recurrent and τ-Recurrent Sets)**.**
A set $S \subseteq \mathbb{R}^n$ is recurrent w.r.t. (1)*, if for any $x \in S$ and $t \geq 0$,*

$$\exists\, t' > t, \quad \text{s.t.} \quad \phi(t', x) \in S.$$

A set $S \subseteq \mathbb{R}^n$ is τ-recurrent w.r.t. (1)*, if for any $x \in S$, and $t \geq 0$,*

$$\exists\, t' > t, \quad \text{with} \quad t' - t \in (0, \tau] \quad \text{s.t.} \quad \phi(t', x) \in S.$$

Note that, while not invariant, such a τ-recurrent set guarantees that solutions starting in this set will visit it back within τ-seconds infinitely often. In particular, by Definition 2, an invariant set is τ-recurrent for any $\tau > 0$. Additionally, a 0-recurrent set is invariant. Thus, Definition 4 generalizes the notion of invariance by allowing the solution $\phi(t, x_0)$ to step outside the set S.

In the rest of this section, we generalize the aforementioned barrier function formulations into recurrent-based versions that certify the τ-recurrence of the super-level set $h_{\geq 0}$. Specifically, in theorems 7, 8, and 9, we present the recurrence-based versions of the integral barrier function formulations of theorems 4, 5, and 6, respectively. The relationships between all barrier functions are summarized in Fig 1. As usual, we start with Nagumo's version.

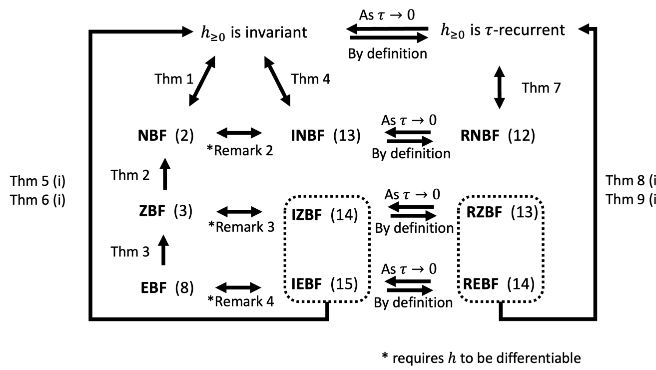

Fig. 1: Relationships among classic barrier functions (Theorems 1-3), integral barrier functions (Theorem 4-6), and recurrent barrier functions (Theorem 7-9).

Theorem 7 (Recurrent Nagumo's Barrier Functions)**.** *Consider a continuous function $h : D \to \mathbb{R}$, then the super-level $h_{\geq 0}$ is τ-recurrent if and only if h satisfies:*

$$\max_{t\in(0,\tau]} h(\phi(t,x)) \geq 0,\ \ \forall x \in h_{=0}, \tag{16}$$

in which case we call it a Recurrent Nagumo's Barrier Function (RNBF).

Proof. ($\Longrightarrow$): This direction follows directly from the definition of a τ-recurrent set. Precisely, the super-level set $h_{\geq 0}$ being τ-recurrent implies that for all $x \in h_{\geq 0}$,

$$\exists t' \in (0,\tau] \text{ s.t. } \phi(t',x) \in h_{\geq 0} \implies \max_{t\in(0,\tau]} h(\phi(t,x)) \geq 0.$$

($\Longleftarrow$): Suppose the closed set $h_{\geq 0}$ is not τ-recurrent. There must exist an initial state $x \in h_{=0}$ such that $\phi(t,x) \notin h_{\geq 0}$ for all $t \in (0,\tau]$. In this case, $h(\phi(t,x)) < h(x) = 0$ for all $t \in (0,\tau]$, which contradict with condition (16). Therefore, $h_{\geq 0}$ is τ-recurrent. □

As in the differential and integral formulations, Nagumo's version of recurrence does not restrict the rate at which the barrier function changes along the trajectories. For that, we will need to introduce the Zeroing formulation.

Definition 5 (Recurrent Zeroing Barrier Functions)**.**
A continuous function $h : D \subset \mathbb{R}^n \to \mathbb{R}$ is a Recurrent Zeroing Barrier Function (RZBF) if there exists an extended class $\mathcal{K}$ function ζ and a super-level set $D_0 := h_{\geq -c}$, $c \geq 0$, such that:

$$\max_{t\in(0,\tau]} h(\phi(t,x)) + \int_0^t \zeta(h(\phi(s,x)))ds \geq h(x),\ \forall x \in D_0. \tag{17}$$

In contrast with the differential and integral formulations of zeroing barrier functions, the recurrent one does not constrain the rates at which the barrier function changes at all time instants of a trajectory, but only on countably infinite many times that are most separated by τ in consecutive steps.

Lemma 2. *Consider a Recurrent Zeroing Barrier Function*

h defined over $D_0 := h_{\geq -c}$ for some $c \geq 0$. Then, for any $x \in D_0$, there exists a sequence of times $\{t_n\}_{n\in\mathbb{N}}$, with $t_0 = 0$,

$$t_{n+1} = \max\{\arg\max_{t\in(t_n,t_n+\tau]} h(\phi(t,x)) + \int_{t_n}^{t} \zeta(h(\phi(s,x)))ds\},$$

$$\lim_{n\to\infty} t_n = \infty, \text{ and } t_{n+1} - t_n \in (0,\tau], \ \ \forall n \in \mathbb{N}, \tag{18}$$

such that for each $x_n := \phi(t_n, x)$, $x_n \in D_0$, and

$$h(x_{n+1}) \geq \max\{h(x_n) + \delta_n, \ [h(x_n)]_-\}, \forall n \in \mathbb{N}, \tag{19}$$

with $\delta_n := -\int_0^{t_{n+1}-t_n} \zeta(h(\phi(s,x_n)))ds$, satisfying $\delta_n > 0$ whenever $h(x_n)$ and $h(x_{n+1}) < 0$.

Proof. See Appendix (Section VIII-A). ☐

The following theorem gives a detailed explanation of the implications of the Recurrent Zeroing Barrier Function.

Theorem 8. *Consider a Recurrent Zeroing Barrier Function h defined over $D_0 := h_{\geq -c}$ for some $c \geq 0$ and let $x_n := \phi(t_n, x)$ be the states along the sequence of times $\{t_n\}_{n\in\mathbb{N}}$ specified in Lemma 2. Then:*

(i) the super-level set $h_{\geq 0}$ is τ-recurrent,

(ii) whenever $x \in h_{<0} \cap D_0$, then as long as $h(x_n) < 0$, $h(x_{n+1})$ must monotonically increase, at least, by a step size of $\delta_n > 0$, if $h(x_{n+1}) < 0$, or $-h(x_n) > 0$, if $h(x_{n+1}) \geq 0$, and

(iii) whenever $h(x_n) > 0$, then $h(x_{n+1})$ may decrease to zero, at most, by a negative step size of $\max\{\delta_n, -h(x_n)\}$.

