A Recurrence-based Direct Method for Stability Analysis and GPU-based Verification of Non-monotonic Lyapunov Functions

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***Abstract*— Lyapunov's direct method is a powerful tool that provides a rigorous framework for stability analysis and control design for dynamical systems. A critical step that enables the application of the method is the existence of a Lyapunov function V—a function whose value monotonically decreases along the trajectories of the dynamical system. Unfortunately, finding a Lyapunov function is often tricky and requires ingenuity, domain knowledge, or significant computational power. At the core of this challenge is the fact that the method requires every sub-level set of V ($V_{\leq c}$) to be forward invariant, thus implicitly coupling the geometry of $V_{\leq c}$ and the trajectories of the system. In this paper, we seek to disentangle this dependence by developing a direct method that substitutes the concept of invariance with the more flexible notion of recurrence. A set is (τ-)recurrent if every trajectory that starts in the set returns to it (within τ seconds). We show that, under mild conditions, the recurrence of sub-level sets $V_{\leq c}$ is sufficient to guarantee stability and introduce the appropriate stronger notions to obtain asymptotic stability and exponential stability. We further provide a GPU-based algorithm to verify whether V satisfies such recurrence conditions up to an arbitrarily small neighborhood of the equilibrium.**

I. INTRODUCTION

Lyapunov stability theory plays a central role in the study of dynamical systems. It provides a rigorous mathematical framework for qualitatively analyzing system solutions and has heavily influenced systems theory and engineering over the past century. A fundamental tool derived from this theory is the so-called Lyapunov direct method, a.k.a. Lyapunov's second method [1], which states mild conditions on a function $V(x)$ (non-increasing along trajectories and proper) that can certify stability of an equilibrium point. Since first proposed in 1892, Lyapunov's direct method has found ubiquitous applications across multiple branches of engineering, including aerospace, electrical, mechanical, and chemical, among others [2]–[4].

A critical step in the application of Lyapunov's direct method is finding the function V that indeed satisfies all the conditions stated by the theory. Unfortunately, while the existence of such a function is known to exist via converse theorems [5], manually finding a Lyapunov function is often tricky and relies on ingenuity and deep domain knowledge. To circumvent this step, a variety of computational methods have been proposed for finding Lyapunov functions [6], e.g., via the use of partial differential equation (PDE) solvers to solve Zubov's Equation [7], [8], linear programs (LPs) to find piece-wise linear Lyapunov functions [9], and semidefinite programs (SDPs) to solve linear matrix inequalities (LMIs) [10] or sum of square (SoSs) problems [11]. However, the computational complexity is known to exponentially increase with not only the dimension of the state space but also the parameterization of the Lyapunov function [6], [12].

This has led to multiple investigations into relaxing the conditions required for V, and in particular, its time derivative $\dot{V}$. Such relaxations can be broadly divided into three groups. The first group seeks LaSalle-Krasovskii type of conditions by relaxing the negative definiteness of $\dot{V}$, i.e., only requiring $\dot{V} \leq 0$; see [4], [13] and its generalization [14], [15]. The second group further relaxes the strict negative definite condition Lyapunov method by allowing $\dot{V} > 0$ on some regions of the state space. This is implicitly done by using generalizations of the comparison lemma [16] to impose conditions on higher order time derivatives of V that still ensure convergence of $V \to 0$ while allowing $\dot{V} > 0$ for some regions of the state space. The third and final group, known as the discretization method, considers a fixed parameter $T > 0$ and leverages the net decrement of V across any trajectory $x(t)$, i.e., $V(x(t+T)) - V(x(t))$, to reason about stability [17], [18]. Unfortunately, despite such efforts, the basic principle can still be traced back to the (indirect) construction of a Lyapunov function whose sub-level sets are invariant [19], [20], which still needs to be verified either analytically or via the solution of a convex program, rendering similar verification challenges as before.

At the core of this challenge is that the Lyapunov direct method implicitly constrains the shape of the function by requiring every sub-level set to be an invariant set. In this paper, we seek to relax this condition by replacing the invariance of sub-level sets with a weaker notion known as recurrence–a set is (τ-)recurrent if every trajectory that starts in the set returns to it (within τ seconds). Such relaxation has been recently shown to provide a powerful mechanism for estimating regions of attractions of stable equilibrium points [21]. Herein, we seek to explore their role in certifying stability. To that end, we derive conditions on a function V that renders its sub-level sets recurrent. We show that, under properly defined progressively stricter conditions, recurrence can be used to show (asymptotic) stability (Theorem 1), and exponential stability (Theorem 2).

