# Closed-Loop Motion Planning for Differentially Flat Systems: A Time-Varying Optimization Framework

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***Abstract*— Motion planning and control are two core components of the robotic systems autonomy stack. The standard approach to combine these methodologies comprises an offline/open-loop stage, *planning*, that designs a feasible and safe trajectory to follow, and an online/closed-loop stage, *tracking*, that corrects for unmodeled dynamics and disturbances. Such an approach generally introduces conservativeness into the planning stage, which becomes difficult to overcome as the model complexity increases and real-time decisions need to be made in a changing environment. This work addresses these challenges for the class of differentially flat nonlinear systems by integrating planning and control into a cohesive closed-loop task. Precisely, we develop an optimization-based framework that aims to steer a differentially flat system to a trajectory implicitly defined via a constrained *time-varying optimization* problem. To that end, we generalize the notion of feedback linearization, which makes non-linear systems behave as linear systems, and develop controllers that effectively transform a differentially flat system into an optimization algorithm that seeks to find the optimal solution of a (possibly time-varying) optimization problem. Under sufficient regularity assumptions, we prove global asymptotic convergence for the optimization dynamics to the minimizer of the time-varying optimization problem. We illustrate the effectiveness of our method with two numerical examples: a multi-robot tracking problem and a robot obstacle avoidance problem.**

***Index Terms*— Time-varying optimization, Differentially flat system, Closed loop systems.**

## I. Introduction

AUTONOMY is the ability of a robot or machine to perform a task without any human intervention. In robotics and transportation, autonomous tasks usually involve gathering information from the environment (sensing), identifying the agent's current position (localization), devising a safe navigation strategy (planning), and compensating for unexpected changes (control) [1]–[3]. A particularly challenging step of this process is the motion planning stage [1]–[4], wherein an agent with uncertain information about its position and environment must devise an admissible and collision-free trajectory to be followed toward a final destination. This highly complex task has received widespread research interest [2]–[11], as it requires a delicate balance between computational complexity and optimality while simultaneously respecting the agent's dynamic capabilities.

Standard approaches to solving this problem can be broadly categorized into three groups: Grid-based search (GBS), Sampling-based Planning (SBP), and Optimization-based (OB). GBS algorithms assign each configuration of the dynamical system to a grid point and use graph search algorithms such as Dijkstra [5], $A^*$ [6], and $D^*$ [7] to find a path. Although GBS algorithms are easy to implement and often provide an acceptable answer, they scale poorly with the number of degrees of freedom of the configuration space [12] and fail to ensure the dynamic feasibility of the path. SBP algorithms [3], such as rapidly-exploring random trees (RRTs) [8], probabilistic roadmap methods (PRMs) [4], and their variants scale better for high-dimensional problems. However, optimality guarantees are usually absent, and path feasibility is only achieved via sufficiently dense sampling of either the configuration or action space [12]. On the other hand, OB algorithms such as direct multiple-shooting [13] and direct collocation [14] explicitly consider the dynamic constraints in the optimization problems, providing by construction dynamically feasible trajectories which can be enforced to avoid collisions [15], [16]. However, OB algorithms suffer from high computational costs, typically requiring solving a nonlinear programming problem without convergence guarantees [11].

Two common features of the above-mentioned solutions are (a) the struggle between the computational complexity of the planning process and the need to enforce dynamic constraints and (b) the open-loop nature of the solution that does not account for unmodeled dynamics or disturbances. Thus, such methodologies are commonly complemented with a motion execution stage that implements a feedback controller that tracks the open-loop trajectory. However, such an approach usually introduces some level of conservativeness in the planning stage to avoid collisions [17], [18], which when explicitly accounted for via robust methods, further increases the computational complexity of the solution. This work aims to explore an alternative approach aiming at integrating planning and control as a unique task while ensuring dynamic feasibility and accounting for changing conditions in real time.

*Contributions of this work:* This work uses time-varying optimization to combine safe motion planning and control in a unique *closed-loop task*. We seek to encode planning goals and safety constraints as a time-varying (TV) constrained

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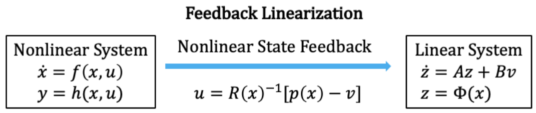


Fig. 1. Feedback linearization transforms a nonlinear system into a linear system via nonlinear state feedback control and coordinate transformation.

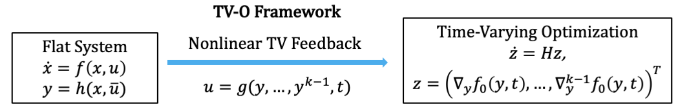


Fig. 2. TV-O framework effectively transforms a differentially flat system into an optimization algorithm.

optimization problem and develop a general methodology to design closed-loop feedback controllers by drawing insights from mathematical optimization. Our methodology combines tools from differential flatness and optimization theory to develop controllers which effectively transform a dynamical system into an optimization algorithm that seeks to track the optimal solution of the aforementioned optimization problem.

The contributions of our work are as follows:

- *Planning and Control as TV Optimization.* We formulate a framework to encode planning and control goals within a time-varying optimization problem, wherein planning goals are implicitly encoded as the (apriori unknown) optimal solution $y^*(t)$ of a TV-Optimization problem. This formulation allows us, in turn, to recast the control design problem as the problem of choosing an optimization algorithm.
- *Flat Systems as Optimization Algorithms.* We provide a general methodology to design control laws that steer the output $y(t)$ of a differentially flat nonlinear system towards $y^*(t)$. Inspired by feedback linearization [19], [20] (Fig. 1), the proposed methodology transforms any flat system of order $k$ into a time-varying optimization algorithm that depends on the first $k$ 1 time derivatives of the objective function's gradient (Fig. 2).
- *Theoretical Guarantees.* Our control design framework can readily provide rigorous theoretical guarantees on the asymptotic behavior of the system. Precisely, we show that under mild conditions, the output $y(t)$ of our differentially flat nonlinear system will converge asymptotically to $y^*(t)$.
- *Extensions for Formation and Collision Avoidance.* We further extend our time-varying feedback optimization framework to allow for (i) the asymptotic satisfaction of time-varying equality constraints, which allow for specification of formation constraints, and (ii) hard inequality constraints, which can model collision avoidance specifications.

A preliminary version of this work appeared in [19]. The present paper extends the work of [19] in many ways, including extensions from feedback linearization to general flat systems, as well as the inclusion of equality and inequality constraints. The technical results of Sections IV , V and VI are new, as well as the numerical validations of Section VII.

*Other related literature:* Our work also broadly aligns with and contributes to the growing literature of *online optimization with feedback loops*, *network systems*, and *algorithm design for time-varying optimization*.

**Online optimization with feedback loops** seeks to design online optimization algorithms to regulate the output of a dynamical system towards the optimal solution of an optimization problem. For the case of LTI systems, numerous works have designed controllers that track the optimal solution of (i) a static optimization problem [21]–[23], and (ii) time-varying convex optimization problems [24], including also input-output constraints [25]. Nonlinear system dynamics are considered for steering a physical system to a steady state that solves a predefined constrained static optimization problem in [26] and an unconstrained time-varying optimization problem in [27].

**Online optimization of network systems** considers the extension of the above framework for problems where systems and computations are distributed. The papers [28], [29] seek to design controllers to regulate the network of agents to the global minimizer of a predefined convex optimization problem. Time-varying versions of this problem are have been considered, including versions with inequality constraints [30], with double-integrator dynamics [31], and with nonlinear dynamics in a strict feedback form [32].

**Time-varying optimization** has been a popular subject of research for online decision making. It provides a computationally frugal optimization framework that produces solutions in "a timely fashion and is essential when input data streams are of large-scale and decisions must be made at high frequency." [31], [33]–[36]. Our work is a direct application of time-varying optimization formalisms in the area of feedback control and motion planning [33]. Online solvers for time-varying optimization problems have been proposed both in continuous time [31], [34] and in discrete time [35], [36].

Our work extends the existing literature by devising controllers for nonlinear differentially flat systems that facilitate tracking the optimal solution of constrained time-varying optimization problems.

*Outline of the paper:* The paper is structured as follows. Section II introduces some preliminary definitions and tools we use. In Section III, we formally introduce the new optimization-based framework, which aims to steer differentially flat systems to trajectories implicitly defined via constrained time-varying optimization problems. We motivate our solution approach using two examples: an integrator and Wheeled Mobile Robot. Furthermore, in Section III-C, we summarize the key features of the proposed framework, which generalize the notion of feedback linearization and transform a flat system into an optimization algorithm. Designing the nonlinear feedback controller is reduced to finding a solution to the ODE system that satisfies the optimization dynamic and system dynamic simultaneously. Our main results are described in Sections IV V, and VI, which deal with unconstrained and constrained time-varying optimization problems, respectively. To illustrate our results, we perform two numerical evaluations in Section VII on a multi-object tracking problem and an obstacle avoidance problem.

*Notation:* We us bold symbols to represent vectors ($\mathbf{x}$) and matrices ($\mathbf{A}$). For $n \in \mathbb{N}$ the set $\{1, \dots, n\}$ is denoted by $[n]$. A square symmetric matrix $\mathbf{A}$ is positive (semi)definite, written as $\mathbf{A} \succ 0$ ($\mathbf{A} \succeq 0$), if and only if all the eigenvalues of $\mathbf{A}$ are positive (nonnegative). We further write $\mathbf{A} \succ \mathbf{B}$ ($\mathbf{A} \succeq \mathbf{B}$) whenever $\mathbf{A} - \mathbf{B} \succ 0$ ($\mathbf{A} - \mathbf{B} \succeq 0$). The symbol $\otimes$ denotes the Kronecker product between two matrices. The Euclidean norm of a vector $\mathbf{x}$ is denoted by $\|\mathbf{x}\|_2$, and the spectral norm of a matrix $\mathbf{A}$ by $\|\mathbf{A}\|_2$. The $k$-th order derivative of $\mathbf{x}$ with respect to time $t$ is denoted as $\mathbf{x}^{(k)}$. For a nonnegative integer $k$ we use the short-hand notation $\mathbf{x}^{[k]} = (\mathbf{x}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)})$.

Given a differentiable function (of sufficient order) $f(\mathbf{x}, t)$ of state $\mathbf{x} \in \mathbb{R}^n$ and time $t \in \mathbb{R}$, the gradient with respect to $\mathbf{x}$ (resp. $t$) is denoted by $\nabla_{\mathbf{x}} f(\mathbf{x}, t)$ (resp. $\nabla_t f(\mathbf{x}, t)$). When $x(t)$ depends on time too, the total derivative (resp. $n$-th total derivative) of $\nabla_{\mathbf{x}} f(\mathbf{x}(t), t)$ with respect to $t$ is denoted by $\dot{\nabla}_{\mathbf{x}} f(\mathbf{x}, t) := \frac{\mathrm{d}}{\mathrm{d}t} \nabla_{\mathbf{x}} f(\mathbf{x}(t), t)$ (resp. $\nabla_{\mathbf{x}}^{(n)} f(\mathbf{x}, t)$). The partial derivatives of $\nabla_{\mathbf{x}} f(\mathbf{x}, t)$ with respect to $\mathbf{x}$ and $t$ are denoted by $\nabla_{\mathbf{x}\mathbf{x}} f(\mathbf{x}, t) := \frac{\partial}{\partial \mathbf{x}} \nabla_{\mathbf{x}} f(\mathbf{x}, t) \in \mathbb{R}^{n \times n}$ and $\nabla_{\mathbf{x}t} f(\mathbf{x}, t) := \frac{\partial}{\partial t} \nabla_{\mathbf{x}} f(\mathbf{x}, t) \in \mathbb{R}^n$, respectively.

## II. Differential Flatness

Over the past several decades, differential flatness theory has been a main direction in the area of nonlinear control for motion planning, trajectory generation, and stabilization [37]–[40]. Roughly speaking, flat systems are those systems that are equivalent to a controllable linear one, namely, a system made of chains of integrators of arbitrary length [38]. More precisely, consider the system

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}). \tag{1}$$

with $\mathbf{x} \in \mathbb{R}^n$ being its state and $\mathbf{u} \in \mathbb{R}^m$ its inputs. The above system is said to be *differentially flat* if we can find positive integers $r$, $k \in \mathbb{N}$, and functions $h$, $\varphi$, and $\alpha$ such that all states and inputs are determined by the *flat outputs* $\mathbf{y} \in \mathbb{R}^m$ and a finite number $k$ of its derivatives [41], i.e.,

$$\mathbf{y} = h(\mathbf{x}, \mathbf{u}^{[r]}), \tag{2}$$

such that

$$\mathbf{x} = \varphi(\mathbf{y}^{[k]}), \quad \mathbf{u} = \alpha(\mathbf{y}^{[k]}). \tag{3}$$

For our purposes, the main advantage of using flat outputs in control system design is that doing so simplifies the process of generating input trajectories that satisfy certain constraints. Instead of designing complex controllers that directly manipulate the control inputs, one can design controllers that manipulate the flat outputs, and then use the algebraic relationships between the flat outputs and the control inputs to generate optimal state and control trajectories. Notably, many commonly used classes of systems in nonlinear control theory are differentially flat, for example fully actuated holonomic systems, mobile robots, and classical $n$-trailer systems. A complete characterization of differential flatness and a catalog of finite dimensional flat systems can be found in [37]–[40].