Proof. (i): Suppose the closed set $h_{\geq 0}$ is not τ-recurrent. Then, there exists an initial condition $x \in h_{=0}$ such that $\phi(t,x) \notin h_{\geq 0}$ for all $t \in (0,\tau]$. In this case, $h(\phi(t,x)) < 0$ and $\zeta(h(\phi(t,x))) < 0$ for all $t \in (0,\tau]$. Note that this contradicts condition (17), which requires its left-hand side to be nonnegative starting from $h(x) = 0$. Therefore, $h_{\geq 0}$ is τ-recurrent.

(ii): Starting from $h(x_n) < 0$, if $h(x_{n+1}) \geq 0$, a positive step size $h(x_{n+1}) - h(x_n) \geq -h(x_n) > 0$ can be ensured automatically. Conversely, if $h(x_{n+1}) < 0$, it follows from Lemma 2 that $h(x_{n+1}) \geq h(x_n) + \delta_n$ with $\delta_n > 0$. Thus, a positive step size $h(x_{n+1}) - h(x_n) \geq \delta_n > 0$ is also guaranteed.

(iii): Given $h(x_n) > 0$, the inequality (19) requires:

$$h(x_{n+1}) \geq h(x_n) + \delta_n \quad \text{and} \quad h(x_{n+1}) \geq 0.$$

Then, by rearranging terms, one can conclude $h(x_{n+1}) - h(x_n) \geq \max\{\delta_n, -h(x_n)\}$. ☐

As before, we end with the exponential formulation which constrains the rates using exponential functions of time. As in the zeroing version, it only constrains the rates at countably infinite time instants.

Definition 6 (Recurrent Exponential Barrier Functions). *A continuous function $h : D \subset \mathbb{R}^n \to \mathbb{R}$ is a Recurrent Exponential Barrier Function (REBF) if there exists parameters $\alpha, \beta > 0$ and a super-level set $D_0 := h_{\geq -c}$, for some $c \geq 0$, such that:*

$$\max_{t\in(0,\tau]} e^{\beta t}[h(\phi(t,x))]_- + e^{\alpha t}[h(\phi(t,x))]_+ \geq h(x), \tag{20}$$

Lemma 3. *Consider a Recurrent Exponential Barrier Function h defined over $D_0 := h_{\geq -c}$ for some $c \geq 0$ with parameters $\alpha, \beta > 0$. Then, for any $x \in D_0$, there exists a sequence of times $\{t_n\}_{n\in\mathbb{N}}$, with $t_0 = 0$,*

$$t_{n+1} = \max\{\arg\max_{t\in(t_n,t_n+\tau]} e^{\beta t}[h(\phi(t,x))]_- + e^{\alpha t}[h(\phi(t,x))]_+\}$$

$$\lim_{n\to\infty} t_n = \infty \text{ and } t_{n+1} - t_n \in (0,\tau], \ \ \forall n \in \mathbb{N}, \tag{21}$$

such that for each state $x_n := \phi(t_n, x)$, we have, $x_n \in D_0$, and

$$h(x_{n+1}) \geq e^{-\beta\Delta t_n}[h(x_n)]_- + e^{-\alpha\Delta t_n}[h(x_n)]_+ , \tag{22}$$

for all $n \in \mathbb{N}$, with $\Delta t_n := t_{n+1} - t_n$.

Proof. See Appendix (Section VIII-B). ☐

We then summarize the implications of the Recurrent Exponential Barrier Function in the next theorem.

Theorem 9. *Consider a Recurrent Exponential Barrier Function h defined over $D_0 := h_{\geq -c}$ for some $c \geq 0$ with parameters $\alpha, \beta > 0$ and let $x_n := \phi(t_n, x)$ be the states along the sequence of times $\{t_n\}_{n\in\mathbb{N}}$ specified in Lemma 3. Then:*

(i) the super-level set $h_{\geq 0}$ is τ-recurrent,

(ii) whenever $x \in h_{<0} \cap D_0$, then as long as $h(x_n) < 0$, $h(x_{n+1})$ must monotonically increase, at least, by a positive step size of $\delta_n = (e^{-\beta\Delta t_n} - 1)h(x_n)$, which implies,

$$h(x_n) \geq h(x)e^{-\beta t_n}, \ \forall n \in \mathbb{N}, x \in h_{<0} \cap D_0, \tag{23}$$

and

(iii) whenever $h(x_n) > 0$, then $h(x_{n+1})$ may decrease to zero, at most, by a negative step size of $\delta_n = (e^{-\beta\Delta t_n} - 1)h(x_n)$, which implies,

$$h(x_n) \geq h(x)e^{-\alpha t_n}, \ \forall n \in \mathbb{N}, x \in h_{\geq 0}. \tag{24}$$

Proof. (i): Suppose the closed set $h_{\geq 0}$ is not τ-recurrent. There exists an initial condition $x \in h_{=0}$ such that $\phi(t,x) \notin h_{\geq 0}$ for all $t \in (0,\tau]$. In this case, $h(\phi(t,x)) < 0$ for all $t \in (0,\tau]$. Note that this contradicts condition (20), which requires $\max_{t\in(0,\tau]} e^{\beta t} h(\phi(t,x)) \geq 0$ starting from $h(x) = 0$. Therefore, $h_{\geq 0}$ is τ-recurrent.

(ii): Given $h(x_n) < 0$, then inequality (22) requires $h(x_{n+1}) \geq e^{-\beta\Delta t_n} h(x_n)$. Thus,

$$h(x_{n+1}) - h(x_n) = (e^{-\beta\Delta t_n} - 1)h(x_n) > 0.$$

Now, starting from $x \in h_{<0} \cap D_0$, we have

$$\begin{aligned} 0 > h(x_n) &\geq e^{-\beta(t_n - t_{n-1})}h(x_{n-1}) \\ &\geq e^{-\beta(t_n - t_{n+1} + t_{n+1} - t_{n-2})}h(x_{n-2}) \geq h(x)e^{-\beta t_n}. \end{aligned}$$

whenever $x_n, ..., x_1 \in h_{<0} \cap D_0$. If there exists a $n' \in \{1, \ldots, n\}$ such that $h(x_{n'}) \geq 0$, (23) still follows since

$$h(x_n) \geq h(x_{n'})e^{-\hat{\alpha}(t_n - t_{n'})} \geq 0 > h(x)e^{-\beta t_n}.$$