Our derived conditions for V are similar in spirit to the ones considered by Karafyllis in [20], which considers robust stability analogs (c.f. Proposition 2.3 and 2.5). Particularly,

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our asymptotic stability conditions seem to be contemplated by [20, Prop. 2.5]. Our stability and exponential stability conditions are, however, new and not present in prior work. More importantly, the focus of our paper is on exploring the connection of such conditions with the recurrence of level sets of V and developing highly parallelizable algorithms that can be implemented on GPUs, whereas [20], focuses on robust stability and provides Matrosov-type conditions that imply such level set recurrence.

The rest of this paper is organized as follows. In Section II, we introduce preliminary definitions for dynamical systems and stability. Section III introduces the concepts of recurrence and τ-recurrence and characterizes properties of a function V that render its sub-level sets recurrent. We then move towards proving stability and asymptotic stability of an equilibrium point in Section IV. A critical step of this effort is showing that under mild conditions, τ-recurrence leads to the boundedness of trajectories, thus making recurrent sets functionally equivalent to invariant sets in proving stability. We further extend our analysis in Section V to exponential stability and develop, in Section VI, an algorithm that can be used to verify it for arbitrarily small neighborhoods of an equilibrium. We conclude in Section VII.

Notation: Throughout the text, we will use $B_r(x)$ to denote the closed ball of radius r around the equilibrium point x, and $\|\cdot\|$ will denote an arbitrary norm.

II. Preliminaries

We consider a continuous time dynamical system

$$\dot{x} = f(x)\,, \tag{1}$$

where $x \in D \subset \mathbb{R}^n$ is the state, and the map $f : D \to \mathbb{R}^n$ is a continuously differentiable and locally Lipschitz function defined over a domain D. Given an initial state x_0, we use $\phi(t, x_0)$ to denote the solution of (1). Whenever the initial condition is understood from the context, we will use $x(t) := \phi(t, x_0)$. We next introduce the core building blocks of Lyapunov Theory.

Definition 1 (Stability)**.** *An equilibrium x^* is stable if for any $\varepsilon > 0$, $\exists\delta > 0$, such that if $\|x_0 - x^*\| \leq \delta$ then $\|\phi(t, x_0) - x^*\| \leq \varepsilon$ $\forall t \geq 0$.*

Definition 2 (Asymptotic Stability)**.** *An equilibrium x^* is asymptotically stable if it is stable, and $\exists\delta > 0$ small enough such that if $\|x_0 - x^*\| \leq \delta$ then $\|\phi(t, x_0) - x^*\| \to 0$ as $t \to \infty$.*

Definition 3 (Exponential Stability)**.** *An equilibrium x^* is exponentially stable with rate α if there exists positive constants δ, k, α such that if $\|x_0 - x^*\| \leq \delta$, then $\|\phi(t, x_0) - x^*\| \leq ke^{-\alpha t}\|x_0 - x^*\|$, $\forall t \geq 0$.*

Definition 4 (ω-Limit Sets)**.** *For a dynamical system f, we say that $x \in D \subset \mathbb{R}^n$ is an **ω-limit point** of f if there exists a sequence $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}_{\geq 0}$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \phi(t_n, y) = x$ for some $y \in D \subset \mathbb{R}^n$. We denote by $\Omega(f)$ the set of all ω-limit points of f, which we call the $\omega-$**limit set of** f.*

Definition 5 (Positively Invariant Set)**.** *A set $S \subseteq \mathbb{R}^n$ is positively invariant w.r.t.* (1) *if and only if:*

$$x_0 \in S \implies \phi(t, x_0) \in S, \quad \forall t \in \mathbb{R}_{\geq 0}.$$

The notion of positive invariance is a fundamental part of Lyapunov theory. By trapping trajectories on compact level sets of a function one can guarantee boundedness of trajectories, stability, and even asymptotic stability via a gradual reduction of the Lyapunov function value. However, invariance is a very strong condition, and is quite restrictive. Thus, we seek to weaken invariance into something more easily satisfied.

III. Recurrence

To relax the notion of invariance, one must allow trajectories to temporarily leave a set. In order to make statements about asymptotic behavior, our first condition requires trajectories to return infinitely often.

Definition 6 (Recurrent Set)**.** *A set $S \subseteq \mathbb{R}^n$ is recurrent w.r.t.* (1)*, if for any $x_0 \in S$, and $t \geq 0$,*

$$\exists\, t' > t, \quad \text{s.t.} \quad \phi(t', x_0) \in S.$$

As we will soon see, Definition 6 will ensure that part of the ω-limit set of x_0 is contained within S. However, for stability analysis purposes we require some control on how far the trajectory may depart from S; this is achieved by the following stronger notion of recurrence.