Another important concept in nonlinear control theory is *(dynamic) feedback linearization*, in which a nonlinear system is transformed into a linear system by a state diffeomorphism and a feedback transformation [20]. Although differential flatness and feedback linearization are related, they are not identical. In particular, while all state feedback-linearizable systems are differentially flat, a differentially flat system need not be state feedback linearizable, and need not be dynamic feedback linearizable everywhere in the state space. Differential flatness is a geometric property of a nonlinear system, independent of coordinate representation, and this can be exploited for nonlinear controller design [42].

## III. Problem Statement

In this section, we formally state the problem of interest, together with some regularity assumptions needed in our derivations, and we introduce two motivating examples. Consider a differentially flat system described as in (1), along with the associated flat output

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}), \quad \mathbf{y} = h(\mathbf{x}, \mathbf{u}^{[r]}). \tag{4}$$

We begin by formulating a time-varying optimization problem in the variable $\mathbf{y}$. To this end, let $t \geq 0$ be a continuous time index, and $f_0 : \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}$ be a time-varying objective function of the flat output $\mathbf{y}$. For $p \in \mathbb{Z}_{\geq 0}$ and $i \in [p]$, the maps $f_i : \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}$ will encode $p$ time-varying inequality constraints on $\mathbf{y}$. We similarly consider $q \in \mathbb{Z}_{\geq 0}$ time-varying linear equality constraints of the form $\mathbf{a}_j(t)^\mathsf{T} \mathbf{y} = b_j(t)$, which we stack into matrix form as $\mathbf{A}(t)\mathbf{y} = \mathbf{b}(t)$, where now $\mathbf{A} : \mathbb{R}_+ \to \mathbb{R}^{q \times m}$ and $\mathbf{b} : \mathbb{R}_+ \to \mathbb{R}^q$; we assume throughout that $q < m$. This leads to the following constrained time-varying optimization problem

$$\begin{aligned} \mathbf{y}^*(t) := \arg\min_{\mathbf{y} \in \mathbb{R}^m} \ & f_0(\mathbf{y}, t) \\ \text{s.t.} \ & f_i(\mathbf{y}, t) \leq 0, \quad i \in [p] \\ & \mathbf{A}(t)\mathbf{y} = \mathbf{b}(t). \end{aligned} \tag{5}$$

We will assume throughout that all functions $f_0, f_1, \dots, f_p$ are infinitely differentiable in both $\mathbf{y}$ and $t$, and that $\mathbf{A}$ and $\mathbf{b}$ are infinitely differentiable in $t$. Additionally, we assume the minimizer $\mathbf{y}^*(t)$ of (5) is unique for all $t \geq 0$ (see Assumption 1). Our goal is to generate a control input $\mathbf{u}(t)$ for (4) such that for some $C > 0$, $\|\mathbf{y}(t) - \mathbf{y}^*(t)\|_2 \leq Ce^{-\lambda t}$ for all $t \geq 0$ and all initial conditions, i.e., global attractivity to $y^*(t)$.

Various motion planning and control tasks can be encoded as instances of the time-varying optimization problem (5). For example, if the positions of a controlled robot and a moving target are denoted by $\mathbf{y}(t)$ and $\mathbf{y}^d(t)$ respectively, then minimizing the objective function $\|\mathbf{y}(t) - \mathbf{y}^d(t)\|$ represents the task of tracking a moving target. Along the same line, if $\mathbf{y}(t)$ denotes the vector of positions of a network of agents in 2 dimensions, represented by complex values $y_i \in \mathbb{C}$, one can impose formation constraints using a constraint of the form $\mathbf{L}\mathbf{y}(t) = 0$, where $\mathbf{L(t)}$ is the complex-valued Laplacian matrix associated with a desired formation [43]. Similarly, to ensure collision avoidance with respect to $O \in \mathbb{N}$ obstacles, a set of inequality constraints can be employed: $\{\mathbf{a}_i(\mathbf{y})^T \mathbf{z} \leq b_i(\mathbf{y}) \leq 0, i \in [O]\}$ [34]. This is elaborated further in Section VII-B.

The following regularity assumptions will be used throughout this paper, and are commonly used in the context of time-varying optimization [33].

**Assumption 1** (Uniform strong convexity)**.** *The objective function $f_0(\mathbf{y}, t)$ is uniformly strongly convex in $\mathbf{y}$, i.e., $\nabla_{\mathbf{yy}} f_0(\mathbf{y}, t) \succeq m_f \mathbf{I}_m$ for some $m_f > 0$, for all $\mathbf{y}$ and for all $t \geq 0$. The inequality constraint functions $f_i(\mathbf{y}, t)$ are convex in $\mathbf{y}$ for all $t \geq 0$ and for all $i \in [p]$.*

**Assumption 2** (Uniform Mangasarian-Fromowitz constraint qualification)**.** *For a global minimum $\mathbf{y}^*(t)$ of (5)*

1) *there exists a uniformly bounded $\mathbf{d}(t) \in \mathbb{R}^m$, i.e., $\|\mathbf{d}(t)\|_2 \leq d$ for some constant $d > 0$, and a constant $\epsilon > 0$ such that for all $t \geq 0$*
$$\nabla_{\mathbf{y}} f_i(\mathbf{y}^*(t), t)^T \mathbf{d}(t) \leq -\epsilon, \qquad i \in \mathrm{I}(\mathbf{y}^*(t)),$$
$$\mathbf{a}_j(t)^{\mathsf{T}} \mathbf{d}(t) = 0, \qquad j \in [q],$$
*where $\mathrm{I}(\mathbf{y}^*(t)) := \{i \in [p] \mid f_i(\mathbf{y}^*(t), t) = 0\}$ denotes the index set associated with active inequality constraints.*
2) *there exist constants $0 < \tau_{\min} \leq \tau_{\max} < +\infty$ such that $\sigma_{\min}(\mathbf{A}(t)) \geq \tau_{\min}$ and $\sigma_{\max}(\mathbf{A}(t)) \leq \tau_{\max}$ for all $t \geq 0$, i.e., the vectors $\{\mathbf{a}_j\}$ for $j \in [q]$ are uniformly linearly independent and uniformly bounded.*

Since the time-varying convex optimization problem has smooth objective and constraints functions, Assumption 1 and Assumption 2 imply that for all $t \geq 0$, the Karush-Kuhn-Tucker (KKT) conditions [44] provide necessary and sufficient conditions for optimality. Notice that these Assumptions are not written in the most familiar way. For example in Assumption 2, we are replacing the traditional $\nabla_{\mathbf{y}} f_i(\mathbf{y}^*(t), t)^T \mathbf{d}(t) < 0$ with $\nabla_{\mathbf{y}} f_i(\mathbf{y}^*(t), t)^T \mathbf{d}(t) \leq -\epsilon$ for some positive constants $\epsilon > 0$. Such modifications are made in order to exclude the possibility that, e.g., $\nabla_{\mathbf{y}} f_i(\mathbf{y}^*(t), t)^T \mathbf{d}(t) \to 0$ as $t \to \infty$. In Section VI, we will show that these assumptions are sufficient for the time-varying optimization problem (5) to be well-defined for all $t \geq 0$, and exclude the possibility that the optimal dual variables escape to infinity exponentially fast, which was merely assumed to hold in prior work [34].

The remainder of this section provides two examples that help motivate both our goals and our solution approach. In Section III-A, we begin with a linear system, an integrator, which is the simplest form of a flat system. We illustrate how to design a control law that steers it to the trajectory implicitly defined by an unconstrained time-varying optimization problem. We relate this case with recent research concerning *Prediction-Correction Methods* and describe a general methodology for control design wherein we match the evolution of the flat output with that of a time-varying gradient descent algorithm that converges to $\mathbf{y}^*(t)$. In Section III-B, we extend the approach for a simple but representative second-order nonholonomic system, the Wheeled Mobile Robot (WMR), to illustrate how to incorporate kinematic or dynamic constraints in the system modeling by matching the flat output to a second order gradient descent algorithm. Lastly, we summarize the key features of the time-varying optimization-based framework to illustrate our solution approach, which effectively transforms a general flat system into an optimization algorithm that achieves asymptotic convergence to the optimal solution.

### A. Example #1: Integrator

We begin the analysis with the simplest possible flat system, the linear integrator
$$\dot{\mathbf{x}} = \mathbf{u}, \qquad \mathbf{y} = \mathbf{x}, \tag{6}$$
where $\mathbf{y}$ is the flat output. For a differentially flat system, one can design controllers that manipulate the flat outputs, and then use the algebraic relationships between the flat outputs and the control inputs to generate optimal state and control trajectories. Indeed, for the linear integrator (6), the relationship is simply
$$\mathbf{u} := \alpha(\mathbf{y}^{[k]}) = \dot{\mathbf{y}}. \tag{7}$$
For our purposes it will be useful to represent the above relationship in the implicit form
$$\mathrm{F}(\dot{\mathbf{y}}, \mathbf{u}) := \dot{\mathbf{y}} - \mathbf{u} = \mathbf{0}. \tag{8}$$
As a first example, consider an *unconstrained* version of the time-varying optimization problem (5), in which our goal is to regulate the output $\mathbf{y}$ of the integrator (6) to asymptotically track the minimizer
$$\mathbf{y}^*(t) = \arg\min_{\mathbf{y} \in \mathbb{R}^m} f_0(\mathbf{y}, t). \tag{9}$$
Under Assumption 1, the minimizer $\mathbf{y}^*(t)$ is characterized by $\nabla_{\mathbf{y}} f_0(\mathbf{y}^*(t), t) = 0$. Consider now the following *target system*
$$\frac{d}{dt}\nabla_{\mathbf{y}} f_0(\mathbf{y}, t) = -\mathbf{P} \nabla_{\mathbf{y}} f_0(\mathbf{y}, t), \quad \mathbf{P} \succ 0. \tag{10}$$
The unique solution of (10) will of course be such that $\nabla_{\mathbf{y}} f_0(\mathbf{y}(t), t) \to 0$ exponentially as $t \to \infty$, i.e., the gradient is driven to zero, and hence by Assumption 1, $\mathbf{y}(t)$ will be driven to $\mathbf{y}^*(t)$. The question then becomes the following: can we design the control input $\mathbf{u}$ for (6) such that the output trajectory $\mathbf{y}$ of (6) satisfies the target system (10)?

We first characterize the required evolution of $\mathbf{y}$ such that (10) holds. Using the chain rule to differentiate the gradient $\nabla_{\mathbf{y}} f_0(\mathbf{y}, t)$ with respect to time yields
$$\frac{d}{dt}\nabla_{\mathbf{y}} f_0(\mathbf{y}, t) = \nabla_{\mathbf{yy}} f_0(\mathbf{y}, t)\dot{\mathbf{y}} + \nabla_{\mathbf{y}t} f_0(\mathbf{y}, t). \tag{11}$$
Substituting (11) into (10), the target system dynamics can be equivalent described via the implicit function $\mathrm{G}(\dot{\mathbf{y}}, \mathbf{y}, t) = 0$, where
$$\mathrm{G}(\dot{\mathbf{y}}, \mathbf{y}, t) = \nabla_{\mathbf{yy}} f_0(\mathbf{y}, t)\dot{\mathbf{y}} + \nabla_{\mathbf{y}t} f_0(\mathbf{y}, t) + \mathbf{P} \nabla_{\mathbf{y}} f_0(\mathbf{y}, t), \tag{12}$$
We seek to simultaneously resolve the two implicit equations $\mathrm{F}(\dot{\mathbf{y}}, \mathbf{u}) = 0$ and $\mathrm{G}(\dot{\mathbf{y}}, \mathbf{y}, t) = 0$ for a solution $(\dot{\mathbf{y}}, \mathbf{u}) = S(\mathbf{y}, t)$. By uniform strong convexity (Assumption 1), the Hessian matrix $\nabla_{\mathbf{yy}} f_0(\mathbf{y}, t)$ is uniformly positive definite. Consequently, one can uniquely solve $\mathrm{G}(\dot{\mathbf{y}}, \mathbf{y}, t) = \mathbf{0}$ for $\dot{\mathbf{y}}$ and then recover $\mathbf{u}$ from (8), yielding
$$\dot{\mathbf{y}} = \mathbf{u} = -\nabla_{\mathbf{yy}}^{-1} f_0(\mathbf{y}, t)[\mathbf{P} \nabla_{\mathbf{y}} f_0(\mathbf{y}, t) + \nabla_{\mathbf{y}t} f_0(\mathbf{y}, t)]. \tag{13}$$
By construction, the feedback control law (13) regulates the output of the integrator system (6) such that it asymptotically tracks the trajectory implicitly defined by the unconstrained time-varying optimization problem (9). Recent studies of time-varying optimization algorithms provide an optimization-based interpretation of this design [34], [35]. The feedback law (13)

for the integrator system (6) consists of two parts:

1) a *prediction term* $-\nabla_{\mathbf{yy}}^{-1} f_0(\mathbf{y},t)\nabla_{\mathbf{y}t} f_0(\mathbf{y},t)$, which accounts for the time-variation of the optimal solution;
2) a *correction term* $-\nabla_{\mathbf{yy}}^{-1} f_0(\mathbf{y},t)\mathbf{P}\nabla_{\mathbf{y}} f_0(\mathbf{y},t)$, which acts as a proportional controller to drive the "error" (i.e., the gradient $\nabla_{\mathbf{y}} f_0(\mathbf{y},t)$) towards zero.