(iii): Whenever $h(x_n) > 0$, then inequality (22) requires $h(x_{n+1}) \geq e^{-\alpha \Delta t_n} h(x_n)$. Thus,

$$h(x_{n+1}) - h(x_n) = (e^{-\alpha \Delta t_n} - 1) h(x_n) < 0.$$

In the case that $x \in h_{\geq 0}$, we have:

$$\begin{aligned} h(x_n) &\geq e^{-\alpha(t_n - t_{n-1})} h(x_{n-1}) \\ &\geq e^{-\alpha(t_n - t_{n+1} + t_{n+1} - t_{n-2})} h(x_{n-2}) \\ &\geq h(x) e^{-\alpha t_n} \geq 0, \end{aligned}$$

i.e., (24) follows. □

V. The Generality of Recurrence Conditions

In the previous section, we introduced a set of novel barrier conditions that relaxed the invariant requirement on the zero super level set of h. We will now show that this relaxation widely decouples the geometry of the vector field with the geometry of the level sets of h. This allows us to characterize a vast family of sets and functions that can be used to certify safety. Our prior work inspires our results, [22], wherein we show that under mild conditions, every set contained within the region of attraction of an equilibrium point is recurrent see, e.g., [22, Cor 2]. In this section, we generalize this idea in the context of certifying safety. We start by introducing mild regularity constraints on h, which we will need later in Theorem 10. This requires us to introduce the notion of sector-bounded functions.

Definition 7 (Sector Containment). *Let $h : D \subset \mathbb{R}^n \to \mathbb{R}$ be continuous. If $\exists \alpha_1, \alpha_2 > 0$ such that*

$$(h(x) - \alpha_1 \mathrm{sd}(x, h_{\leq 0}))(h(x) - \alpha_2 \mathrm{sd}(x, h_{\leq 0})) \leq 0, \quad (25)$$

for all $x \in D$, we say that h is ***sector-contained****.*

Remark 5. *Given parameters $\alpha_2 \geq \alpha_1 > 0$, (25) is satisfied if and only if for all $x \in D$:*

$$\alpha_2 \mathrm{sd}(x, h_{\leq 0}) \geq h(x) \geq \alpha_1 \mathrm{sd}(x, h_{\leq 0}) \geq 0 \ \textbf{\textit{if}}\ h(x) \geq 0, \quad (26)$$

$$0 \geq \alpha_1 \mathrm{sd}(x, h_{\leq 0}) \geq h(x) \geq \alpha_2 \mathrm{sd}(x, h_{\leq 0}) \ \textbf{\textit{if}}\ h(x) < 0. \quad (27)$$

In the following theorem, we show that the existence of a sector-contained IEBF h is sufficient to make the signed distance to the zero sub-level set of h a REBF. As such, this theorem illustrates the generality of our recurrent condition.

Theorem 10. *Let h be an Integral Exponential Barrier Function with exponential rates α and β, defined over $D_0 := h_{\geq -c}$ for some $c \geq 0$. Then, if h is sector-contained with parameters α_1 and α_2, the function $\hat{h}(\cdot) = \mathrm{sd}(\cdot, h_{\leq 0})$ is a Recurrent Exponential Barrier Function, i.e., the following conditions hold:*

$$\max_{t \in (0, \hat{\tau}]} e^{\hat{\beta} t} [\hat{h}(\phi(t, x))]_- + e^{\hat{\alpha} t} [\hat{h}(\phi(t, x)]_+ \geq \hat{h}(x), \quad (28)$$

for all $x \in D_0$ and any $\hat{\alpha}, \hat{\beta}, \hat{\tau} > 0$ satisfying $\hat{\alpha} > \alpha$, $\hat{\beta} < \beta$, and

$$\hat{\tau} \geq \max\{\frac{\log(\alpha_2/\alpha_1)}{\hat{\alpha} - \alpha}, \frac{\log(\alpha_2/\alpha_1)}{\beta - \hat{\beta}}\}. \quad (29)$$

Proof. Let us first consider the case that $x \in h_{<0} \cap D_0$. In this case, (28) is automatically satisfied if there is a time t' with $h(\phi(t', x)) \geq 0$. Therefore, we focus on the case where $h(\phi(t, x)) < 0$ for all $t \in (0, \hat{\tau}]$. In this case, we have:

$$\begin{aligned} 0 &> \max_{t \in (0, \hat{\tau}]} e^{\hat{\beta} t} \mathrm{sd}(\phi(t, x), h_{\leq 0}) \\ &\geq \max_{t \in (0, \hat{\tau}]} e^{\hat{\beta} t} \frac{1}{\alpha_1} h(\phi(t, x)) && (30) \\ &\geq \max_{t \in (0, \hat{\tau}]} e^{(\hat{\beta} - \beta) t} \frac{1}{\alpha_1} h(x) && (31) \\ &\geq e^{(\hat{\beta} - \beta)\hat{\tau}} \frac{\alpha_2}{\alpha_1} \mathrm{sd}(x, h_{\leq 0}) && (32) \\ &\geq \mathrm{sd}(x, h_{\leq 0}), && (33) \end{aligned}$$

where (31) comes from the definition of the IEBF (15) and (30) and (32) are based on the sector containment assumption (27). Note that $\mathrm{sd}(x, h_{\leq 0}) < 0$, and thus (33) is true whenever $e^{(\hat{\beta} - \beta)\hat{\tau}} \frac{\alpha_2}{\alpha_1} \leq 1$, which is achieved by choosing $\hat{\tau} \geq \log(\alpha_2/\alpha_1)/(\beta - \hat{\beta})$.

Next, starting from $x \in h_{\geq 0}$, observe that $h(\phi(t, x)) \geq 0, \forall t \geq 0$, since Theorem 6 ensures $h_{\geq 0}$ is an invariant set. Therefore, we proceed similarly:

$$\begin{aligned} &\max_{t \in (0, \hat{\tau}]} e^{\hat{\alpha} t} \mathrm{sd}(\phi(t, x), h_{\leq 0}) \\ &\qquad \geq \max_{t \in (0, \hat{\tau}]} e^{\hat{\alpha} t} \frac{1}{\alpha_2} h(\phi(t, x)) && (34) \\ &\qquad \geq \max_{t \in (0, \hat{\tau}]} e^{(\hat{\alpha} - \alpha) t} \frac{1}{\alpha_2} h(x) && (35) \\ &\qquad \geq e^{(\hat{\alpha} - \alpha)\hat{\tau}} \frac{\alpha_1}{\alpha_2} \mathrm{sd}(x, h_{\leq 0}) && (36) \\ &\qquad \geq \mathrm{sd}(x, h_{\leq 0}) \geq 0, && (37) \end{aligned}$$

where (35) comes from the definition of the IEBF (15), (34) and (36) come from the sector containment assumption (26). Now, since $\mathrm{sd}(x, h_{\leq 0}) \geq 0$, (37) is true whenever $e^{(\hat{\alpha} - \alpha)\hat{\tau}} \frac{\alpha_1}{\alpha_2} \geq 1$, which can be achieved by choosing $\hat{\tau} \geq \log(\alpha_1/\alpha_2)/(\alpha - \hat{\alpha})$. Consequently, by choosing $\hat{\tau}$ as specified in (29), we guarantee (28) in all possible scenarios. □

While the REBF in Theorem 10 has a simple definition, it still requires the knowledge of the lower-level set of the IEBF that is assumed to exist. In the following theorem, we relax this assumption and only require the knowledge of any set that contains the super-level set of the IEBF and is contained in the domain that satisfies the IEBF conditions.