Definition 7 (τ-Recurrent Set)**.** *A set $S \subseteq \mathbb{R}^n$ is τ-recurrent w.r.t.* (1)*, if for any $x_0 \in S$, and $t \geq 0$,*

$$\exists\, t' > t, \quad \text{with} \quad t' - t \in (0, \tau] \quad \text{s.t.} \quad \phi(t', x_0) \in S.$$

We say that S is strictly τ-recurrent, *if for any $x_0 \in S$, and $t \geq 0$,*

$$\exists\, t' > t, \quad \text{with} \quad t' - t \in (0, \tau] \quad \text{s.t.} \quad \phi(t', x_0) \in S \backslash \partial S.$$

We now introduce Lyapunov-like functions which form the basis of our stability results. In contrast to their standard counterpart, they are not required to decrease monotonically over trajectories. Rather, we allow τ units of time to elapse before the function is required to go below its current value; this will make their sub-level sets τ-recurrent. Definitions are given below, after some notation.

Let $V : D \subset \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, V continuous. Given $\tau > 0$ we denote:

$$\Lambda_{\min}^{(t,t+\tau]} V(x) := \begin{cases} \min_{s \in (t,t+\tau]} V(\phi(s, x)), & \text{if min exists;} \\ \infty\,, & \text{otherwise.} \end{cases}$$

Definition 8 (τ-Decreasing Functions)**.** *Let $V : D \subset \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and fix $\tau > 0$. We say that V is **τ-decreasing** over the set $D_0 \subset D$ if*

$$\Lambda_{\min}^{(0,\tau]} V(x) \leq V(x), \quad \forall x \in D_0. \tag{2}$$

If the inequality holds strictly for all $x \in D_0 \backslash \Omega(f)$, i.e.,

$$\Lambda_{\min}^{(0,\tau]} V(x) < V(x), \quad \forall x \in D_0 \backslash \Omega(f),$$

*we say that V is **strictly τ-decreasing**.*

The following Lemma establishes basic properties of τ-decreasing functions and their connection with τ-recurrent sets.

Lemma 1. *Let $V : D \subset \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ continuous, and $D_0 = V_{\leq c} := \{x \in D : V(x) \leq c\}$, $c > 0$ be a sub-level set, assumed compact. Assume V is τ-decreasing over D_0. Then:*

(i) Given $x_0 \in D_0$, there exists a sequence $\{t_n\}_{n \in \mathbb{N}}$, with

$$\lim_{n \to \infty} t_n = \infty \quad \text{and} \quad t_{n+1} - t_n \in (0, \tau] \quad \forall n, \tag{3}$$

such that

$$V(\phi(t_{n+1}, x_0)) \leq V(\phi(t_n, x_0)) \leq V(x_0) \quad \forall n. \tag{4}$$

(ii) D_0 is τ-recurrent.

Furthermore, if V is strictly *τ-decreasing, then the strict version of the first inequality in* (4) *holds for $x_0 \in D_0 \backslash \Omega(f)$, and D_0 is strictly τ-recurrent.*

Proof. Given $x_0 \in D_0$, we build the sequence $\{t_n\}_{n \in \mathbb{N}}$ satisfying (3) and (4) by induction. For the base case, let $t_0 = 0$ so that $\phi(t_0, x_0) = x_0$ and $V(\phi(t_0, x_0)) = V(x_0)$, and choose

$$t_1 = \max\{\arg\min_{t \in (0,\tau]} V(\phi(t, x_0))\};$$

note that the minimum exists by hypothesis (2), and is no larger than $V(x_0)$; if there are multiple minimizing times, t_1 is defined as the largest. By construction, $t_1 - t_0 \in (0, \tau]$ and $V(\phi(t_1, x_0)) \leq V(x_0) \leq c$, therefore $x_1 := \phi(t_1, x_0) \in D_0$.

The inductive construction proceeds in a similar manner: given $t_1 < t_2 < \cdots t_n$, with $x_n := \phi(t_n, x_0) \in D_0$, define

$$t_{n+1} - t_n = \max\{\text{argmin}_{s \in (0,\tau]} V(\phi(s, x_n))\}. \tag{5}$$

Note that $t_{n+1} - t_n \in (0, \tau]$ as required. Also,

$$x_{n+1} := \phi(t_{n+1}, x_0) = \phi(t_{n+1} - t_n, x_n)$$

satisfies $V(x_{n+1}) \leq V(x_n)$ by the τ-recurrence condition at $x_n \in D_0$, so we verify (4).

It remains to show that $t_n \to \infty$, which we argue by contradiction. If, instead, the strictly increasing sequence of times was bounded, we would have $t_n \uparrow t^*$. Note that $x_n = \phi(t_n, x_0)$ remains bounded in D_0, compact, so $\phi(t^*, x_0)$ is well defined and satisfies by continuity:

$$v_n := V(\phi(t_n, x_0)) \to V(\phi(t^*, x_0)) =: v^*.$$

Since $\{v_n\}$ is non-increasing we conclude that $v^* \leq v_n$ for all $n \in \mathbb{N}$. Now pick n such that $t_n \geq t^* - \tau$. This means that $s^* := t^* - t_n \in (0, \tau]$ is in the feasible set for the minimization in (5), which by definition gives as minimum v_{n+1}, achieved at $t_{n+1} - t_n$.