The proposed design generalizes the idea of feedback linearization, in that it transforms the original dynamical system into an optimization algorithm that converges to the optimizer of a time-varying optimization problem. Precisely, the nonlinear feedback control law (13) transforms the integrator into the linear gradient system $\dot{\mathbf{w}} = -\mathbf{P}\mathbf{w}$ where $\mathbf{w} = \nabla_{\mathbf{y}} f_0(\mathbf{y},t)$ and $\mathbf{P} \succ 0$.

## B. Example #2: Wheeled Mobile Robot

We now show how the previous approach extends to a more involved example, where we aim to control a nonholonomic flat system, the wheeled mobile robot (WMR) [42]:

$$\begin{aligned} \dot{x}_1 &= \cos(x_3)u_1, & \dot{x}_2 &= \sin(x_3)u_1, \\ \dot{x}_3 &= u_2, & \mathbf{y} &= (x_1, x_2). \end{aligned} \tag{14}$$

The states $(x_1, x_2) \in \mathbb{R}^2$ represent the position, and $x_3$ is the angular position of the WMR. The control inputs $(u_1, u_2)$ are the positional and angular velocities, respectively, and the position vector $\mathbf{y}$ is the flat output.

Consider again the unconstrained time-varying optimization problem (9), where our goal is now to regulate the output vector $\mathbf{y}$ of the WMR to asymptotically track the time-varying minimizer. We will again make use of the algebraic relationships between the flat outputs and the control inputs. For a WMR (14), this relationship can be determined to be

$$\mathbf{u} := \alpha(\mathbf{y}^{[k]}) = \begin{bmatrix} \sqrt{\dot{y}_1^2 + \dot{y}_2^2} \\ (\dot{y}_1\ddot{y}_2 - \ddot{y}_1\dot{y}_2)/(\dot{y}_1^2 + \dot{y}_2^2) \end{bmatrix}, \tag{15}$$

which we express implicitly as

$$\mathsf{F}(\mathbf{y},\dot{\mathbf{y}},\ddot{\mathbf{y}},\mathbf{u}) := \begin{bmatrix} u_1 - \sqrt{\dot{y}_1^2 + \dot{y}_2^2} \\ u_2 - (\dot{y}_1\ddot{y}_2 - \ddot{y}_1\dot{y}_2)/(\dot{y}_1^2 + \dot{y}_2^2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Accordingly, we generalize the first-order target system (10) to the second-order target system

$$\frac{d}{dt}\begin{bmatrix} \nabla_{\mathbf{y}} f_0(\mathbf{y},t) \\ \dot{\nabla}_{\mathbf{y}} f_0(\mathbf{y},t) \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{I}_m \\ -k_{\mathrm{p}}\mathbf{I}_m & -k_{\mathrm{d}}\mathbf{I}_m \end{bmatrix}\begin{bmatrix} \nabla_{\mathbf{y}} f_0(\mathbf{y},t) \\ \dot{\nabla}_{\mathbf{y}} f_0(\mathbf{y},t) \end{bmatrix}, \tag{16}$$

where $k_{\mathrm{p}}, k_{\mathrm{d}} > 0$. Of course, (16) now defines an exponentially stable linear system with state $\mathbf{z} := (\nabla_{\mathbf{y}} f_0(\mathbf{y},t), \dot{\nabla}_{\mathbf{y}} f_0(\mathbf{y},t))$. To determine the required target evolution of $\mathbf{y}$, we differentiate the gradient term $\nabla_{\mathbf{y}} f_0(\mathbf{y},t)$ with respect to time twice, yielding

$$\ddot{\nabla}_{\mathbf{y}} f_0(\mathbf{y},t) = \nabla_{\mathbf{yy}} f_0(\mathbf{y},t)\ddot{\mathbf{y}} + \dot{\nabla}_{\mathbf{yy}} f_0(\mathbf{y},t)\dot{\mathbf{y}} + \dot{\nabla}_{\mathbf{y}t} f_0(\mathbf{y},t).$$

Equating this with the second row of (16), we again obtain the implicit function

$$\begin{aligned} \mathsf{G}(\mathbf{y},\dot{\mathbf{y}},\ddot{\mathbf{y}},t) :=\ & \nabla_{\mathbf{yy}} f_0(\mathbf{y},t)\ddot{\mathbf{y}} + \dot{\nabla}_{\mathbf{yy}} f_0(\mathbf{y},t)\dot{\mathbf{y}} \\ & + \dot{\nabla}_{\mathbf{y}t} f_0(\mathbf{y},t) + k_{\mathrm{p}}\nabla_{\mathbf{y}} f_0(\mathbf{y},t) + k_{\mathrm{d}}\dot{\nabla}_{\mathbf{y}} f_0(\mathbf{y},t) = 0, \end{aligned} \tag{17}$$

which describes the solution trajectory of the time-varying optimization problem.

Again, we seek to simlutaneously resolve the two implicit equations $\mathsf{F}(\mathbf{y},\dot{\mathbf{y}},\ddot{\mathbf{y}},\mathbf{u}) = 0$ and $\mathsf{G}(\mathbf{y},\dot{\mathbf{y}},\ddot{\mathbf{y}},t) = 0$ for a solution $(\ddot{\mathbf{y}},\mathbf{u}) = S(\mathbf{y},\dot{\mathbf{y}},t)$. In this case, we may explicitly solve (17) for $\ddot{\mathbf{y}}$ and substitute into (15), yielding the solution

$$\begin{aligned} \ddot{\mathbf{y}} :=\ & g(\mathbf{y},\dot{\mathbf{y}}) = -\nabla_{\mathbf{yy}}^{-1} f_0(\mathbf{y},t)\big(\dot{\nabla}_{\mathbf{yy}} f_0(\mathbf{y},t)\dot{\mathbf{y}} + \dot{\nabla}_{\mathbf{y}t} f_0(\mathbf{y},t) \\ & + k_{\mathrm{p}}\nabla_{\mathbf{y}} f_0(\mathbf{y},t) + k_{\mathrm{d}}\dot{\nabla}_{\mathbf{y}} f_0(\mathbf{y},t)\big) \\ u_1 =\ & \|\dot{\mathbf{y}}\|_2, \\ u_2 =\ & \frac{1}{\|\dot{\mathbf{y}}\|_2^2}\, g(\mathbf{y},\dot{\mathbf{y}})^{\mathsf{T}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\dot{\mathbf{y}} \end{aligned}$$

The above equations define the feedback control law, which again generalizes the notion of feedback linearization, in that the above nonlinear feedback control law transforms the WMR (14) into the optimization algorithm

$$\dot{\mathbf{z}} = \mathbf{H}\mathbf{z}, \qquad \mathbf{z} = (\nabla_{\mathbf{y}} f_0(\mathbf{y},t), \dot{\nabla}_{\mathbf{y}} f_0(\mathbf{y},t))^{\mathsf{T}},$$

where $\mathbf{H}$ is the Hurwitz matrix from (16). This optimization algorithm seeks to find the optimal solution of the unconstrained version of the time-varying optimization problem (5).

Figure 3 illustrates the behavior of the proposed solution. Notably, even when $\mathbf{y}^*(t)$ is not a feasible trajectory due to, e.g., a mismatch on the initial conditions, the control law forces the trajectory to asymptotically converge to $\mathbf{y}^*(t)$.

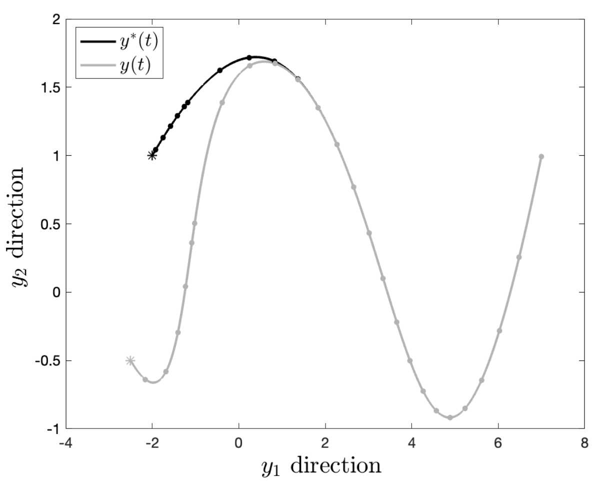


Fig. 3. Plot of a robot tracking an object, where $\mathbf{y}^*(t)$ (5) is the minimizer and $\mathbf{y}(t)$ represents the real trajectory of the robot, where the dots represent their locations at different $t$. Due to mismatching initial conditions (highlighted using asterisk), we design a control law that converges asymptotically to $\mathbf{y}^*(t)$.

## C. Key Features

We end this section by highlighting the key features that enable the success of our framework in the previous two motivating examples. In general, we consider simultaneously solve two implicit functions $\mathsf{F}(\mathbf{y}^{[k]},\mathbf{u}^{[k]}) = 0$ and $\mathsf{G}(\mathbf{y}^{[k]},t) = 0$ for a solution $(\mathbf{y}^{[k]},\mathbf{u}^{[k]}) = S(\mathbf{y}^{[k]-1},t)$, consisting of

1) the *system dynamics* or *tracking component* $\mathsf{F}(\mathbf{y}^{[k]},\mathbf{u}^{[k]})$, which is an implicit function derived from the dynamical system characterizing the system input-output relationship;

2) the *optimization dynamics* or *planning component* $\mathsf{G}(\mathbf{y}^{[k]}, t)$, which is an implicit function derived from a set of target dynamics, characterizing the desired convergence to the minimizer of the time-varying optimization problem.

Finding the desired feedback controller can be reduced to the problem of finding a solution to this system of implicit equations. In the context of a flat system, the system dynamics term can be easily expressed using (3), resulting in $\mathsf{F}(\mathbf{y}^{[k]}, \mathbf{u}) := \mathbf{u} - \alpha(\mathbf{y}^{[k]}) = 0$. As a result, our focus in the subsequent sections shifts towards the optimization dynamic term. In the two illustrative examples, the optimization dynamics $\mathsf{G}(\mathbf{y}^{[k]}, t) = 0$ that we considered are obtained from an exponentially stable linear system. This system is characterized by a state comprising the gradient $\nabla_{\mathbf{y}} f_0(\mathbf{y}, t)$ and its higher-order time derivatives as given in (17).

In the rest of this paper, we seek to generalize this approach to tackle the specific problem: for an arbitrary differentially flat system (1) and a time-varying convex optimization problem (5), define an implicit function of the form $\mathsf{G}(\mathbf{y}^{[k]}, t) = 0$, such that its solutions globally asymptotically converge to the minimizer of a general time-varying constrained convex optimization problem. A key to the success of this effort is the design of general target systems of the form

$$\begin{aligned} \dot{\mathbf{w}} &= \mathbf{H}\mathbf{w}, \\ \mathbf{w} &= (\nabla_{\mathbf{z}} L(\mathbf{z}, t), \ldots, \nabla_{\mathbf{z}}^{k-1} L(\mathbf{z}, t))^T, \end{aligned} \tag{18}$$

where $L$, $\mathbf{z}$, and $\mathbf{H}$, will be properly chosen to guarantee the asymptotic convergence of $\mathbf{y}(t)$ to the optimal solution of a general *constrained* time-varying optimization problem of the form of (5).

## IV. Unconstrained Time-varying Optimization Framework

In this section, we first consider the case where our goal is to regulate a differentially flat system (1), to the minimizer $\mathbf{y}^*(t)$ of the unconstrained time-varying optimization problem

$$\mathbf{y}^*(t) := \arg\min_{\mathbf{y} \in \mathbb{R}^m} f_0(\mathbf{y}, t). \tag{19}$$

Recall that for a differentially flat system (1), the inputs are determined by the flat outputs and a finite number of their derivatives according to (3). That is, the implicit function $\mathsf{F}(\mathbf{y}^{[k]}, \mathbf{u}) := \mathbf{u} - \alpha(\mathbf{y}^{[k]}) = 0$ represents the *system dynamics*, which is a function of up to $k$-th order derivatives of the flat output $\mathbf{y}$. The time-varying optimization problem is assumed to be uniformly convex (Assumption 1) and the objective smooth in both $\mathbf{y}$ and $t$. Building on the previously considered target systems (10) and (16), we now consider the $k$-th order target system

$$\begin{bmatrix} \nabla_{\mathbf{y}} f_0(\mathbf{y}, t) \\ \vdots \\ \nabla_{\mathbf{y}}^{(k)} f_0(\mathbf{y}, t) \end{bmatrix} = \mathbf{H} \begin{bmatrix} \nabla_{\mathbf{y}} f_0(\mathbf{y}, t) \\ \vdots \\ \nabla_{\mathbf{y}}^{(k-1)} f_0(\mathbf{y}, t) \end{bmatrix}, \tag{20}$$

where

$$\mathbf{H} = \hat{\mathbf{H}} \otimes \mathbf{I}_m, \quad \hat{\mathbf{H}} := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{k-1} \end{bmatrix}$$

is Hurwitz. Equation (20) is a natural generalization of (10) and (16). The need to increase the order of the target system as the order of the flat system increases is evidenced by the following lemma.