Theorem 11. *Let h be an Integral Exponential Barrier Function with exponential rates α and β defined over $D_0 := h_{\geq -c}$ for some $c \geq 0$. If h is sector-contained with parameters α_1 and α_2, then, for any closed set S satisfying $h_{\geq 0} \subset S \subseteq D_0 = h_{\geq -c}$ and $\partial S \cap h_{=0} = \emptyset$, the function*

$$\hat{h}(x) := -\mathrm{sd}(x, S)$$

is a Recurrent Exponential Barrier, i.e., the following conditions hold:

$$\max_{t \in (0, \hat{\tau}]} \{e^{\hat{\beta} t} [\hat{h}(\phi(t, x))]_- + e^{\hat{\alpha} t} [\hat{h}(\phi(t, x))]_+\} \geq \hat{h}(x) \quad (38)$$

for all $x \in \hat{h}_{\geq -\hat{c}}$ with $\hat{c} \geq 0$ such that $\hat{h}_{\geq -\hat{c}} \subseteq D_0$, any

$\hat{\alpha}, \hat{\beta}, \hat{\tau} > 0$ *satisfying* $\hat{\alpha} > \alpha$*,* $\hat{\beta} < \beta$*,* $\hat{\beta} \leq \hat{\alpha}$*, and*

$$\hat{\tau} \geq \max\{\frac{\log(\alpha_2/\alpha_1)}{\hat{\alpha}-\alpha}, \frac{\log(\alpha_2/\alpha_1)}{\beta-\hat{\beta}}\} + \frac{\log(\bar{\delta}/\underline{\delta})}{\min\{\hat{\alpha},\hat{\beta}\}},$$

with

$$\bar{\delta} := \sup_{x\in D_0} (\mathrm{sd}(x, h_{\geq 0}) - \mathrm{sd}(x, S)), \tag{39a}$$

$$\underline{\delta} := \inf_{x\in D_0} (\mathrm{sd}(x, h_{\geq 0}) - \mathrm{sd}(x, S)). \tag{39b}$$

Proof. We first note that since $h_{\geq 0} \subset S$ and $\partial S \cap h_{=0} = \emptyset$, the inequality $\bar{\delta} \geq \underline{\delta} > 0$ naturally holds. On top of this, definitions (39a) and (39b) together imples:

$$0 < \underline{\delta} \leq \mathrm{sd}(x, h_{\geq 0}) - \mathrm{sd}(x, S) \leq \bar{\delta},\ \forall\, x \in D_0.$$

Consequently, we have

$$0 > -\underline{\delta} \geq \mathrm{sd}(x, h_{\leq 0}) - \hat{h}(x) \geq -\bar{\delta},\ \forall\, x \in D_0, \tag{40}$$

since $\mathrm{sd}(x, h_{\leq 0}) = -\mathrm{sd}(x, h_{\geq 0})$ and $\hat{h}(x) = -\mathrm{sd}(x, S)$.

Since h is a sector contained IEBF, Theorem 10 ensures that the function $\mathrm{sd}(\cdot, h_{\leq 0})$ is a REBF satisfying (28) with any $\hat{\tau} \geq \tau^* := \max\{\frac{\log(\alpha_2/\alpha_1)}{\hat{\alpha}-\alpha}, \frac{\log(\alpha_2/\alpha_1)}{\beta-\hat{\beta}}\}$. Theorem 9 part (ii-iii) further establishes a sequence $\{t_n\}_{n\in\mathbb{N}}$ with $t_0 = 0$ such that for each state $x_n := \phi(t_n, x)$, we have

$$\mathrm{sd}(x_n, h_{\leq 0}) \geq \begin{cases} \mathrm{sd}(x, h_{\leq 0})e^{-\hat{\beta}t_n} \text{ if } x \in h_{<0} \cap D_0 & (41a) \\ \mathrm{sd}(x, h_{\leq 0})e^{-\hat{\alpha}t_n} \text{ if } x \in h_{\geq 0} & (41b) \end{cases}$$

for all $n \in \mathbb{N}$.

Now, starting from any initial state $x \in \hat{h}_{<0} \cap D_0$, we have $h(x) < 0$ since $\hat{h}_{\geq 0} = S \supset h_{\geq 0}$. Therefore,

$$e^{\hat{\beta}t_n}\hat{h}(\phi(t_n, x)) \geq e^{\hat{\beta}t_n}(\mathrm{sd}(\phi(t_n, x), h_{\leq 0}) + \underline{\delta}) \tag{42}$$

$$\geq \mathrm{sd}(x, h_{\leq 0}) + e^{\hat{\beta}t_n}\underline{\delta} \tag{43}$$

$$\geq \hat{h}(x) - \bar{\delta} + e^{\hat{\beta}t_n}\underline{\delta} \tag{44}$$

$$\geq \hat{h}(x) \tag{45}$$

where (42) and (44) are based on (40), (43) comes from (41a), and (45) follows whenever $t_n \geq \log(\bar{\delta}/\underline{\delta})/\hat{\beta}$.

Next, starting from any $x \in \hat{h}_{\geq 0}$, we have:

$$e^{\hat{\alpha}t_n}\hat{h}(\phi(t_n, x)) \geq e^{\hat{\alpha}t_n}(\mathrm{sd}(\phi(t_n, x), h_{\leq 0}) + \underline{\delta}) \tag{46}$$

$$\geq \mathrm{sd}(x, h_{\leq 0}) + e^{\hat{\alpha}t_n}\underline{\delta} \tag{47}$$

$$\geq \hat{h}(x) - \bar{\delta} + e^{\hat{\alpha}t_n}\underline{\delta} \tag{48}$$

$$\geq \hat{h}(x) \geq 0, \quad \text{if } h(x) \geq 0, \tag{49}$$

or

$$e^{\hat{\alpha}t_n}\hat{h}(\phi(t_n, x)) \geq e^{\hat{\beta}t_n}\hat{h}(\phi(t_n, x)) \tag{50}$$

$$\geq \hat{h}(x) \geq 0, \quad \text{if } h(x) < 0. \tag{51}$$

where (46) and (48) are based on (40), (47) comes from (41b), and (49) follows whenever $t_n \geq \log(\bar{\delta}/\underline{\delta})/\hat{\alpha}$. In the case that $h(x) < 0$, (50) and (51) are based on (45) and the assumption that $\hat{\alpha} \geq \hat{\beta}$.