Now, since $v^* = V(s^*, x_n) \leq v_{n+1}$, this means s^* also qualifies as a minimizer, and in fact $s^* = t^* - t_n > t_{n+1} - t_n$. This contradicts the definition of $t_{n+1} - t_n$ given in (5), because it would not be the largest minimizing time. Thus the sequence must be divergent, establishing claim (i).

As for (ii), note that by construction, the sequence $\{t_n\}$ has an element in any interval $[t, t + \tau)$ of length τ, with $\phi(t_n, x_0) \in D_0$; these are precisely the conditions for τ-recurrence of $D_0 = V_{\leq c}$.

The proof of the strict case follows analogously. □

IV. Stability and Asymptotic Stability

From the preceding results, it follows that if one finds a τ-decreasing function V, trajectories must visit its sublevel sets infinitely often, which is a certain indication of stability. To formalize a result with the standard notions of Lyapunov stability, we must control the behavior of trajectories between subsequent visits to the sublevel set. Since these excursions last at most τ, we will proceed to bound the distance a point can deviate from a set within τ units of time.

We recall here that the vector field is assumed locally Lipschitz: i.e. for any point $z \in D$, there exists a neighborhood U_z around z and constant L_z such that $\forall x, y \in U_z$,

$$\|f(y) - f(x)\| \leq L_z \|y - x\|.$$

We note that under these conditions, a uniform Lipschitz constant can be defined over any compact set.

In what follows we will use $F_r := \max_{B_r(x^*)} \|f(x)\|$. Note that $F_r \to 0$ as $r \to 0$, by continuity of the vector field.

Lemma 2 (Containment Lemma)**.** *Consider a ball $S' = B_{r'}(x^*)$ for some $r' > 0$ around an equilibrium point x^*. Let $\tau > 0$, $L := \max_{z \in S'} L_z$, and let $\varphi(\ell) = \ell + F_\ell \tau e^{L\tau}$. There exists an r satisfying*

$$0 < r \leq \varphi^{-1}(r'). \tag{6}$$

Furthermore, for any r satisfying (6) *and any $x \in S := B_r(x^*)$ the following holds:*

$$\sup_{t \in (0,\tau]} d(\phi(t, x), S) \leq F_r \tau e^{L\tau},$$

where $d(y, S) := \min_{s \in S} \|y - s\|$. In particular, $\phi(t, x) \in S'$ for $t \in [0, \tau]$.

Proof. Let $r' > 0$ be given. To see that r satisfying (6) exists, observe that $\varphi(0) = 0$, and φ is a monotonically strictly increasing function such that $\varphi(\ell) > \ell$ for $\ell > 0$. Thus, we can select $r = \varphi^{-1}(r')$ uniquely as the largest such ℓ.

Now, consider an initial point $x \in S := B_r(x^*)$. Let us define $P_S[y] := \text{argmin}_{x \in S} \|y - x\|$. Let us also define $a(t) := d(\phi(t, x), S)$. Observe that

$$a(t) \leq \|\phi(t, x) - x\| = \left\| \int_0^t f(\phi(\sigma, x)) d\sigma \right\| \tag{7}$$

$$\leq \int_0^t \|f(\phi(\sigma, x)) - f(P_S(\phi(\sigma, x)))\| + \|f(P_S(\phi(\sigma, x)))\| d\sigma \tag{8}$$

$$\leq \int_0^t (L \cdot d(\phi(\sigma, x), S) + F_r) d\sigma = F_r t + \int_0^t L a(\sigma) d\sigma \tag{9}$$

for any $t \leq \tau$, wherein (7) follows from $x \in S$, (8) follows from the Triangle Inequality, and (9) follows from the Lipschitz constant L and the definition of F_r, Now, applying Grönwall's inequality (c.f [4], Lemma 2.1 with $\lambda = F_r t, \mu = L, y(t) = a(t)$), we have

$$a(t) = d(\phi(t, x), S) \leq F_r t e^{Lt}.$$

and thus $\|x(t) - x^*\| \leq r + F_r \tau e^{L\tau} = \varphi(r) = r'$, which implies that $x(t) \in S'$ for any $t \leq \tau$, as desired. □

The combination of Lemma 1 and Lemma 2, applied to a ball around equilibrium, forms the basis of our local stability analysis, which is now presented.