**Lemma 1** ($k$-th order time derivative of $\nabla_{\mathbf{y}} f_0(\mathbf{y}, t)$)**.** *Differentiating the gradient $\nabla_{\mathbf{y}} f_0(\mathbf{y}, t)$ with respect to time $k$ times yields*

$$\begin{aligned} \nabla_{\mathbf{y}}^{(k)} f_0(\mathbf{y}, t) = &\sum_{m=0}^{k-1} \binom{k-1}{m} \nabla_{\mathbf{yy}}^{(m)} f_0(\mathbf{y}, t) \mathbf{y}^{(k-m)} \\ &+ \nabla_{\mathbf{y}t}^{(k-1)} f_0(\mathbf{y}, t), \end{aligned} \tag{21}$$

*where $\binom{k-1}{m}$ denotes the binomial coefficient.*

*Proof.* See Appendix A. □

Lemma 1 shows that $k$ is the smallest number of differentiations of $\nabla_{\mathbf{y}} f_0(\mathbf{y}, t)$ needed such that $\mathbf{y}^{(k)}$ appears; by previous discussion, this provides all necessary information to compute the control for flat systems. Thus, now combining (20) and (21), for a general flat system and unconstrained time-varying optimization problem, we obtain the implicit equation

$$\begin{aligned} \mathsf{G}_{\text{unc}}(\mathbf{y}^{[k]}, t) := &\sum_{m=0}^{k-1} \binom{k-1}{m} \nabla_{\mathbf{yy}}^{(m)} f_0(\mathbf{y}, t) \mathbf{y}^{(k-m)} \\ &+ \nabla_{\mathbf{y}t}^{(k-1)} f_0(\mathbf{y}, t) + \sum_{i=0}^{k-1} a_i \nabla_{\mathbf{y}}^{(i)} f_0(\mathbf{y}, t) = 0. \end{aligned} \tag{22}$$

which specifies the desired *optimization dynamics*. The next result states that if certain regularity conditions are met, the output $\mathbf{y}$, which follows the optimization dynamics described in equation (22), will asymptotically converge globally to the minimizer $\mathbf{y}^*(t)$ of equation (19).

**Theorem 2** (Convergence of optimization dynamics (22))**.** *Let Assumption 1 hold. Then for any initial condition, the trajectory $t \mapsto \mathbf{y}(t)$ of system $\mathsf{G}_{\text{unc}}(\mathbf{y}^{[k]}, t) = 0$ defined in* (22) *globally asymptotically converges to the optimal solution $\mathbf{y}^*(t)$ of* (19)*. Moreover, the estimates*

$$\begin{aligned} \|\mathbf{y}(t) - \mathbf{y}^*(t)\|_2 &\le Ce^{-\alpha t}, \\ f_0(\mathbf{y}(t), t) - f_0(\mathbf{y}^*(t), t) &\le m_f C^2 e^{-2\alpha t} \end{aligned}$$

*hold, where*

$$C = \left( \frac{c^2}{m_f^2} \sum_{j=0}^{k-1} \|\nabla_{\mathbf{y}}^{(j)} f_0(\mathbf{y}(0), 0)\|_2^2 \right)^{\frac{1}{2}} < \infty,$$

*for some constant $c > 0$, $-\alpha := \max_{\lambda \in \mathrm{spec}(\mathbf{H})} \Re[\lambda] + \epsilon_H$, for some $\epsilon_H > 0$ sufficiently small.*

*Proof.* See Appendix B. □

The above theorem states that the solution trajectory of

$\mathsf{G}_{\mathsf{unc}}(\mathbf{y}^{[k]}, t) = 0$ from (22) converges asymptotically to the minimizer $\mathbf{y}^*(t)$ of (19). It remains to show that one can indeed simultaneously resolve the system of equations $\mathsf{F}(\mathbf{y}^{[k]}, \mathbf{u}) = 0$ and $\mathsf{G}_{\mathsf{unc}}(\mathbf{y}^{[k]}, t) = 0$ for a solution $(\mathbf{y}^{(k)}, \mathbf{u}) = S(\mathbf{y}^{[k-1]}, t)$. By uniform strong convexity (see Assumption 1), the Hessian matrix $\nabla_{\mathbf{yy}} f_0(\mathbf{y}, t)$ is uniformly positive definite, which allows us to solve $(\mathbf{y}^{(k)}, \mathbf{u})$ by solving for first for $\mathbf{y}^{(k)}$ using (22), and subsequently solving for $\mathbf{u}$ using (3). The following theorem summarizes these findings.

**Theorem 3** (TVO-based control for system (1))**.** *Let Assumption 1 hold and consider the differentially flat system* (1) *with the feedback controller*

$$\mathbf{u} := \alpha(\mathbf{y}, \ldots, \mathbf{y}^{(k-1)}, g_{\mathsf{unc}}(\mathbf{y}^{[k-1]}))$$

*where*

$$g_{\mathsf{unc}}(\mathbf{y}^{[k-1]}) = -\nabla_{\mathbf{yy}}^{-1} f_0(\mathbf{y}, t) \Bigg[ \sum_{i=0}^{k-1} a_i \nabla_{\mathbf{y}}^{(i)} f_0(\mathbf{y}, t) + \nabla_{\mathbf{y}t}^{(k-1)} f_0(\mathbf{y}, t) + \sum_{m=1}^{k-1} \binom{k-1}{m} \nabla_{\mathbf{yy}}^{(m)} f_0(\mathbf{y}, t)\mathbf{y}^{(k-m)} \Bigg]$$

*Then, for any initial condition, the flat output of* (1) *globally asymptotically converges to the optimal solution* $\mathbf{y}^*(t)$ *of* (19).

Notably, the above nonlinear feedback control effectively transforms the differentially flat system into an optimization algorithm (18) that seeks to converge to the optimal solution of the time-varying optimization problem.

## V. Equality Constrained Time-varying Optimization Framework

In this section, we consider the equality-constrained version of the time-varying optimization problem (5), written as

$$\begin{aligned} \mathbf{y}^*(t) := \arg\min_{\mathbf{y}\in\mathbb{R}^m} \;& f_0(\mathbf{y}, t) \\ \text{s.t.} \;& \mathbf{A}(t)\mathbf{y} = \mathbf{b}(t). \end{aligned} \tag{23}$$

Define the Lagrangian $L : \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}_+ \to \mathbb{R}$ associated with the problem as

$$L(\mathbf{y}, \lambda, t) = f_0(\mathbf{y}, t) + \lambda^{\mathsf{T}}(\mathbf{A}(t)\mathbf{y} - \mathbf{b}(t)), \tag{24}$$

and where $\lambda_i$ as the Lagrange multiplier associated with the $i$th equality constraint $\mathbf{a}_i(t)^{\mathsf{T}}\mathbf{y} = b_i(t)$. The Assumption 2 on $\mathbf{A}(t)$ implies that there are fewer equality constraints than primal variables, and that the equality constraints are uniformly independent. Additionally, the time-varying optimization problem is uniformly convex (see Assumption 1), and the KKT conditions are necessary and sufficient for the points to be primal and dual optimal. The optimal trajectory $\mathbf{z}^*(t) = \mathsf{col}(\mathbf{y}^*(t), \lambda^*(t)) \in \mathbb{R}^{m+q}$ is therefore characterized by the KKT conditions $\nabla_{\mathbf{z}} L(\mathbf{z}^*(t), t) = 0$, or more explicitly

$$\begin{aligned} 0 &= \nabla_{\mathbf{y}} L(\mathbf{y}^*(t), \lambda^*(t), t) = \nabla_{\mathbf{y}} f_0(\mathbf{y}^*(t), t) + \mathbf{A}(t)^{\mathsf{T}} \lambda^*(t) \\ 0 &= \nabla_{\lambda} L(\mathbf{y}^*(t), \lambda^*(t), t) = \mathbf{A}(t)\mathbf{y}^*(t) - \mathbf{b}(t). \end{aligned}$$

Building upon (20), the idea is to ensure that $z(t)$ converges to the optimal primal-dual trajectory $\mathbf{z}^*(t)$ by designing a linear target linear system $\dot{\mathbf{w}} = \mathbf{H}\mathbf{w}$ with state $\mathbf{w} = \mathsf{col}(\nabla_{\mathbf{z}} L(\mathbf{z}, t), \ldots, \nabla_{\mathbf{z}}^{(k-1)} L(\mathbf{z}, t))$, i.e.,

$$\begin{bmatrix} \nabla_{\mathbf{z}} L(\mathbf{z}, t) \\ \vdots \\ \nabla_{\mathbf{z}}^{(k)} L(\mathbf{z}, t) \end{bmatrix} = \mathbf{H} \begin{bmatrix} \nabla_{\mathbf{z}} L(\mathbf{z}, t) \\ \vdots \\ \nabla_{\mathbf{z}}^{(k-1)} L(\mathbf{z}, t) \end{bmatrix}, \tag{25}$$

where $\mathbf{H} = \hat{\mathbf{H}} \otimes \mathbf{I}_{m+q}$ is Hurwitz. As a result of Lemma 1, differentiating the gradient $\nabla_{\mathbf{z}} L(\mathbf{z}, t)$ with respect to time $k$ times yields

$$\nabla_{\mathbf{z}}^{(k)} L(\mathbf{z}, t) = \sum_{m=0}^{k-1} \binom{k-1}{m} \nabla_{\mathbf{zz}}^{(m)} L(\mathbf{z}, t)\mathbf{z}^{(k-m)} + \nabla_{\mathbf{z}t}^{(k-1)} L(\mathbf{z}, t). \tag{26}$$

In particular $\nabla_{\mathbf{zz}} L(\mathbf{z}, t)$ is the KKT matrix [44]

$$\nabla_{\mathbf{zz}} L(\mathbf{z}, t) = \begin{bmatrix} \nabla_{\mathbf{yy}} f_0(\mathbf{y}, t) & \mathbf{A}(t)^{\mathsf{T}} \\ \mathbf{A}(t) & \mathbf{0}_{q\times q} \end{bmatrix},$$

which is nonsingular because $\mathsf{rank}(\nabla_{\mathbf{yy}} f_0(\mathbf{y}, t)) = m$ and $\mathsf{rank}(\mathbf{A}(t)) = q$ for all $t \geq 0$ by Assumptions 1 and 2. This also means the optimal primal-dual pair $(\mathbf{y}^*(t), \lambda^*(t))$ is unique at each $t \geq 0$. In our convergence analysis, we will need, however, a uniform lower bound on the eigenvalues of $\nabla_{\mathbf{zz}} L(\mathbf{z}, t)$, in order to ensure a uniform bound on $\|\nabla_{\mathbf{zz}}^{-1} L(\mathbf{z}, t)\|_2 \leq K^{-1}$. Though at first, the need for such a bound is unclear, in several algorithms, such as the Newton method or interior point method, a bound of the form $\|\nabla_{\mathbf{zz}}^{-1} L(\mathbf{z}, t)\|_2 \leq K^{-1}$ on the inverse KKT matrix, for some $K > 0$, plays the same role as the strong convexity (Assumption 1) for unconstrained settings [44]–[47]. In our time-varying optimization setting, the following Uniform Lipschitz Continuous Gradient assumption (together with Assumption 1) helps us establish the uniform boundedness of the inverse KKT matrix.

**Assumption 3** (Uniform Lipschitz Continuous Gradient)**.** *The objective function* $f_0(\mathbf{y}, t)$ *has uniform Lipschitz Continuous Gradient, i.e.,*

$$\|\nabla_{\mathbf{y}} f_0(\mathbf{y}_1, t) - \nabla_{\mathbf{y}} f_0(\mathbf{y}_2, t)\|_2 \leq L\|\mathbf{y}_1 - \mathbf{y}_2\|_2 \tag{27}$$

*for some* $L > 0$*, for all* $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^m$ *and for all* $t \geq 0$.

We will also use the following lemma to characterize the uniform boundedness of the eigenvalues of the KKT matrix. The lemma is stated for fixed (time-invariant) matrices, but will be later generalized for time-varying settings.

**Lemma 4** (Eigenvalues of KKT matrix)**.** *[48, Lemma 2.1] Suppose a matrix* $\mathbf{A}$ *takes the form*

$$\mathbf{A} := \begin{bmatrix} \mathbf{M} & \mathbf{B}^{\mathsf{T}} \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)},$$

*where* $\mathbf{M} \in \mathbb{R}^{m\times m}$ *is symmetric and positive definite,* $\mathbf{B} \in \mathbb{R}^{q\times m}$ *with* $q < m$ *is full rank. Let* $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m > 0$ *be the eigenvalues of* $\mathbf{M}$*, let* $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_q > 0$ *be the singular values of* $\mathbf{B}$*, and denote by* $\lambda(\mathbf{A})$ *the spectrum of the matrix. Then* $\lambda(\mathbf{A}) \subset I^- \cup I^+$*, where* $I^- = [\frac{1}{2}(\mu_m - \sqrt{\mu_m^2 + 4\sigma_1^2}), \frac{1}{2}(\mu_1 - \sqrt{\mu_1^2 + 4\sigma_q^2})]$ *and* $I^+ = [\mu_m, \frac{1}{2}(\mu_1 + \sqrt{\mu_1^2 + 4\sigma_1^2})]$.