Finally, by combining these conditions, we can verify (38) whenever $t_n \geq \hat{\delta} := \log(\bar{\delta}/\underline{\delta})/\min\{\hat{\alpha}, \hat{\beta}\}$. Note that,

$$\lim_{n\to\infty} t_n = \infty \text{ and } t_{n+1} - t_n \in (0, \tau^*],\ \forall n \in \mathbb{N}.$$

Therefore, by choosing $\hat{\tau} \geq \tau^* + \hat{\delta}$, there must be a $t_n \in [\hat{\delta}, \hat{\tau}]$ ensures (38), and the result follows. □

VI. Safety Assessment

A τ-recurrent set S outside of the known unsafe region does not immediately imply safety, as is the case with an invariant one. To practically employ the notion of recurrence for ensuring safety, the following result is pivotal: it demonstrates that a τ-recurrent set, along with the states visited by the finite-time trajectories starting from it, i.e., the finite-time reachable set, constitute an invariant set. Consequently, this process certifies an invariant set in a manner akin to classical methodologies, which can be used to verify safety.

Theorem 12. *Consider a closed set S that is τ-recurrent. Then the finite time reachable set*

$$\mathcal{R}_\tau(S) := \bigcup_{x\in S, t\in[0,\tau]} \phi(t, x) \tag{52}$$

is invariant.

Proof. Suppose that $\mathcal{R}_\tau(S)$ is not invariant, there must exist a $y \in \mathcal{R}_\tau(S)$ and a $t_1 > 0$ such that $\phi(t_1, y) \notin \mathcal{R}_\tau(S)$. By the definition of the reachable set, there also exists a $x \in S$ and a $t_2 \in [0, \tau)$, such that $\phi(t_2, x) = y$. Since $\mathcal{R}_\tau(S) \supseteq S$, we conclude $\phi(t_1, y) = \phi(t_1 + t_2, x) \notin S$.

We then use t' to denote the last time the trajectory $\phi(t, x)$ stays within the closed set S, i.e.,

$$x' := \phi(t', x) \in S \quad \text{and} \quad \phi(t, x) \notin S \ \ \forall t \in (t', t_1 + t_2].$$

Note that we must have $t_1+t_2-t' \leq \tau$ since S is τ-recurrent. This contradicts the assumption that $\phi(t_1, y) \notin \mathcal{R}_\tau(S)$ since $\phi(t_1, y) = \phi(t_1 + t_2 - t', x') \in \mathcal{R}_\tau(S)$. □

We have identified sufficient conditions that guarantee a super-level set of a recurrent-based barrier function is τ-recurrent, which in turn implies its bounded-time reachable set with bound τ is invariant.

Now, one needs to additionally make sure its τ-seconds reachable set, defined in (52), does not intersect with the known unsafe. However, characterizing such a finite-time reachable set is non-trivial, as it may require adaptations to accommodate the distinct trajectories of usually undecidable nonlinear systems.

Yet, under mild conditions, trajectories originating from a τ-recurrent set S are restricted from straying too far from S, as they can only leave S for at most τ-seconds. Consequently, it is possible to over-approximate the τ-seconds reachable set and effectively certify safety if the resulting over-approximation does not intersect the unsafe set.

Remark 6. *Since the vector field is assumed to be locally Lipschitz, it is also locally one-sided Lipschitz [30, Page 70], i.e., for any point $z \in D$, there exists a neighborhood U_z around z and a constant L_z such that $\forall x, y \in U_z$:*

$$(y - x)^T(f(y) - f(x)) \leq L_z\|y - x\|^2$$

We note that a uniform one-sided Lipschitz constant can be defined under these conditions over any (bounded) subset of D. In what follows, we will use:

$$F(S) := \sup_{z\in S} \|f(z)\|; \quad L(S) := \sup_{z\in S} L_z,$$

for a set $S \subseteq D$. These terms will be used in the following lemma that bounds how far the τ-recurrent trajectories can stray from the recurrent set in τ seconds. The lemma is an extension of Lemma 1 in [24].

Lemma 4 (Containment Lemma)**.** *Let $S \subseteq D$ be a closed set that is τ-recurrent and define:*

$$c_1 = F(\mathcal{R}_\tau(S))\tau, \quad c_2 = F(\partial S)\tau e^{L(\mathcal{R}_\tau(S))\tau}, \\ c = \min\{c_1, c_2\}. \tag{53}$$

Then, starting from any $x \in S$, the trajectory satisfies:

$$\mathrm{sd}(\phi(t,x), S) \le c, \quad \forall t \ge 0. \tag{54}$$

Proof. Starting from a point $x \in S$, we use d^* to denote the maximum signed distance from the trajectory $\phi(t,x)$ to the set S within τ seconds, and t^* to denote the time this maximum distance is achieved, i.e.,

$$d^* := \max_{t\in[0,\tau]} \mathrm{sd}(\phi(x,t), S),\ t^* := \arg\max_{t\in[0,\tau]} \mathrm{sd}(\phi(x,t), S).$$

Note that the assumption that the set S is τ-recurrent implies that any trajectory starting from S can leave S for at most τ seconds. Therefore, we only need to consider the maximum signed distance within τ seconds, which would be equivalent to the maximum distance of any trajectory starting from S over unbounded time.