Theorem 1 (Stability Analysis via τ-decreasing Lyapunov Functions)**.** *Consider the system* (1)*, with a locally Lipschitz field. Let $x^* \in D$ be an equilibrium point of* (1)*, and let $V : D \subset \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be positive definite around x^*, i.e.,*

$$V(x^*) = 0, \qquad V(x) > 0 \;\; \textit{for} \;\; x \in D \backslash \{x^*\}. \tag{10}$$

If V is τ-decreasing over $D_0 = V_{\leq c} \subset D$, for some $c > 0$, then x^* is stable. Furthermore, if V is strictly τ-decreasing over D_0, then x^* is asymptotically stable.

Proof. Given any $\varepsilon > 0$, choose $0 < r \leq \varepsilon$ s.t. $B_r(x^*) \subset D$, let $L = \max_{z \in B_r(x^*)} L_z$ and find $\varepsilon' > 0$ small enough such that

$$\varepsilon' + \tau F_{\varepsilon'} e^{\tau L} < r \leq \varepsilon.$$

Now let $\alpha = \min_{\varepsilon' \leq \|x - x^*\| \leq r} V(x)$. Note that by construction $\alpha > 0$ due to (10) and continuity of V. Select β such that $0 < \beta < \min(\alpha, c)$ and introduce the set

$$\Omega_\beta := \{x \in B_{\varepsilon'}(x^*) : V(x) \leq \beta\}. \tag{11}$$

Consider an initial condition $x_0 \in \Omega_\beta$, and apply the construction of in the proof of Lemma 1 a sequence $\{t_n\}$. In particular, by the τ-recurrence hypothesis there exists a time $t_1 \in (0, \tau]$ such that $x_1 := \phi(t_1, x_0)$ satisfies $V(x_1) \leq V(x_0) \leq \beta$. Also, by Lemma 2, we have that the distance $d(x_1, B_{\varepsilon'}(x^*)) \leq \tau F_{\varepsilon'} e^{\tau L}$, hence

$$\|x_1 - x^*\| \leq \varepsilon' + \tau F_{\varepsilon'} e^{\tau L} < r.$$

Note, however, that $x_1 \notin \{x : \varepsilon' \leq \|x - x^*\| \leq r\}$ because $V(x_1) < \alpha$; therefore we must have $\|x_1 - x^*\| < \varepsilon'$ and $x_1 \in \Omega_\beta$. The construction can be repeated inductively as in Lemma 1, generating a sequence of times $t_n \to \infty$, $t_{n+1} - t_n \leq \tau$, and such that $x_n = \phi(t_n, x_0) \in \Omega_\beta$. Now, invoking the Containment Lemma (Lemma 2) once again, we can bound the behavior at intermediate times $t \in (t_n, t_{n+1}]$,

$$d(\phi(t, x_0), B_{\varepsilon'}(x^*)) \leq \tau F_{\varepsilon'} e^{\tau L} \Longrightarrow \|\phi(t, x_0) - x^*\| < r \leq \varepsilon.$$

In summary, a trajectory which starts at $x_0 \in \Omega_\beta$ stays within $B_\varepsilon(x^*)$ for all time. Finally, there exists $\delta > 0$ small enough such that $B_\delta(x^*) \subset \Omega_\beta$, and thus we satisfy the conditions of stability in Definition 1.

We now turn to asymptotic stability, under the condition that V is *strictly* τ-decreasing. The stability requirement is already established, we must prove the convergence to equilibrium of a trajectory $\phi(t, x_0)$ with initial condition $x_0 \in B_\delta \subset \Omega_\beta$.

Reviewing again the construction from Lemma 1, the points $x_n = \phi(t_n, x_0) \in \Omega_\beta$ now can be taken to make $v_n := V(x_n)$ a *strictly* decreasing sequence. Let $\bar{v}$ be its limit, we have $v_n > \bar{v}$ for all n.

Since $\{x_n\}$ is bounded in $\mathbb{R}^n$, we may take a convergent subsequence $x_{n_k} \xrightarrow{k \to \infty} \bar{x} \in \Omega_\beta$. By continuity, $\bar{v} = V(\bar{x})$. Suppose $\bar{v} > 0$, so $\bar{x} \neq x^*$. Then, by strict τ-recurrence there exists $\bar{s} \in (0, \tau]$ satisfying $V(\phi(\bar{s}, \bar{x})) < \bar{v}$. Note that

$$V(\phi(\bar{s}, x_{n_k})) \xrightarrow{k \to \infty} V(\phi(\bar{s}, \bar{x})) < \bar{v}. \tag{12}$$

However, by construction of the sequence according to Lemma 1, we have that

$$v_{n_k+1} = \min_{s \in (0,\tau]} V(\phi(s, x_{n_k})) \leq V(\phi(\bar{s}, x_{n_k})),$$

which contradicts (12) because $\bar{v} < v_{n_k+1}$.