As mentioned before, the above Lemma bounds the spectrum of any fixed KKT matrix to be within two intervals, a negative $I^-$ and positive interval $I^+$. Notably, when $\mu_m > 0$ and $\sigma_q > 0$, the entire spectrum of the KKT matrix is bounded away from zero. A simple generalization of this argument for time-varying matrices establishes the uniform boundedness (for all times) of the inverse KKT matrix $\nabla_{\mathbf{zz}} L(\mathbf{z}, t)$.

**Lemma 5** (Uniform boundeness of inverse KKT matrix)**.** *Let Assumptions 1, 2 and 3 hold, then for all $t \geq 0$ and $\mathbf{z}$, we have $\|\nabla_{\mathbf{zz}}^{-1} L(\mathbf{z}, t)\|_2 \leq 1/\min\{m_f, \frac{1}{2}(\sqrt{L^2 + 4\tau_{\min}} - L)\}$.*

*Proof.* See Appendix C. □

We are now ready to extend our framework to the equality-constrained problem (23). Combining (25) and (26), we use the following implicit function to define the *optimization dynamics*, when time-varying equality constraints are included:

$$\begin{aligned}\mathsf{G}_{\text{eq}}(\mathbf{z}^{[k]}, t) := &\sum_{m=0}^{k-1} \binom{k-1}{m} \nabla_{\mathbf{zz}}^{(m)} L(\mathbf{z}, t) \mathbf{z}^{(k-m)} \\ &+ \nabla_{\mathbf{z}t}^{(k-1)} L(\mathbf{z}, t) + \sum_{i=0}^{k-1} a_i \nabla_{\mathbf{z}}^{(i)} L(\mathbf{z}, t) = 0. \end{aligned} \tag{28}$$

The following theorem states that if certain regularity conditions are met, the output $\mathbf{z}$, which follows the optimization dynamics described in equation (28), will globally asymptotically converge to the optimal primal solution $\mathbf{y}^*(t)$ and the optimal equality constraint dual solution $\lambda^*(t)$ of (23).

**Theorem 6** (Convergence of equality constrained optimization dynamics (28))**.** *Let Assumptions 1, 2 and 3 hold. Then for any initial condition, the trajectory $t \mapsto \mathbf{z}(t)$ of system $\mathsf{G}_{\text{eq}}(\mathbf{z}^{[k]}, t) = 0$ defined in* (28) *globally asymptotically converges to the optimal solution of $\mathbf{z}^*(t) = \mathrm{col}(\mathbf{y}^*(t), \lambda^*(t))$ of the time-varying equality constrained optimization problem* (23)*. Moreover, the following bound holds*

$$\|\mathbf{z}(t) - \mathbf{z}^*(t)\|_2 \leq C e^{-\alpha t},$$

*where*

$$0 < C = \left(\frac{c^2}{K^2} \sum_{j=0}^{k-1} \|\nabla_{\mathbf{z}}^{(j)} L(\mathbf{z}(0), 0)\|_2^2\right)^{\frac{1}{2}} < \infty,$$

*for some constant $c > 0$, $\alpha := -\max_{\lambda \in \mathrm{spec}(\mathbf{H})} \Re[\lambda] + \epsilon_H$, for some $\epsilon_H > 0$ small enough and $K = 1/\min\{m_f, \frac{1}{2}(\sqrt{L^2 + 4\tau_{\min}} - L)\}$.*

*Proof.* See Appendix D. □

Lastly, It remains to show that one can indeed simultaneously resolve the system of implicit equations $\mathsf{F}(\mathbf{z}^{[k]}, \mathbf{u}) := \mathbf{u} - \alpha_z(\mathbf{z}^{[k]}) = 0$ and $\mathsf{G}_{\text{eq}}(\mathbf{z}^{[k]}, t) = 0$ for the pair $(\mathbf{z}^{(k)}, \mathbf{u}) = S(\mathbf{z}^{[k-1]}, t)$, where $\alpha_z(\mathbf{z}^{[k]}) := \alpha(\mathbf{y}^{[k]})$. According to Lemma 5, the KKT matrix is uniformly well-defined and bounded, which allows to solve $(\mathbf{z}^{(k)}, \mathbf{u})$ by solving for first for $\mathbf{z}^{(k)}$ using (28) and subsequently solving for $\mathbf{u}$ using (3). The following result summarizes these findings.

**Theorem 7** (Equality constrained TVO-based control for system (1))**.** *Let Assumption 1, 2 and 3 hold and consider the differentially flat system* (1) *with the feedback controller*

$$\mathbf{u} = \alpha_z(\mathbf{z}, \ldots, \mathbf{z}^{(k-1)}, g_{\text{eq}}(\mathbf{z}^{[k-1]}))$$

*where*

$$\begin{aligned} g_{\text{eq}}(\mathbf{z}^{[k-1]}) := &-\nabla_{\mathbf{zz}}^{-1} L(\mathbf{z}, t) \left[\sum_{i=0}^{k-1} a_i \nabla_{\mathbf{z}}^{(i)} L(\mathbf{z}, t) \right. \\ &\left. + \nabla_{\mathbf{z}t}^{(k-1)} L(\mathbf{z}, t) + \sum_{m=1}^{k-1} \binom{k-1}{m} \nabla_{\mathbf{yy}}^{(m)} L(\mathbf{z}, t) \mathbf{z}^{(k-m)}\right]. \end{aligned}$$

*Then, for any initial condition, the flat output of* (1) *globally asymptotically converges to the optimal solution $\mathbf{y}^*(t)$ of* (23).

## VI. Inequality Constrained Time-varying Optimization Framework

In Section V, we showed how to incorporate equality constraints by Lagrangian duality in our framework. In this section, we now consider the time-varying inequality-constrained optimization problem

$$\begin{aligned} \mathbf{y}^*(t) = \arg\min_{\mathbf{y} \in \mathbb{R}^m} &\ f_0(\mathbf{y}, t) \\ \text{s.t.} &\ f_i(\mathbf{y}, t) \leq 0, \quad \forall i \in [p]. \end{aligned} \tag{29}$$

Define the Lagrangian $L : \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}_+ \to \mathbb{R}$ associated with the problem (29) as

$$L(\mathbf{y}, \lambda, t) = f_0(\mathbf{y}, t) + \sum_{i=1}^{p} \lambda_i f_i(\mathbf{y}, t)$$

where $\lambda_i$ is the Lagrange multiplier associated with the $i$th inequality constraint $f_i(\mathbf{y}, t) \leq 0$. Under our previous assumptions, the time-varying optimization problem is uniformly strongly convex, and the KKT conditions are necessary and sufficient for optimality [44], [49]. Precisely, for any $t \geq 0$ we have the following KKT conditions

$$\begin{aligned} &\nabla_{\mathbf{y}} f_0(\mathbf{y}^*(t), t) + \sum_{i=1}^{p} \lambda_i^*(t) \nabla_{\mathbf{y}} f_i(\mathbf{y}^*(t), t)) = 0 \\ &\lambda_i^*(t) \geq 0, \qquad \forall i \in [p], \\ &\lambda_i^*(t) f_i(\mathbf{y}^*(t), t)) = 0, \quad \forall i \in [p]. \end{aligned} \tag{30}$$

where the primal feasibility conditions automatically hold for the global minimum $\mathbf{y}^*(t)$ and are dropped. Motivated by [34], [44], in this section, we will use an interior-point algorithm – the barrier method – to enforce the inequality constraints in (29). The goal of the barrier method is to approximately formulate the time-varying inequality-constrained problem as a time-varying unconstrained problem. Towards this goal, we first rewrite the problem (29) as

$$\mathbf{y}^*(t) = \arg\min_{\mathbf{y} \in \mathbb{R}^m} f_0(\mathbf{y}, t) + \sum_{i=1}^{p} \mathrm{I}_-(f_i(\mathbf{y}, t)), \tag{31}$$

where $\mathrm{I}_- : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is the indicator function for the nonpositive reals: $\mathrm{I}_-(u) = 0$ for $u \leq 0$ and $\mathrm{I}_-(u) = +\infty$ for $u > 0$. The constraints have been moved to the objective function, but the objective function is now extended-real valued and non-differentiable. To approximate the indicator function $\mathrm{I}_{-}$, we use a continuously differentiable logarithmic barrier function given by: $\hat{\mathrm{I}}_{-}(u,t) = -\frac{1}{c(t)}\log(-u)$, where $c(t) > 0$ is a parameter that ensures the accuracy of the approximation improves as $t$ increases. Specifically, the coefficient $c(t)$ is required to be monotonically increasing, asymptotically converging to infinity, and bounded in finite time; a convenient choice is

$$c(t) = c_0 e^{\alpha_c t}, \qquad \text{where } \alpha_c, c_0 > 0. \tag{32}$$

Substituting $\mathrm{I}_{-}$ with $\hat{\mathrm{I}}_{-}$ approximates the objective function of (31) as

$$\Phi(\mathbf{y},t) = f_0(\mathbf{y},t) + \sum_{i=1}^{p} -\frac{1}{c(t)}\log(-f_i(\mathbf{y},t)). \tag{33}$$

A limitation of (33) is that it requires a starting point that satisfies all the constraints. If such a point is not known a priori, a time-varying slack variable denoted as $s(t)$ may be introduced, leading to the perturbed approximation of (33) given by

$$\hat{\Phi}(\mathbf{y},t) := f_0(\mathbf{y},t) - \frac{1}{c(t)}\sum_{i=1}^{p}\log(s(t) - f_i(\mathbf{y},t)). \tag{34}$$

The idea is that the slack variable should be initially large enough to ensure feasibility, and then should shrink as time increases to obtain feasibility of the original desired constraints. For instance, one may take $s(t) = s_0 e^{-\alpha_s t}$ with $\alpha_s > 0$ and

$$s_0 = \begin{cases} 0 & \text{if } \max_i f_i(\mathbf{y}(0),0) \le 0 \\ \max_i f_i(\mathbf{y}(0),0) + \epsilon_s & \text{if } \max_i f_i(\mathbf{y}(0),0) > 0 \end{cases} \tag{35}$$

for some $\epsilon_s > 0$. By incorporating the inequality constraints into the objective function with logarithmic barrier functions as in (34), the above constrained time-varying optimization problem can be approximated by:

$$\hat{\mathbf{y}}^*(t) := \arg\min_{\mathbf{y}\in\mathbb{R}^m} \hat{\Phi}(\mathbf{y},t). \tag{36}$$

The following result [34, Lemma 1] provides an upper bound on the duality gap associated with $\mathbf{y}^*(t)$ and the Lagrange multiplier $\lambda^*(t)$. It also confirms that with proper choices of $s(t)$ and $c(t)$, the approximated optimal solution $\hat{\mathbf{y}}^*(t)$ converges to the optimal solution $\mathbf{y}^*(t)$ as $t \to +\infty$, provided that the Lagrange multipliers $\lambda^*(t)$ remain bounded.

**Lemma 8** (Approximation error [34, Lemma 1])**.** *Consider the inequality-constrained time-varying optimization problem* (29) *let and $\mathbf{y}^*(t)$ be the optimal solution. Let $\lambda^*(t)$ be the Lagrange multiplier associated with inequality constraints and $\hat{\mathbf{y}}^*(t)$ be the optimal solution of the perturbed approximation* (36)*. Under Assumptions 1 and 2, it holds for all $t \ge 0$ that*

$$|f_0(\hat{\mathbf{y}}^*(t),t) - f_0(\mathbf{y}^*(t),t)| \le \frac{p}{c(t)} + \sum_{i=1}^{p}\lambda_i^*(t)s(t). \tag{37}$$

Lemma 8 not only provides a uniform bound for the optimality error, but it also suggests appropriate selections of $s(t)$, $c(t)$. In particular, choosing $c(t)$ as in (32) ensures that the first term in (37) goes to zero. Thus, if one were to further guarantee that $\sum_{i=1}^{p}\lambda_i^*(t)s(t) \to 0$ as $t \to +\infty$, then this would readily imply that the optimal solution $\hat{\mathbf{y}}^*(t)$ of (36) converges to the optimal solution $\mathbf{y}^*(t)$ of (5). Since we are interested in asymptotic convergence, roughly speaking, this requires that the optimization problem does not have exponentially unbounded optimal dual variables. We will prove next that under sufficient regularity conditions, one can provide a uniform constant bound on the value of the multipliers $\lambda^*(t)$, thus making the approximation $\hat{\mathbf{y}}^*(t)$ of (36) converge asymptotically to $\mathbf{y}^*(t)$. One notable contribution of this paper is the direct establishment of regularity conditions that ensure the uniform boundedness of the Lagrange multipliers. This contributes to the literature of time-varying optimization [34], [35], wherein asymptotic boundedness of multipliers is assumed. For a static nonconvex optimization problem, *Mangasarian-Fromowitz constraint qualification* (MFCQ) is shown to be necessary and sufficient to have the set of Lagrange multipliers being nonempty and bounded [50]. For a general time-varying optimization problem (5), where both equality and inequality constraints are considered, the following Lemma provides a sufficient condition for the uniform boundedness of the set of Lagrange multipliers.