If $d^* \le 0$, it follows that $d^* \le 0 \le c$, and the result follows trivially. We then consider the case that $d^* > 0$. In this case, we further use t' to denote the last time before t^* such that the trajectory $\phi(x,t) \in S$, i.e.,

$$\phi(t', x) \in S \quad \text{and} \quad \phi(t,x) \notin S \ \ \forall t \in (t', t^*].$$

To show $\mathrm{sd}(\phi(t,x), S) \le c_1$, we have:

$$\begin{aligned} &\mathrm{sd}(\phi(t,x), S) \le d^* = \mathrm{sd}(\phi(t^*,x), S) \\ &\le \|\phi(t^*,x) - \phi(t',x)\| = \int_{t'}^{t^*} \|f(\phi(s,x))\| ds \\ &\le F(\mathcal{R}_\tau(S))(t^* - t') \\ &\le F(\mathcal{R}_\tau(S))\tau. \end{aligned}$$

To show $\mathrm{sd}(\phi(t,x), S) \le c_2$, we have:

$$\begin{aligned} &\mathrm{sd}(\phi(t,x), S) \le d^* = \mathrm{sd}(\phi(t^*,x), S) \\ &\le \int_{t'}^{t^*} \|f(\phi(s,x))\| ds \\ &\le \int_{t'}^{t^*} \|f(\phi(s,x)) - F(\mathrm{P}_S(\phi(s,x)))\| + \|F(\mathrm{P}_S(\phi(s,x)))\| ds \\ &\le \int_{t'}^{t^*} \mathrm{sd}(\phi(s,x), S) L(\mathcal{R}_\tau(S)) + F(\partial S) ds \\ &= \int_{t'}^{t^*} \mathrm{sd}(\phi(s,x), S) L(\mathcal{R}_\tau(S)) ds + F(\partial S)t \end{aligned}$$

Then, by applying the Grönwall's inequality [31, Lemma 2.1] with $\lambda = F(\partial S)t, \mu = L(\partial S), y(t) = \mathrm{sd}(\phi(t,x), S)$, we have:

$$\mathrm{sd}(\phi(t,x), S) \le F(\partial S)\tau e^{L(\mathcal{R}_\tau(S))\tau}, \forall x \in S.$$

A combination of these two conditions implies (54). □

Note that Lemma 4 provides necessary regularity conditions for trajectories starting of a τ-recurrent set $h_{\ge 0}$. Building on this foundation, we present the following theorem, which practically leverages the concept of τ-recurrence to characterize a safe state-space region of the system.

Theorem 13. *Consider a continuous function $h : D \to \mathbb{R}$ and a set $\mathcal{X}_u$ of unsafe states. If the super-level set $h_{\ge 0}$ is τ-recurrent and $\{h_{\ge 0} + \mathcal{B}_c\} \cap \mathcal{X}_u = \emptyset$, where the '+' stands for the Minkovski sum, $\mathcal{B}_c$ is a closed ball of radius r around the origin, and the constant c is defined in* (53)*, then $h_{\ge 0}$ is a safe state space region.*

Proof. Given the closed τ-recurrent set $h_{\ge 0}$, note first that Theorem 12 implies its τ-seconds reachable set $\mathcal{R}_\tau(h_{\ge 0})$ is invariant. Then, Lemmma 4 further ensures the invariant set $\mathcal{R}_\tau(h_{\ge 0}) \subseteq \{h_{\ge 0} + \mathcal{B}_c\}$ since trajecties starting from $h_{\ge 0}$ cannot depart from it more then c. Finally, we have

$$\phi(t,x) \in \mathcal{R}_\tau(h_{\ge 0}) \subseteq \{h_{\ge 0} + \mathcal{B}_c\},$$

for all $x \in h_{\ge 0}$ and $t \ge 0$. This, together with the fact that $\{h_{\ge 0} + \mathcal{B}_c\} \cap \mathcal{X}_u = \emptyset$, further implies $\phi(t,x) \notin \mathcal{X}_u$. Therefore, result follows. □

VII. Conclusions

In this paper, we seek to relax the notion of set invariance in the context of characterizing the safe state space region. To this end, we systematically relax the classic differential barrier conditions into integral conditions and further into recurrent conditions. We also thoroughly explore the interconnections between these conditions. Finally, we establish sufficient conditions under which a τ-recurrent set, induced by the recurrent conditions, can be utilized to confirm safety.

References

[1] C. Tomlin, I. Mitchell, A. Bayen, and M. Oishi, "Computational techniques for the verification of hybrid systems," *Proceedings of the IEEE*, p. 986–1001, Jun 2003.

[2] S. Glavaski, A. Papachristodoulou, and K. Ariyur, "Safety verification of controlled advanced life support system using barrier certificates," in *Hybrid Systems: Computation and Control*, 2005, pp. 306–321.

[3] Y. Chen, A. Singletary, and A. D. Ames, "Guaranteed obstacle avoidance for multi-robot operations with limited actuation: A control barrier function approach," *IEEE Control Systems Letters*, vol. 5, no. 1, pp. 127–132, 2021.

[4] A. Alan, A. J. Taylor, C. R. He, A. D. Ames, and G. Orosz, "Control barrier functions and input-to-state safety with application to automated vehicles," *IEEE Transactions on Control Systems Technology*, vol. 31, no. 6, pp. 2744–2759, 2023.

[5] J. Garcıa and F. Fernández, "A comprehensive survey on safe reinforcement learning," *Journal of Machine Learning Research*, vol. 16, no. 1, pp. 1437–1480, 2015.

[6] T. Gurriet, A. Singletary, J. Reher, L. Ciarletta, E. Feron, and A. Ames, "Towards a framework for realizable safety critical control through active set invariance," in *2018 ACM/IEEE 9th International Conference on Cyber-Physical Systems*, 2018, pp. 98–106.

[7] S. Prajna and A. Jadbabaie, "Safety verification of hybrid systems using barrier certificates," in *International Workshop on Hybrid Systems: Computation and Control*. Springer, 2004, pp. 477–492.

[8] S. Prajna, A. Jadbabaie, and G. J. Pappas, “A framework for worst-case and stochastic safety verification using barrier certificates,” *IEEE Transactions on Automatic Control*, vol. 52, no. 8, pp. 1415–1428, 2007.

[9] X. Xu, P. Tabuada, J. W. Grizzle, and A. D. Ames, “Robustness of control barrier functions for safety critical control,” *IFAC-PapersOnLine*, vol. 48, no. 27, p. 54–61, 2015.

[10] M. NAGUMO, “Uber die lage der integralkurven gewouml;hnlicher differentialgleichungen,” *Proceedings of the Physico-Mathematical Society of Japan. 3rd Series*, vol. 24, pp. 551–559, 1942.

[11] S. Kolathaya and A. D. Ames, “Input-to-state safety with control barrier functions,” *IEEE control systems letters*, vol. 3, no. 1, pp. 108–113, 2018.

[12] P. Wieland and F. Allgöwer, “Constructive safety using control barrier functions,” *IFAC Proceedings Volumes*, vol. 40, no. 12, pp. 462–467, 2007, 7th IFAC Symposium on Nonlinear Control Systems.

[13] E. D. Sontag, “A ‘universal’ construction of artstein’s theorem on nonlinear stabilization,” *Systems and Control Letters*, vol. 13, no. 2, pp. 117–123, 1989.

[14] M. Z. Romdlony and B. Jayawardhana, “Uniting control lyapunov and control barrier functions,” in *53rd IEEE Conference on Decision and Control*, 2014, pp. 2293–2298.