Therefore we have shown that $v_n = V(\phi(t_n, x_0)) \xrightarrow{n \to \infty} 0$. An immediate consequence of (10) is that $x_n = \phi(t_n, x_0) \xrightarrow{n \to \infty} x^*$. Indeed, $x_n \in B_{\varepsilon'}(x^*)$ $\forall n$, and for any $\tilde{\varepsilon} < \varepsilon'$ the minimum of $V(x)$ in $\{\tilde{\varepsilon} \leq \|x - x^*\| \leq \varepsilon'\}$ is positive, so x_n must exit such a set in a finite number of steps, satisfying $\|x_n - x^*\| < \tilde{\varepsilon}$ afterward.

Let $r_n := \|x_n - x^*\| \leq \varepsilon'$. Applying the Containment Lemma to B_{r_n}, we have that for any $t \in (t_n, t_{n+1}]$,

$$\|\phi(t, x_0) - x^*\| \leq r_n + F_{r_n} \tau e^{L\tau}.$$

Since the right-hand side goes to zero in n, we have that $\lim_{t \to \infty} \phi(t, x_0) = x^*$. □

V. Exponential Stability

In the previous section, we showed that strictly τ decreasing functions were able to sequentially contain trajectories via finding progressively smaller level sets. We now move towards finding conditions on function V that ensure the exponential stability of an equilibrium point.

To that end, we will use the following notation:

$$\Lambda_\alpha^{(t,t+\tau]} V(x) := \begin{cases} \min\limits_{s \in (t,t+\tau]} e^{\alpha s} V(\phi(s, x)), & \text{if min exists,} \\ +\infty, & \text{otherwise.} \end{cases}$$

Definition 9 (α-Exponentially τ-Decreasing Functions)**.** *Let $V : D \subset \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and fix $\alpha, \tau > 0$.*

*We say that V is **exponentially τ-decreasing with rate** α over the set $D_0 \subset D$ if*

$$\Lambda_\alpha^{(0,\tau]} V(x) \leq V(x), \quad \forall x \in D_0 \backslash \Omega(f). \tag{13}$$

For short, we will often refer to V as α-exponentially τ-decreasing over D_0.

Note that an α-exponentially τ-decreasing function is always strictly τ-decreasing, but not the other way around. We will use (13) to control (exponentially decreasing) upper and lower bounds of $V(\phi(t, x))$ as $t \to \infty$.

A standard approach to prove the exponential stability of an equilibrium point using a Lyapunov function first shows that for some c and any $x \in V_{\leq c}$ the trajectories $\phi(t, x)$ satisfy

$$V(\phi(t, x)) \leq k_1 V(x) e^{-\alpha t} \tag{14}$$

for some positive constants k_1 and α. Then, one proceeds to assume certain regularity conditions, such as the ones described next.

Definition 10 (Linear Containment)**.** *Let $V : D \subset \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be continuous. If $\exists \alpha_1, \alpha_2 > 0$ such that*

$$\alpha_1 \|x - x^*\| \leq V(x) \leq \alpha_2 \|x - x^*\|, \quad \forall x \in D, \tag{15}$$

*we say that V is **linearly contained**.*

It follows from (14) that a linearly contained V satisfies

$$\begin{aligned}\|\phi(t,x) - x^*\| &\le \frac{1}{\alpha_1} V(\phi(t,x)) \le \frac{k_1}{\alpha_1} e^{-\alpha t} V(x) \\ &\le k_1 \frac{\alpha_2}{\alpha_1} e^{-\alpha t} \|x - x_0\|\end{aligned}$$

for all $x \in V_{\le c}$. Thus choosing δ small enough s.t. $\|x - x^*\| \le \delta \implies V(x) \le c$, and $k = k_1 \frac{\alpha_2}{\alpha_1}$ leads to exponential stability according to Definition 3.

Condition (13) is actually weaker than (14), so obtaining an exponential stability result is not as immediate. Nevertheless, we will see it suffices.

Theorem 2 (Exponential Stability)**.** *Consider the system* (1), *with locally Lipschitz field. Let x^* be an equilibrium point* (1), *and let $V : D \subset \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ be linearly contained, i.e., V satisfies* (15). *If V is α-exponentially τ-decreasing over $D_0 = V_{\le c}$, for some $c, \alpha, \tau > 0$, then x^* is exponentially stable with rate α. In particular, there exists $\delta > 0$ small enough such that*

$$\|\phi(t,x) - x^*\| \le \left(\frac{\alpha_2}{\alpha_1}\right) e^{\alpha\tau}(1+\tau L e^{\tau L}) e^{-\alpha t} \|x - x^*\|, \quad \forall x \in B_\delta(x^*) \subset D_0, t \in \mathbb{R}^+. \tag{17}$$

Proof. Choose $r > 0$ s.t. $B_r(x^*) \subset D$, and let $L = \max_{z \in B_r(x^*)} L_z$. First, since our hypothesis is stronger than that in Theorem 1, we can introduce the set Ω_β as in (11), such that for any $x \in \Omega_\beta$, we can guarantee $\phi(t,x) \in B_r(x^*)$ for all $t \in \mathbb{R}_+$. In particular, for such x the trajectory remains within the region of validity of the Lipschitz constant L.