**Lemma 9.** *(Uniform boundedness of Lagrange multipliers) Let $\mathbf{y}^*(t)$ be the optimal solution of* (29)*, and suppose that Assumptions 1, 2, and 3 are satisfied. Then the set of Lagrange multipliers $\lambda^*(t) \in \mathbb{R}^p$ satisfying the KKT conditions* (30) *is nonempty and uniformly bounded, i.e.,*

$$\|\lambda^*(t)\|_1 \le \frac{Ld}{\epsilon}, \qquad \forall t \ge 0,$$

*where $\|\cdot\|_1$ denotes the $l_1$ vector norm, $d, \epsilon$ are defined in Assumption 2, $L$ represents the L-Lipschitz gradient.*

*Proof.* See Appendix E. □

Lemmas 8 and 9 provide the necessary ingredients to ensure that the approximation error (37) asymptotically goes to zero. In particular, we will choose $c(t) = c_0 e^{\alpha_c t}$ and $s(t) = s_0 e^{-\alpha_s t}$ to ensure a vanishing approximation error (see Theorem 10 below). Having ensure the asymptotically exactness of the approximation, we now consider the following target system as a natural extension of (20) when inequality constraints are included

$$\begin{bmatrix} \nabla_{\mathbf{y}}\hat{\Phi}(\mathbf{y},t) \\ \vdots \\ \nabla_{\mathbf{y}}^{(k)}\hat{\Phi}(\mathbf{y},t) \end{bmatrix} = \mathbf{H}\begin{bmatrix} \nabla_{\mathbf{y}}\hat{\Phi}(\mathbf{y},t) \\ \vdots \\ \nabla_{\mathbf{y}}^{(k-1)}\hat{\Phi}(y,t) \end{bmatrix}, \tag{38}$$

where $\mathbf{H} = \hat{\mathbf{H}} \otimes \mathbf{I}_m$ being Hurwitz.

Analogously, combining (38) and Lemma 1, we can use the following implicit function to define the *optimization dynamics*, when inequality constraints are included:

$$\begin{aligned} \mathrm{G}_{\mathrm{ineq}}(\mathbf{y}^{[k]},t) := & \sum_{m=0}^{k-1}\binom{k-1}{m}\nabla_{\mathbf{yy}}^{(m)}\hat{\Phi}(\mathbf{y},t)\mathbf{y}^{(k-m)} \\ & + \nabla_{\mathbf{y}t}^{(k-1)}\hat{\Phi}(\mathbf{y},t) + \sum_{i=0}^{k-1} a_i \nabla_{\mathbf{y}}^{(i)}\hat{\Phi}(\mathbf{y},t) = 0. \end{aligned} \tag{39}$$

Under sufficient regularity conditions, the output $\mathbf{y}(t)$ satisfying the optimization dynamics (39) globally converges to the minimizer $\mathbf{y}^*(t)$ of the time-varying inequality constrained optimization problem (29).

**Theorem 10** (Convergence of inequality constrained optimization dyanmics (39))**.** *Let Assumptions 1, 2 and 3 hold, with $c(t)$ given by* (32)*, and $s(t) = s_0 e^{-\gamma_s t}$ with $s_0$ as in* (35)*. Then for any initial condition, the trajectory $t \mapsto \mathbf{y}(t)$ of system $\mathsf{G}_{\mathsf{ineq}}(\mathbf{y}^{[k]}, t) = 0$ defined in* (39) *globally asymptotically converges to the optimal solution$\mathbf{y}^*(t)$ of the time-varying inequality constrained optimization problem* (29)*. Moreover, the following bounds hold*

$$\|\mathbf{y}(t) - \hat{\mathbf{y}}^*(t)\|_2 \le Ce^{-\alpha t},$$

$$|f_0(\mathbf{y}(t), t) - f_0(\mathbf{y}^*(t), t)| \le LCe^{-\alpha t} + pc_0e^{-ct} + \frac{Lds_0}{\epsilon}e^{-\gamma_s t},$$

*where*

$$0 < C = \left(\frac{c^2}{m_f^2} \sum_{j=0}^{k-1} \|\nabla_{\mathbf{y}}^{(j)}\hat{\varphi}(\mathbf{y}(0), 0)\|_2^2\right)^{\frac{1}{2}} < \infty,$$

*for some constant $c > 0$, $\alpha := \max_{\lambda \in \mathrm{spec}(\mathbf{H})} \Re[\lambda] + \epsilon_H$, for some $\epsilon_H > 0$ small enough.*

*Proof.* See Appendix F. □

Theorem 10 states that the solution trajectory of the implicit function $\mathsf{G}_{\mathsf{ineq}}(\mathbf{y}^{[k]}, t)$ from (39) converges to the minimizer $\mathbf{y}^*(t)$ of (29). It remains to show that one can indeed simultaneously resolve the system of implicit equations $\mathsf{F}(\mathbf{y}^{[k]}, \mathbf{u}) = 0$ and $\mathsf{G}_{\mathsf{ineq}}(\mathbf{y}^{[k]}, t) = 0$ for the pair $(\mathbf{y}^{(k)}, \mathbf{u}) = S(\mathbf{z}^{[k-1]}, t)$. Since, from the proof of Theorem 10, we have $\|\nabla_{\mathbf{yy}}^{-1}\hat{\varphi}(\mathbf{y}, t)\|_2 \le m_f^{-1}$ (see Appendix F), we can solve $(\mathbf{y}^{(k)}, \mathbf{u})$ by solving first for $\mathbf{y}^{(k)}$ using (39), and subsequently solving for $\mathbf{u}$ using (3). The following theorem summarizes these findings.

**Theorem 11** (Inequality constrained TVO-based control for system (1))**.** *Let Assumptions 1, 2 and 3 hold, with $c(t)$ given by* (32)*, and $s(t) = s_0 e^{-\gamma_s t}$ with $s_0$ as in* (35)*. Consider the differentially flat system* (1) *with the feedback controller*

$$\mathbf{u} := \alpha(\mathbf{y}, \ldots, \mathbf{y}^{(k-1)}, g_{\mathsf{ineq}}(\mathbf{y}^{[k-1]}))$$

*where*

$$g_{\mathsf{ineq}}(\mathbf{y}^{[k-1]}) := -\nabla_{\mathbf{yy}}^{-1}\hat{\varphi}(\mathbf{y}, t)\left[\sum_{i=0}^{k-1} a_i \nabla_{\mathbf{y}}^{(i)}\hat{\varphi}(\mathbf{y}, t) + \nabla_{\mathbf{y}t}^{(k-1)}\hat{\varphi}(\mathbf{y}, t) + \sum_{m=1}^{k-1}\binom{k-1}{m}\nabla_{\mathbf{yy}}^{(m)}\hat{\varphi}(\mathbf{y}, t)\mathbf{y}^{(k-m)}\right].$$

*Then, for any initial condition, the flat output of* (1) *globally asymptotically converges to the optimal solution $\mathbf{y}^*(t)$ of* (29)*.*

## VII. Numerical Examples

In this section, we use two numerical examples arising in multi-robot coordination to illustrate the effectiveness of our solution approach. As our time-varying feedback optimization framework automatically guarantees asymptotic satisfaction of time-varying equality and inequality constraints, we apply the method for the specification of formation constraints (Section VII-A) and to enforce collision avoidance (Section VII-B).

### *A. Multi-robot Navigation with formation constraints*

In this numerical example, two WMRs (14) are required to track two moving objects respectively, but the maximum distance between two agents is limited (e.g., due to communication or formation constraints). Let $\mathbf{y}_1(t), \mathbf{y}_2(t) \in \mathbb{R}^2$ denote the position of each WMR, with $\mathbf{y}_1^d(t), \mathbf{y}_2^d(t)$ denoting the positions of the two moving objects. To model the above objectives, consider the time-varying optimization problem

$$\begin{aligned} \min_{\mathbf{y}_1, \mathbf{y}_2} & \|\mathbf{y}_1 - \mathbf{y}_1^d(t)\|_2^2 + \|\mathbf{y}_2 - \mathbf{y}_2^d(t)\|_2^2 \\ s.t. \ & \|\mathbf{y}_1 - \mathbf{y}_2\|_2 \le d(t), \end{aligned} \tag{40}$$

where $d(t)$ denotes the maximum (Euclidean) separation allowed between the two robots at time $t$. The trajectories of the moving objects, $\mathbf{y}_1^d(t)$ and $\mathbf{y}_2^d(t)$, are designed using a time-parametric representation (Section 2.4 [42]). More specifically, we parametrize the trajectories $\mathbf{y}_1^d(t)$ and $\mathbf{y}_2^d(t)$ by

$$\mathbf{y}_i^d(t) = \sum_{j=1}^{N} \mathbf{A}_{ij} t^j, \tag{41}$$

where the coefficients $\mathbf{A}_{ij}$ can be calculated from the initialization.

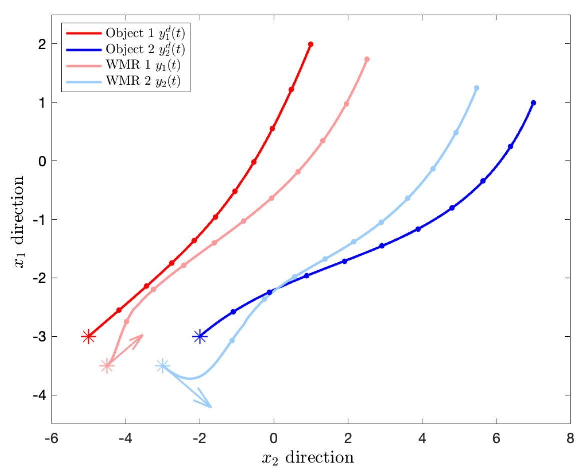


Fig. 4. Trajectories of two moving objects $\mathbf{y}_1^d(t), \mathbf{y}_2^d(t)$ (dark) and two WMRs $\mathbf{y}_1, \mathbf{y}_2$ (light). WMRs succeed in tracking two moving objects while satisfying distance constraints between them.

The simulation results are illustrated in Figure 4 and Figure 5. The dark red and blue curves in Figure 4 represent two moving object trajectories which are generated using the time parametric representation (41). More specifically, the randomly picked starting states (using asterisk) are $[-5; -3; 0.5]$ and $[-2; -3; 0.5]$. As for the robots, they are positioned around the starting position with random perturbations, which are $[-4.5, -3.5; 0.5]$ and $[-3, -3.5; -0.5]$ (using asterisk). The two WMRs' trajectories are represented using light red and blue curves. And the arrows represent the positional velocity vector at each position $\mathbf{y}_i$. The total simulation time is

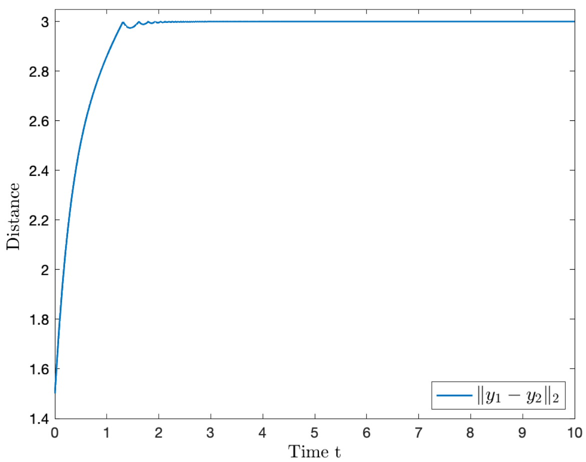


Fig. 5. Euclidean distance between two WMRs $\|\mathbf{y_1} - \mathbf{y_2}\|_2$. The maximum distance constraint is satisfied for all $t \leq T$.

$T = 10s$ and the maximum distance allowed between two robots $d(t) = 3$. For calculation simplicity, we choose the logarithmic barrier coefficient $c_0 = 1$. For this implementation, the differential equations are solved using MATLAB standard ODE solver based on an explicit Runge-Kutta (4,5) formula (ode45). In Figure 4, two robots starting from arbitrary positions, by seeking to find the minimizer of (40), succeed in tracking respective moving targets while not exceeding a predefined total maximum mutual distance. Furthermore, in Figure 5 we plot the (euclidean) distance between two WMRs $\|\mathbf{y}_1 - \mathbf{y}_2\|_2^2$ and we conclude that the time-varying inequality constraints are not violated using our solution approach, i.e., $\|\mathbf{y}_1 - \mathbf{y}_2\|_2^2 \leq 3$.

## B. Robot tracking and obstacle avoidance

In this section, we aim to solve the problem of navigating a disk-shaped wheeled mobile robot (WMR) to track a moving target without colliding with spherical obstacles in the environment. Our construction follows the concept of a *collision-free local workspace* with *projected goals* [51], and particularly [34], wherein a robot navigation problem (for integrator dynamics (6)) is formulated as time-varying convex optimization problem. We first introduce some preliminary concepts and then show how to generalize the results of [34] to our framework.

Consider a closed and convex *workspace* $W \subseteq \mathbb{R}^2$, which is populated with $O$ non-intersecting spherical obstacles, where the center and radius of the $i$th obstacle are denoted by $\mathbf{y}_i \in W$ and $r_i > 0$, respectively. Suppose the wheeled mobile robot (WMR) of radius $r > 0$ is defined as in (14), where the flat output, namely the position vector of the center of mass of the WMR, is given by $\mathbf{y}_c = (x_1, x_2)$. We define the *free space*, denoted by $F$, as the set of configurations in the workspace in which the robot does not collide with any obstacle, i.e.,

$$F = \{\mathbf{y} \in W : \bar{B}(\mathbf{y}, r) \subseteq W \setminus \bigcup_{i=1}^{O} B(\mathbf{y}_i, r_i)\} \tag{42}$$

where $B(\mathbf{y}, r)$ is the 2-dimensional open ball centered at $\mathbf{y}$ with radius $r$, and $\bar{B}(\mathbf{y}, r)$ denotes its closure. Given the moving target $\mathbf{y}^d(t) \in F$ for all $t \geq 0$, we aim to solve for the control input $\mathbf{u}(t)$ such that $\mathbf{y}_c(t) \in F$ for all $t \geq 0$ with initialization $\mathbf{y}_c(0) \in F$. Moreover, we wish to achieve $\lim_{t\to\infty} \mathbf{y}_c(t) = \mathbf{y}^d(t)$ if the obstacles were to allow it, i.e., global asymptotic convergence of the WMRs towards obstacle free targets.