[15] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, “Control barrier function based quadratic programs for safety critical systems,” *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3861–3876, 2017.

[16] A. D. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, “Control barrier functions: Theory and applications,” in *2019 18th European Control Conference*, 2019, pp. 3420–3431.

[17] S. Prajna and A. Rantzer, “Convex programs for temporal verification of nonlinear dynamical systems,” *SIAM Journal on Control and Optimization*, p. 999–1021, Dec 2006.

[18] S. Prajna, A. Papachristodoulou, and P. Parrilo, “Introducing sostools: a general purpose sum of squares programming solver,” in *Proceedings of the 41st IEEE Conference on Decision and Control, 2002.*, vol. 1, 2002, pp. 741–746 vol.1.

[19] H. Kong, F. He, X. Song, W. N. N. Hung, and M. Gu, “Exponential-condition-based barrier certificate generation for safety verification of hybrid systems,” 2013.

[20] W. Xiao and C. Belta, “High-order control barrier functions,” *IEEE Transactions on Automatic Control*, vol. 67, no. 7, pp. 3655–3662, 2021.

[21] X. Tan, W. S. Cortez, and D. V. Dimarogonas, “High-order barrier functions: Robustness, safety, and performance-critical control,” *IEEE Transactions on Automatic Control*, vol. 67, no. 6, pp. 3021–3028, 2021.

[22] Y. Shen, M. Bichuch, and E. Mallada, “Model-free learning of regions of attraction via recurrent sets,” in *2022 IEEE 61st Conference on Decision and Control*, 2022, pp. 4714–4719.

[23] R. Siegelmann, Y. Shen, F. Paganini, and E. Mallada, “A recurrence-based direct method for stability analysis and gpu-based verification of non-monotonic lyapunov functions,” in *2023 62nd IEEE Conference on Decision and Control*, 2023, pp. 6665–6672.

[24] H. Sibai and E. Mallada, “Recurrence of nonlinear control systems: Entropy and bit rates,” in *Proceedings of the 27th ACM International Conference on Hybrid Systems: Computation and Control*, 2024, pp. 1–9.

[25] M. Soyer, S. Olaru, and Z. Fang, “From constraint satisfactions to periodic positive invariance for discrete-time systems,” in *2020 59th IEEE Conference on Decision and Control*, 2020, pp. 4547–4552.

[26] R. Comelli, A. H. González, A. Ferramosca, S. Olaru, M. M. Seron, and E. Kofman, “Simplified design of practically stable mpc schemes,” *Systems & Control Letters*, vol. 180, p. 105626, 2023.

[27] F. Blanchini, “Set invariance in control,” *Automatica*, vol. 35, no. 11, pp. 1747–1767, 1999.

[28] R. Kannan and C. Krueger, *Advanced Analysis: On the Real Line*, ser. Universitext - Springer-Verlag. Springer New York, 1996.

[29] T. M. Apostol, *One-variable calculus, with an introduction to linear algebra*. Blaisdell Waltham, Toronto, London, 1967, vol. 1.

[30] F. Bullo, *Contraction Theory for Dynamical Systems*. Kindle Direct Publishing, 2023.

[31] H. K. Khalil, “Nonlinear systems; 3rd ed.” 2002.

VIII. Appendix

A. Proof of Lemma 2

Proof. Given $x \in D_0$, we build the time sequence $\{t_n\}_{n\in\mathbb{N}}$ satisfying (18) and (19), following an inductive method similar to that detailed in our previous work [23, Lem 1].
[Base case]: For the base case, we have $t_0 = 0$, $x_0 = x \in D_0$, and define t_1 as follows:

$$t_1 = \max\{\underset{t\in(0,\tau]}{\arg\max}\{h(\phi(t,x_0)) + \int_0^t \zeta(h(\phi(s,x_0)))ds\}\}; \quad (55)$$

note that the second maximum exists by condition (17), and is no smaller than $h(x_0)$; if there are multiple maximizing times, t_1 is defined as the largest. By construction, $t_1 - t_0 \in (0,\tau]$, and the function h evaluated at $x_1 := \phi(t_1, x_0)$ satisfies:

$$h(x_1) \geq h(x_0) - \int_0^{t_1} \zeta(h(\phi(s,x_0)))ds = h(x_0) + \delta_0,$$

thus confirming (19) with the left argument on the $\max$ of the right-hand side.

To prove the right argument on the $\max$ of equation (19), one need to first show that:

$$h(x_{n+1}) \begin{cases} > h(x_n) & \textbf{if } x_n \in h_{<0} \cap D_0 & (56a) \\ \geq 0 & \textbf{if } x_n \in h_{\geq 0}, & (56b) \end{cases}$$

Let us first consider the case $x_0 \in h_{<0} \cap D_0$. If $h(x_1) \geq 0$, then (56a) follows trivially. If $h(x_1) < 0$, we argue that $h(\phi(t,x_0)) < 0$ and $\zeta(h(\phi(t,x_0))) < 0$ for all $t \in [0, t_1]$; otherwise, t_1 would not maximize (55). Hence, $\delta_0 > 0$, and we verified $h(x_1) \geq h(x_0) + \delta_0 > h(x_0)$, thereby satisfying (56a).

In the other case that $x_0 \in h_{\geq 0}$, we demonstrate that $h(x_1) \geq 0$ by contradiction. Suppose $h(x_0) \geq 0$ and $h(x_1) < 0$, we use t' to denote the last time the trajectory $\phi(t, x_0)$ stays within the closed set $h_{\geq 0}$ before t_1, i.e.,

$$x' := \phi(t', x_0) \in h_{=0} \quad \text{and} \quad h(\phi(t,x_0)) < 0, \;\; \forall t \in (t', t_1].$$

This contradicts with the fact that t_1 is a maximizer of (55), since $h(x') = 0 > h(x_1)$ and $\int_0^{t'} \zeta(h(\phi(s,x_0)))ds > \int_0^{t_1} \zeta(h(\phi(s,x_0)))ds$. Therefore, we have $h(x_1) \geq 0$, and thus (56b) follows.
[Inductive step]: Note that condition (19) further implies $x_1 = \phi(t_1, x_0) \in D_0$, since $h(x_0) \geq -c$. Thus, the inductive construction proceeds in a similar manner: given $t_1 < t_2 < \cdots t_n$, with $x_n := \phi(t_n, x) \in D_0$, define $t_{n+1} - t_n$ as:

$$\max\{\underset{t\in(0,\tau]}{\arg\max}\, h(\phi(t,x_n)) + \int_0^t \zeta(h(\phi(s,x_n)))ds\}. \quad (57)$$