Starting with $x \in \Omega_\beta$, using similar reasoning as in Lemma 1, we can recursively define a sequence $\{t_n\}_{n \in \mathbb{N}}$, with $t_0 = 0$, $\lim_{n \to \infty} t_n = \infty$ and $t_{n+1} - t_n \in (0, \tau]$, $\forall n$, such that

$$e^{\alpha t_{n+1}} V(\phi(t_{n+1}, x)) \le e^{\alpha t_n} V(\phi(t_n, x)) \le V(x), \ n \ge 1. \tag{18}$$

Using (15) and (18) we deduce that, for $n \ge 1$ we

$$\|\phi(t_n, x) - x^*\| \le \frac{V(\phi(t_n, x))}{\alpha_1} \le \frac{e^{-\alpha t_n}}{\alpha_1} V(x).$$

Denote now $r_n = \min\left(r, \frac{e^{-\alpha t_n}}{\alpha_1} V(x)\right)$. For any $t \in (t_n, t_{n+1}]$, we can apply the argument in the Containment Lemma to deduce that

$$\|\phi(t,x) - x^*\| \le r_n + F_{r_n} \tau e^{L\tau}.$$

Furthermore, noting that f is L-Lipschitz on $B_{r_n}(x^*) \subset B_r(x^*)$ and $f(x^*) = 0$ we can further bound $F_{r_n} \le L r_n$, leading to

$$\|\phi(t,x) - x^*\| \le r_n(1+\tau L e^{L\tau}) \le \frac{e^{-\alpha t_n}}{\alpha_1}(1+\tau L e^{L\tau}) V(x).$$

Note, further, that $t \le t_{n+1} \le t_n + \tau$, therefore $-t_n \le \tau - t$ so $e^{-\alpha t_n} \le e^{\alpha\tau} e^{-\alpha t}$, leading to

$$\|\phi(t,x) - x^*\| \le e^{\alpha\tau} \frac{e^{-\alpha t}}{\alpha_1}(1 + \tau L e^{L\tau}) V(x).$$

Now applying the upper bound $V(x) \le \alpha_2 \|x - x^*\|$ and we establish the bound (17) for any $x \in \Omega_\beta$. Finally, choose $\delta > 0$ s.t. $B_\delta(x^*) \subset \Omega_\beta$.

□

VI. Numerical Methods

In this section, inspired by the Theorem 2, we will develop an algorithm that, given a function V, seeks to certify a region $D_0 \subset D$ for which V is α-exponentially τ-decreasing. We aim to leverage the highly parallelizable processing units to certify many points simultaneously. It should be noted, however, that numerically testing whether V is α-exponentially τ-decreasing over a dense set D_0 is impossible. Instead, we will seek to check whether a point x satisfies a stricter condition, to simultaneously certify a neighborhood of x (Proposition 1 and 2). Leveraging these results, we then develop, in Section VI-B, a parallelizable algorithm that allows us to certify regions around an equilibrium point. We finally validate this algorithm in Section VI-C.

A. Verification of a Ball

We now show how, by requiring a stricter condition than α-exponentially τ-decreasing on a trajectory (c.f. (19)), we can certify close enough trajectories to be exponentially τ-decreasing with rate κ, with $\kappa \ge 0$.

Proposition 1. *Consider the dynamical system* (1), *with locally Lipschitz field. Let S' be a compact set contained in D and $L := \max_{z \in S'} L_z$, and consider $S \subset S'$ s.t. for all $x_0 \in S$ and $t \in [0, t]$, $\phi(t, x_0) \in S'$. Let $V : D \to \mathbb{R}_{\ge 0}$ be linearly contained, i.e.,* (15), *and assume that there is $x \in S$ such that*

$$\Lambda_\alpha^{(0,\tau]} V(x) \le \mu \left(\frac{\alpha_1}{\alpha_2}\right)^2 V(x) \tag{19}$$

for some $\mu \in (0,1)$ and $\alpha \ge 0$. If there exists $\kappa \ge 0, r > 0$ such that

$$0 < r \le \min\left\{f_1(\mu), f_2(\mu, \kappa, \alpha)\right\} \tag{20}$$

with

$$f_1(\mu) := \frac{V(x)}{\alpha_2}\left(\frac{1-\mu}{1+\frac{\alpha_2}{\alpha_1}}\right), \text{ and} \tag{21a}$$

$$f_2(\mu, \kappa, \alpha) := \frac{V(x)}{\alpha_2}\left(\frac{1 - \mu e^{-(\alpha-\kappa)\tau}}{1+\frac{\alpha_2}{\alpha_1} e^{(L+\kappa)\tau}}\right), \tag{21b}$$

then, we have $\Lambda_\kappa^{(0,\tau]} V(y) \le V(y)$, $\forall y \in B_r(x) \cap S$. That is, V is κ-exponentially τ-decreasing over the set $B_r(x) \cap S$.