Define the *local workspace* as in [34], [51],

$$LW(\mathbf{y}_c) = \{\mathbf{y} \in W : \|\mathbf{y} - \mathbf{y}_c\|_2^2 - r^2 \leq \|\mathbf{y} - \mathbf{y}_i\|_2^2 - r_i^2, \forall i\}.$$

According to [51][Prop 1], the local workspace $LW(\mathbf{y}_c)$ defines a polytope such that $\mathbf{y}_c \in F \iff B(\mathbf{y}_c, r) \subseteq LW(\mathbf{y}_c)$. Therefore, to determine a collision-free neighborhood of the robot, define the robot's *collision-free local workspace* by removing the volume of the robot body sweeping along the boundary $\partial LW(\mathbf{y}_c)$, i.e.,

$$LF(\mathbf{y}_c) := LW(\mathbf{y}_c) \setminus (\partial LW(\mathbf{y}_c) \oplus B(0, r)). \tag{43}$$

Following [51], the collision-free local workspace $LF$ is equivalent to the following set of inequality constraints:

$$LF(\mathbf{y}_c) = \{\mathbf{y} \in W : a_i(\mathbf{y}_c)^\top \mathbf{y} - b_i(\mathbf{y}_c) \leq 0, i \in [O]\},$$

where

$$a_i(\mathbf{y}_c) = \mathbf{y}_i - \mathbf{y}_c, \qquad \theta_i(\mathbf{y}_c) = \frac{1}{2} - \frac{r_i^2 - r^2}{2\|\mathbf{y}_i - \mathbf{y}_c\|^2},$$

$$b_i(\mathbf{y}_c) = (\mathbf{y}_i - \mathbf{y}_c)^\top \left(\theta_i \mathbf{y}_i + (1 - \theta_i)\mathbf{y}_c + r\frac{\mathbf{y}_c - \mathbf{y}_i}{\|\mathbf{y}_c - \mathbf{y}_i\|}\right),$$

we refer the reader to [51] for a detailed derivation of these terms.

Using the definition of collision-free local workspace, [51] and [34] find a robot navigation strategy that steers the robot $\mathbf{y} \in F$ towards the global goal $\mathbf{y}^d(t)$ through a safe local target location, called the *projected goal* by solving the following optimization problem:

$$\mathbf{y}^*(t) := \operatorname*{argmin}_{\mathbf{y}\in\mathbb{R}^2} \frac{1}{2}\|\mathbf{y} - \mathbf{y}^d(t)\|^2$$
$$\text{s.t. } a_i(\mathbf{y}_c)^\top \mathbf{y} - b_i(\mathbf{y}_c) \leq 0, \ i \in [m].$$

In [51] they consider the problem of tracking a static goal $\mathbf{y}^d$, and in [34] the authors extend the problem to track a moving target $\mathbf{y}^d(t)$ for a linear integrator (6). Although in the case of tracking a moving target, there are no theoretical guarantees, the simulation results show the proposed method successfully tracks the moving target without colliding with obstacles in the environment. We further extend the above results by navigating a WMR (14) to track a moving target $\mathbf{y}^d(t)$ in the environment. The simulation results are showed in Figure 6.

Specifically, the red and blue curves represent the real-time trajectories of the WMR $\mathbf{y}(t)$ and moving target $\mathbf{y}_d(t)$ respectively. Likewise, the moving target trajectory is generated using the time parametric representation (41). The randomly picked starting states (illustrated using asterisk) are $[-6; -3; 20]$ and $[-5; -5; 0.5]$ for the WMR and moving target. The black disks represent four random nonintersecting spherical obstacles and the red disks represent the robot configurations at each time instant (the WMR radius $r = 0.2$).

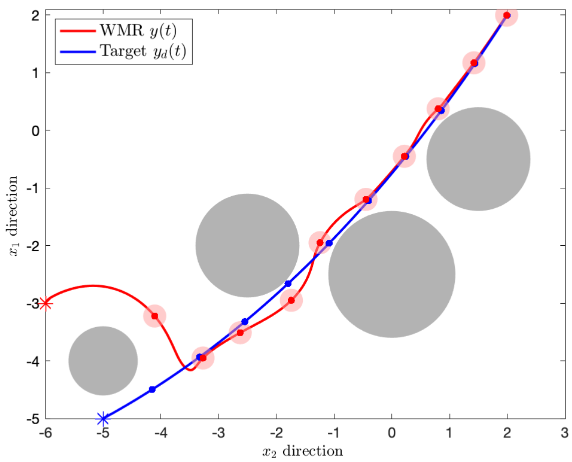


Fig. 6. Trajectory of the WMR (red curve and red disks, starting at the red asterisk) tracking a moving target (blue curve, starting at the blue asterisk) while avoiding the obstacles (grey disks), where the dots represent their locations at different $t$.

For this implementation, the ODEs are also solved using MATLAB ode45. We observe the robot succeeds in tracking the moving target while avoiding the spherical obstacles.

## VIII. Conclusions and Future Work

In this paper, we investigate the problem of steering in real time a differentially flat system to the minimizer of a time-varying constrained optimization problem. We develop a time-varying optimization-based framework composed of the system dynamic, which is an implicit function describing the input-output relationship, and the optimization dynamics, which serves as an online approximation of the minimizer. Such nonlinear control effectively transforms a differentially flat system into an optimization algorithm, which seeks to find the optimal solution to the time-varying optimization problem. Under mild assumptions, we show that the proposed control law asymptotically converges to the optimal solution of the (possibly constrained) time-varying optimization problem. Lastly, the effectiveness of our method is illustrated in two numerical examples: a multi-robot navigation problem and an obstacle avoidance problem. Future work includes generalizations to integral objectives, as in optimal control, and accounting for the effect of discrete algorithm updates.

## Appendix

### A. Proof of Lemma 1

We prove this by mathematical induction. First, we consider when $k = 1$ and $2$.

$$\begin{aligned}\dot\nabla_{\mathbf{y}} f_0(\mathbf{y},t) &= \frac{\partial \nabla_{\mathbf{y}} f_0(\mathbf{y},t)}{\partial \mathbf{y}}\dot{\mathbf{y}} + \frac{\partial \nabla_{\mathbf{y}} f_0(\mathbf{y},t)}{\partial t}\\ &= \nabla_{\mathbf{yy}} f_0(\mathbf{y},t)\dot{\mathbf{y}} + \nabla_{\mathbf{y}t} f_0(\mathbf{y},t)\\ \ddot\nabla_{\mathbf{y}} f_0(\mathbf{y},t) &= \frac{d}{dt}(\nabla_{\mathbf{yy}} f_0(\mathbf{y},t)\dot{\mathbf{y}} + \nabla_{\mathbf{y}t} f_0(\mathbf{y},t))\\ &= \nabla_{\mathbf{yy}} f_0(\mathbf{y},t)\ddot{\mathbf{y}} + \dot\nabla_{\mathbf{yy}} f_0(\mathbf{y},t)\dot{\mathbf{y}} + \dot\nabla_{\mathbf{y}t} f_0(\mathbf{y},t)\end{aligned}$$

We want to show that for every $k \geq k_0$, $k_0 \geq 2$, if the statement holds for $k$, then it holds for $k+1$.

$$\begin{aligned}\nabla_{\mathbf{y}}^{(k)} f_0(\mathbf{y},t) = &\sum_{m=0}^{k-1}\binom{k-1}{m}\nabla_{\mathbf{yy}}^{(m)} f_0(\mathbf{y},t)\mathbf{y}^{(k-m)}\\ &+ \nabla_{\mathbf{y}t}^{(k-1)} f_0(\mathbf{y},t)\end{aligned}$$

Using the binomial theorem we obtain:

$$\begin{aligned}\nabla_{\mathbf{y}}^{(k+1)} f_0(\mathbf{y},t) &= \frac{d}{dt}\Big(\sum_{m=0}^{k-1}\binom{k-1}{m}\nabla_{\mathbf{yy}}^{(m)} f_0(\mathbf{y},t)\mathbf{y}^{(k-m)}\Big)\\ &\quad+ \frac{d}{dt}(\nabla_{\mathbf{y}t}^{(k-1)} f_0(\mathbf{y},t))\\ &= \sum_{m=0}^{k}\binom{k}{m}\nabla_{\mathbf{yy}}^{(m)} f_0(\mathbf{y},t)\mathbf{y}^{(k+1-m)}\\ &\quad+ \nabla_{\mathbf{y}t}^{(k)} f_0(\mathbf{y},t),\end{aligned}$$

which completes the proof.

### B. Proof of Theorem 2

According to Lemma 1, the trajectory $\mathbf{y}(t)$ of system $\mathsf{G}_{\text{unc}}(\mathbf{y}^{[k]},t) = 0$ (22) satisfy the optimization dynamics as in (20), with $\mathbf{H}$ being the designed Hurwitz matrix. And the solution to ODE system (20) is:

$$\begin{bmatrix}\nabla_{\mathbf{y}} f_0(\mathbf{y},t)\\ \vdots\\ \nabla_{\mathbf{y}}^{(k-1)} f_0(\mathbf{y},t)\end{bmatrix} = e^{\mathbf{H}t}\begin{bmatrix}\nabla_{\mathbf{y}} f_0(\mathbf{y}(0),0)\\ \vdots\\ \nabla_{\mathbf{y}}^{(k-1)} f_0(\mathbf{y}(0),0)\end{bmatrix}$$

where $\mathbf{y}(0) \in R^m$ is the initial point. By taking the Euclidean norms of both sides we obtain

$$\sum_{j=0}^{k-1}\left\|\nabla_{\mathbf{y}}^{(j)} f_0(\mathbf{y}(t),t)\right\|_2^2 \leq c^2 e^{-2\alpha t}\sum_{j=0}^{k-1}\left\|\nabla_{\mathbf{y}}^{(j)} f_0(\mathbf{y}(0),0)\right\|_2^2 \tag{44}$$

for some constant $c > 0$, $-\alpha := \max_{\lambda\in\text{spec}(\mathbf{H})}\Re[\lambda] + \epsilon_H$ for some $\epsilon_H > 0$ small enough.

Next, we use the mean-value theorem to expand $\nabla_{\mathbf{y}} f_0(\mathbf{y},t)$ with respect to $\mathbf{y}$ as follows, where $\xi(t)$ is a convex combination of $\mathbf{y}(t)$ and $\mathbf{y}^*(t)$. Additionally using the fact that $\nabla_{\mathbf{y}} f_0(\mathbf{y}^*(t),t) = 0$ for all $t \geq 0$, we obtain:

$$\mathbf{y}(t) - \mathbf{y}^*(t) = \nabla_{\mathbf{yy}}^{-1} f_0(\xi(t),t)\nabla_{\mathbf{y}} f_0(\mathbf{y}(t),t).$$

It follows from Assumption 1, that $\|\nabla_{\mathbf{yy}}^{-1} f_0(\mathbf{y},t)\|_2 \leq m_f^{-1}$. Taking the norm on both sides together with equation (44) we have:

$$\begin{aligned}&\|\mathbf{y}(t) - \mathbf{y}^*(t)\|_2 \leq Ce^{-\alpha t},\\ &0 \leq C = \Big(\frac{c^2}{m_f^2}\sum_{j=0}^{k-1}\left\|\nabla_{\mathbf{y}}^{(j)} f_0(\mathbf{y}(0),0)\right\|_2^2\Big)^{\frac{1}{2}} < \infty.\end{aligned}$$

On the other hand, convexity of $f_0(\mathbf{y},t)$ implies that for each $t \geq 0$

$$0 \leq f_0(\mathbf{y}(t),t) - f_0(\mathbf{y}^*(t),t) \leq \nabla_{\mathbf{y}} f_0(\mathbf{y}(t),t)^{\mathsf{T}}(\mathbf{y}(t) - \mathbf{y}^*(t))$$

By applying Cauchy-Swhartz inequality on the right-hand side we obtain;

$$0 \leq f_0(\mathbf{y}(t),t) - f_0(\mathbf{y}^*(t),t) \leq m_f C^2 e^{-2\alpha t}$$

which completes the proof.

### C. Proof of Lemma 5

For all $t \geq 0$, let $\mu_1(t) \geq \mu_2(t) \geq \cdots \geq \mu_n(t) > 0$ be the eigenvalues of $\nabla_{\mathbf{yy}} f_0(\mathbf{y},t)$. Given Assumption 1 and 3, for all $t \geq 0$, we have $\mu_n(t) \geq m_f > 0$ and $\mu_1(t) \leq L < \infty$. Given Assumption 2, we have $\sigma_{\min}(\mathbf{A}(t)) \geq \tau_{\min}$ and $\sigma_{\max}(\mathbf{A}(t)) \leq \tau_{\max}$ for all $t \geq 0$.