Note that $t_{n+1} - t_n \in (0, \tau]$ as required. Also, similar to the base case,

$$x_{n+1} := \phi(t_{n+1}, x_0) = \phi(t_{n+1} - t_n, x_n)$$

satisfies the conditions in (19).
[Divergence of t_n]: It remains to show that $t_n \to \infty$, which we argue by contradiction. If, instead, the strictly increasing sequence of times was bounded, we would have $t_n \uparrow t^*$. Note that $\phi(t^*, x_0)$ is well defined since the dynamical system (1)

is forward complete. Also, by the continuity of $\phi(\cdot, x)$:

$$v_n := h(\phi(t_n, x_0)) + \int_0^{t_n} \zeta(h(\phi(s, x_0)))ds$$
$$\to v^* := h(\phi(t^*, x_0)) + \int_0^{t^*} \zeta(h(\phi(s, x_0)))ds.$$

Note that it follows from the first inequality of (19) that:

$$\begin{aligned} h(x_{n+1}) &\geq h(x_n) + \delta_n \\ &\geq h(x_n) - \int_0^{t_{n+1}-t_n} \zeta(h(\phi(s, x_n)))ds \\ &= h(x_n) - \int_{t_n}^{t_{n+1}} \zeta(h(\phi(s, x_0)))ds \\ \geq h(x_n) - &\int_0^{t_{n+1}} \zeta(h(\phi(s, x_0)))ds + \int_0^{t_n} \zeta(h(\phi(s, x_0)))ds \\ \implies v_{n+1} &\geq v_n, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Therefore, $\{v_n\}$ is non-decreasing, and we further conclude that $v^* \geq v_n$ for all $n \in \mathbb{N}$. Now pick n such that $t_n \geq t^* - \tau$. This means that $s^* := t^* - t_n \in (0, \tau]$ is in the feasible set for the maximization in (57), which by definition gives as maximum v_{n+1}, achieved at $t_{n+1} - t_n$.

Now, since $v^* = h(s^*, x_n) + \int_0^{s^*} \zeta(h(\phi(s, x_n)))ds \geq v_{n+1}$, this means s^* also qualifies as a maximizer, and in fact $s^* = t^* - t_n > t_{n+1} - t_n$. This contradicts the definition of $t_{n+1} - t_n$ given in (57), because it would not be the largest maximizing time. Thus, the sequence must be divergent, establishing claim (ii). □

B. *Proof of Lemma 3*

Proof. Given $x_0 = x \in D_0$, we build the time sequence $\{t_n\}_{n \in \mathbb{N}}$ satisfying (21) and (22) again by induction.

[Base case]: For the base case, we have $t_0 = 0$ and define t_1 as follows:

$$t_1 = \max\{\arg\max_{t \in (0,\tau]}\{e^{\beta t}[h(\phi(t, x_0))]_- + e^{\alpha t}[h(\phi(t, x_0))]_+\}\}$$

note that the second maximum exists by condition (20), and is no smaller than $h(x_0)$; if there are multiple maximizing times, t_1 is defined as the largest. By construction, $t_1 - t_0 \in (0, \tau]$, and the function h evaluated at $x_1 := \phi(t_1, x_0)$ satisfies:

$$e^{\beta t_1}[h(x_1)]_- + e^{\alpha t_1}[h(x_1)]_+ \geq h(x_0). \tag{58}$$

Note that whenever $h(x_0) \geq 0$, (58) requires $h(x_1) \geq 0$ and thus

$$h(x_1) \geq e^{-\alpha t_1} h(x_0).$$

In the case that $h(x_0) < 0$, we have:

$$h(x_1) \geq e^{-\beta t_1} h(x_0).$$

Therefore, (22) follows. Finally, $h(x_0) \geq -c$ and by (22) $h(x_1) > h(x_0)$, we have $x_1 = \phi(t_1, x_0) \in D_0$, which finishes the proof of the base case of the induction.

[Inductive step]: The inductive step construction proceeds in a similar manner: given $t_1 < t_2 < \cdots t_n$, with $x_k := \phi(t_k, x_0) \in D_0$, $0 \leq k \leq n$. Now, define $t_{n+1} - t_n$ as:

$$\max\{\arg\max_{t \in (0,\tau]}\{e^{\beta t}[h(\phi(t, x_n))]_- + e^{\alpha t}[h(\phi(t, x_n))]_+\}\} \tag{59}$$

Note that $t_{n+1} - t_n \in (0, \tau]$ as required. A similar proof to the base case then shows that

$$x_{n+1} := \phi(t_{n+1}, x_0) = \phi(t_{n+1} - t_n, x_n) \in D_0$$

and (22) is satisfied.

[Divergence of t_n]: It remains to show that $t_n \to \infty$, which we argue by contradiction. If, instead, the strictly increasing sequence of times was bounded, we would have $t_n \uparrow t^*$. Note that $\phi(t^*, x_0)$ is well defined since the dynamical system (1) is forward complete. Also, by the continuity of $\phi(\cdot, x)$:

$$v_n := e^{\beta t_n}[h(\phi(t_n, x_0))]_- + e^{\alpha t_n}[h(\phi(t_n, x_0))]_+$$
$$\to v^* := e^{\beta t^*}[h(\phi(t^*, x_0))]_- + e^{\alpha t^*}[h(\phi(t^*, x_0))]_+.$$

Note that it follows from (22) that:

$$\begin{aligned} e^{\beta t_{n+1}}&[h(\phi(t_{n+1}, x_0))]_- + e^{\alpha t_{n+1}}[h(\phi(t_{n+1}, x_0))]_+ \\ \geq e^{\beta t_n}&[h(\phi(t_n, x_0))]_- + e^{\alpha t_n}[h(\phi(t_n, x_0))]_+ \\ &\implies v_{n+1} \geq v_n, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Therefore, $\{v_n\}$ is non-decreasing, and we further conclude that $v^* \geq v_n$ for all $n \in \mathbb{N}$. Now pick n such that $t_n \geq t^* - \tau$. This means that $s^* := t^* - t_n \in (0, \tau]$ is in the feasible set for the maximization in (59), which by definition gives as maximum v_{n+1}, achieved at $t_{n+1} - t_n$.

Now, since

$$v^* = e^{\beta s^*}[h(\phi(s^*, x_n))]_- + e^{\alpha s^*}[h(\phi(s^*, x_n))]_+ \geq v_{n+1},$$

this means s^* also qualifies as a maximizer, and in fact $s^* = t^* - t_n > t_{n+1} - t_n$. This contradicts the definition of $t_{n+1} - t_n$ given in (59), since it would not be the largest maximizing time. Thus, the sequence must be divergent, establishing claim (ii). □