According to Lemma 4, for the positive eigenvalues of $\nabla_{\mathbf{zz}} L(\mathbf{z},t)$, we have $m_f$ being their uniform lower bound. Also, for the negative eigenvalues of $\nabla_{\mathbf{zz}} L(\mathbf{z},t)$, we have $\frac{1}{2}(L - \sqrt{L^2 + 4\tau_{\min}})$ being their uniform upper bound. Consequently, for all $t \geq 0$, we have for all eigenvalues of the KKT matrix

$$|\lambda_i(\nabla_{\mathbf{zz}} L(\mathbf{z},t))| \geq \min\{m_f, -\frac{1}{2}(\sqrt{L^2 + 4\tau_{\min}} - L)\}, \quad (45)$$

which leads to

$$\|\nabla_{\mathbf{zz}}^{-1} L(\mathbf{z},t)\|_2 \leq \frac{1}{\min\{m_f, \frac{1}{2}(\sqrt{L^2 + 4\tau_{\min}} - L)\}}.$$

### D. Proof of Theorem 6

The structure of proof is similar to the proof of Theorem 2. According to Lemma 1, the trajectory $\mathbf{z}(t)$ of system (28) satisfy the optimization dynamics as in (25), with $\mathbf{H}$ being the designed Hurwitz matrix. Similarly, the solution to this ODE satisfies the following inequality:

$$\sum_{j=0}^{k-1} \left\| \nabla_{\mathbf{z}}^{(j)} L(\mathbf{z}(t),t) \right\|_2^2 \leq c^2 e^{-2\alpha t} \sum_{j=0}^{k-1} \left\| \nabla_{\mathbf{z}}^{(j)} L(\mathbf{z}(0),0) \right\|_2^2$$

for some constant $c > 0$, $-\alpha := \max_{\lambda\in\mathrm{spec}(\mathbf{H})} \Re[\lambda] + \epsilon_H$ for some $\epsilon_H > 0$ small enough. Next, using the mean-value theorem to expand $\nabla_{\mathbf{z}} L(\mathbf{z}(t),t)$, where $\xi(t)$ is a convex combination of $\mathbf{z}(t)$ and $\mathbf{z}^*(t)$ yields:

$$\mathbf{z}(t) - \mathbf{z}^*(t) = \nabla_{\mathbf{zz}}^{-1} L(\xi(t),t)\nabla_{\mathbf{z}} L(\mathbf{z}(t),t).$$

It follows from Corollary 5 $that \|\nabla_{\mathbf{zz}}^{-1} L(\mathbf{z},t)\|_2 \leq K^{-1}$ for some $K > 0$ and therefore,

$$\|\mathbf{z}(t) - \hat{\mathbf{z}}^*(t)\|_2 \leq Ce^{-\alpha t},$$

$$0 < C = \left(\frac{c^2}{K^2}\sum_{j=0}^{k-1}\|\nabla_{\mathbf{z}}^{(j)} L(\mathbf{z}(0),0)\|_2^2\right)^{\frac{1}{2}} < \infty.$$

### E. Proof of Lemma 9

The proof follows from [50] and considers a time-varying inequality constrained optimization problem. For all $t \geq 0$, we assume that uniform MFCQ holds at $\mathbf{y}^*(t)$ (see Assumption 2). For any $\mathbf{d}(t) \in \mathbb{R}^m$ given by uniform MFCQ, define a point $\mathbf{y}(s,t)$ sufficiently close to $\mathbf{y}^*(t)$ by:

$$\mathbf{y}(s,t) = \mathbf{y}^*(t) + s\mathbf{d}(t).$$

For all active inequality constraint functions, that is $i \in \mathrm{I}(\mathbf{y}^*(t))$, we apply Taylor's theorem:

$$\begin{aligned} f_i(\mathbf{y}(s,t),t) &= f_i(\mathbf{y}^*(t) + s\mathbf{d}(t),t) \\ &= f_i(\mathbf{y}^*(t),t) + \nabla_{\mathbf{y}} f_i(\mathbf{y}^*(t),t)^\mathsf{T} s\mathbf{d}(t) \\ &\quad + \mathbf{R}(\mathbf{y}^*(t), s\mathbf{d}(t)) \\ &= s\nabla_{\mathbf{y}} f_i(\mathbf{y}^*(t),t)^\mathsf{T}\mathbf{d}(t) + \mathbf{R}(\mathbf{y}^*(t), s\mathbf{d}(t)), \end{aligned}$$

where $\mathbf{R}(\mathbf{y}^*(t), s\mathbf{d}(t))$ is the remainder satisfying

$$\frac{\mathbf{R}(\mathbf{y}^*(t), s\mathbf{d}(t))}{\|s\mathbf{d}(t)\|} \to 0 \quad as \quad s\mathbf{d}(t) \to 0.$$

From part 1) of uniform MFCQ it follows immediately that for $s$ sufficiently small, $\mathbf{y}(s,t)$ is feasible for (5). Thus, for $s$ sufficiently small,

$$f_0(\mathbf{y}^*(t),t) = f_0(\mathbf{y}(0,t),t) \leq f_0(\mathbf{y}(s,t),t)$$

and

$$\begin{aligned} \nabla_{\mathbf{y}} f_0(\mathbf{y}^*(t),t)^\mathsf{T}\mathbf{d}(t) &= \nabla_{\mathbf{y}} f_0(\mathbf{y}(0),t)^\mathsf{T}\mathbf{d}(t) \geq 0 \implies \\ & -\nabla_{\mathbf{y}} f_0(\mathbf{y}^*(t),t)^\mathsf{T}\mathbf{d}(t) \leq 0. \end{aligned}$$

Next, we consider the linear program:

$$\begin{aligned} \max_{\mathbf{d}} \quad & -\nabla_{\mathbf{y}} f_0(\mathbf{y}^*(t),t)^\mathsf{T}\mathbf{d} \\ \text{s.t.} \quad & \nabla_{\mathbf{y}} f_i(\mathbf{y}^*(t),t)^\mathsf{T}\mathbf{d} \leq -1, \quad i \in \mathrm{I}(\mathbf{y}^*(t)) \\ & \mathbf{d} \quad \text{unrestricted}. \end{aligned}$$

Any optimization variable $\mathbf{d}$ satisfying these constraint functions also satisfy the uniform MFCQ, and the value of the objective function is upper bounded by $-\nabla_{\mathbf{y}} f_0(\mathbf{y}^*(t),t)^\mathsf{T}\mathbf{d} \leq 0$ based on previous analysis. Besides, the feasibility of this linear program is also guaranteed by Assumption 2 (uniform MFCQ), since there exists $\|\mathbf{d}(t)\|_2 \leq d$ for some constant $d > 0$, and a constant $\epsilon > 0$ such that,

$$\begin{aligned} \nabla_{\mathbf{y}} f_i(\mathbf{y}^*(t),t)^\mathsf{T}\mathbf{d}(t) &\leq -\epsilon, \quad i \in \mathrm{I}(\hat{\mathbf{y}}(t)) \implies \\ \nabla_{\mathbf{y}} f_i(\mathbf{y}^*(t),t)^\mathsf{T}\frac{\mathbf{d}(t)}{\epsilon} &\leq -1, \quad i \in \mathrm{I}(\hat{\mathbf{y}}(t)), \end{aligned}$$

which means that a feasible $\mathbf{d}$ is given by $\frac{\mathbf{d}(t)}{\epsilon}$. Furthermore, using Assumption 3 and Cauchy-Schwarz inequality we have $-\nabla_{\mathbf{y}} f_0(\mathbf{y}^*(t),t)^\mathsf{T}\mathbf{d} \leq \frac{Ld}{\epsilon}$. Together, we showed that this linear program is feasible and bounded, with $-\frac{Ld}{\epsilon} \leq -\nabla_{\mathbf{y}} f_0(\mathbf{y}^*(t),t)^\mathsf{T}\mathbf{d} \leq 0$ holds for all $t \geq 0$. Its dual problem:

$$\begin{aligned} \min \quad & \sum_{i\in\mathrm{I}(\mathbf{y}^*(t))} \lambda_i \\ \text{s.t.} \quad & \lambda_i \geq 0, \quad i \in \mathrm{I}(\mathbf{y}^*(t)) \\ & \nabla_{\mathbf{y}} f_0(\mathbf{y}^*(t),t) + \sum_{i=1}^{p} \lambda_i \nabla_{\mathbf{y}} f_i(\mathbf{y}^*(t),t) = 0 \end{aligned}$$

is also feasible and bounded since strong duality holds. That is, the set of feasible $\lambda$ vectors is nonempty and bounded

$$0 \leq \sum_{i\in\mathrm{I}(\mathbf{y}^*(t))} \lambda_i^*(t) \leq \frac{Ld}{\epsilon}$$

for all $t \geq 0$, which completes the proof.

### F. Proof of Theorem 10

The structure of proof is similar to the proof of Theorem 2. According to Lemma 1, the trajectory $\mathbf{y}(t)$ of system (39) satisfy the optimization dynamics as in (38), with $\mathbf{H}$ being the designed Hurwitz matrix. Similarly, the solution to this ODE satisfies the following inequality:

$$\sum_{j=0}^{k-1} \left\| \nabla_{\mathbf{y}}^{(j)} \hat{\Phi}(\mathbf{y}(t), t) \right\|_2^2 \le c^2 e^{-2\alpha t} \sum_{j=0}^{k-1} \left\| \nabla_{\mathbf{y}}^{(j)} \hat{\Phi}(\mathbf{y}(0), 0) \right\|_2^2$$

for some constant $c > 0$, $\alpha := \max_{\lambda \in \mathrm{spec}(\mathbf{H})} \Re[\lambda] + \epsilon_H$ for some $\epsilon_H > 0$ small enough.

Next, we use the mean-value theorem to expand $\nabla_{\mathbf{y}} \hat{\Phi}(\mathbf{y}, t)$, where $\xi(t)$ is a convex combination of $\mathbf{y}(t)$ and $\hat{\mathbf{y}}^*(t)$:

$$\mathbf{y}(t) - \hat{\mathbf{y}}^*(t) = \nabla_{\mathbf{yy}}^{-1} \hat{\Phi}(\xi(t), t) \nabla_{\mathbf{y}} \hat{\Phi}(\mathbf{y}(t), t). \tag{46}$$

Notice that the Hessian $\nabla_{yy} \hat{\Phi}(y, t)$ is given by:

$$\frac{1}{c(t)} \sum_{i=1} \left[ \frac{\nabla_{\mathbf{y}} f_i(\mathbf{y}(t), t) \nabla_{\mathbf{y}} f_i(\mathbf{y}(t), t)^\top}{[s(t) - f_i(\mathbf{y}(t), t)]^2} + \frac{\nabla_{\mathbf{yy}} f_i(\mathbf{y}(t), t)}{s(t) - f_i(\mathbf{y}(t), t)} \right]$$
$$+ \nabla_{\mathbf{yy}} f_0(\mathbf{y}(t), t)$$

It follows from Assumption 1 and [52, Corollary 4.3.12], that $\left\| \nabla_{\mathbf{yy}}^{-1} \hat{\Phi}(\mathbf{y}, t) \right\|_2 \le \left\| \nabla_{\mathbf{yy}}^{-1} f_0(\mathbf{y}, t) \right\|_2 \le m_f^{-1}$. Taking the norm on both sides of equation (46) we have:

$$\|\mathbf{y}(t) - \hat{\mathbf{y}}^*(t)\|_2 \le C e^{-\alpha t},$$
$$0 \le C = \frac{c^2}{m_f^2} \left( \sum_{j=0}^{k-1} \left\| \nabla_{\mathbf{y}}^{(j)} \hat{\Phi}(\mathbf{y}(0), 0) \right\|_2^2 \right)^{\frac{1}{2}} < \infty.$$

On the other hand, convexity of $f_0(\mathbf{y}, t)$ implies that for each $t \ge 0$

$$f_0(\mathbf{y}(t), t) - f_0(\hat{\mathbf{y}}^*(t), t) \le \nabla_{\mathbf{y}} f_0(\mathbf{y}(t), t)^\top (\mathbf{y}(t) - \hat{\mathbf{y}}^*(t))$$

By applying Cauchy-Swhartz inequality on the right-hand side and using Assumption 3 we obtain:

$$|f_0(\mathbf{y}(t), t) - f_0(\hat{\mathbf{y}}^*(t), t)| \le LC e^{-\alpha t}, \tag{47}$$

Lastly, a direct application of Lemma 8 and Lemma 9 yields:

$$|f_0(\hat{\mathbf{y}}^*(t), t) - f_0(\mathbf{y}^*(t), t)| \le p c_0 e^{-\gamma_c t} + \frac{Ld}{\epsilon} s_0 e^{-\gamma_s t} \tag{48}$$

It follows from (47) ,(48) , and the triangular inequality that:

$$|f_0(\mathbf{y}(t), t) - f_0(\mathbf{y}^*(t), t)| \le LC e^{-\alpha t} + p c_0 e^{-\gamma_c t} + \frac{Ld}{\epsilon} s_0 e^{-\gamma_s t}$$

which completes the proof.

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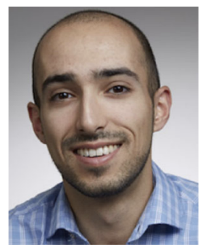

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