Proof. The proof is omitted due to page limits. □

Proposition 1 gives us a condition to verify whether a ball $B_r(x)$ is κ-exponentially τ-decreasing. That is, by finding constants $\alpha > 0$, $\mu \in (0,1)$, and $\kappa \ge 0$ satisfying (19) and (20). However, it does not provide a method to find such constants that can be easily translated into an algorithm. Fortunately, a closer look at (21a) and (21b) allows us to provide more tenable requirements based on an univariate equation.

Proposition 2. *Consider the dynamical system* (1), *with locally Lipschitz field. Let S' be a compact set contained in D and $L := \max_{z \in S'} L_z$, and consider $S \subset S'$ s.t. for* *all $x_0 \in S$ and $t \in [0, t]$, $\phi(t, x_0) \in S^0$. For $r > 0, x \in S$, define*

$$\alpha(\mu) := \max \left\{ \alpha \in \mathrm{R}\ \ _0 \,|\, \Lambda_\alpha^{(0,\tau]} V(x) - \mu \left(\frac{\alpha_1}{\alpha_2}\right)^2 V(x) \leq 0 \right\},$$

$$\mu_{\max} := 1 - r \left(1 + \frac{\alpha_2}{\alpha_1}\right) \frac{\alpha_2}{V(x)}, \;\; \mu_{\min} := \frac{\Lambda_0^{(0,\tau]} V(x)}{V(x)} \left(\frac{\alpha_2}{\alpha_1}\right)^2.$$

Then, if $\mu_{\min} \leq \mu_{\max}$ and $\exists \mu$ such that

$$0 < r \leq f_2(\mu, 0, \alpha(\mu)), \text{ and } \mu \in [\mu_{\min}, \mu_{\max}], \tag{22}$$

then $r, x, \mu, \alpha(\mu)$ satisfy the statement of Proposition 1 with

$$\kappa(\mu, r) = \frac{1}{\tau} \ln \left(\frac{1 - (\alpha_2 / V(x)) r}{\alpha_2^2 r e^{L\tau} / (\alpha_1 V(x)) + \mu e^{\ \ \alpha(\mu)\tau}} \right), \tag{23}$$

i.e., V is exponentially τ-decreasing over the set $B_r(x) \cap S$ with rate given by (23).

Proof. The proof is omitted due to page limits. □

We will now explain how to verify a neighborhood of a point algorithmically. In this section, we assume that the radius $r > 0$ to verify is given. Thus, leveraging Proposition 2, one is only required to search along a single variable, $\mu \in [\mu_{\min}, \mu_{\max}]$, that can guarantee $f_2(\mu, 0, \alpha(u)) > r$. We summarize how to do that next.

Note first that since $\Lambda_\alpha^{(0,\tau]} V(x)$ is an increasing function of α, given μ, $\alpha(\mu)$ can be computed via binary search. To find a μ satisfying (22), in our algorithm, we first verify that $\mu_{\min} \leq \mu_{\max}$, as otherwise $[\mu_{\min}, \mu_{\max}]$ is empty and the algorithm failed (returns false). Then, if $f_2(\mu_{\min}, 0, \alpha(\mu_{\min}))$ and $f_2(\mu_{\max}, 0, \alpha(\mu_{\max}))$ are both at most r, there is no guarantee it will be greater than r at any point in the range. In such a case, we conservatively declare failure. Else, if $f_2(\mu_{\max}, 0, \alpha(\mu_{\max})) \geq r = f_1(\mu_{\max})$, by continuity of f_1, f_2, they must intersect at some point in $[\mu_{\min}, \mu_{\max}]$, (as $f_1(\mu_{\min}) > f_2(\mu_{\min}, 0, \alpha(\mu_{\min}))$), so we can apply a binary search to find an intersecting point and set μ to the point of equality. Otherwise, we can simply select $\mu = \mu_{\min}$ as $r \leq f_2(\mu_{\min}, 0, \alpha(\mu_{\min}))$. See Algorithm 1.

Algorithm 1: VerifyBall(x,r,τ, L,V,α_1, α_2)

Input $x \in \mathrm{R}^n, r > 0, \tau > 0, L > 0, V : \mathrm{R}^n \to \mathrm{R}\ \ _0, \alpha_2 \geq \alpha_1 \geq 0$

Two possible conditions of failure

if $\mu_{\min} > \mu_{\max}$ **then**

  | **return** (False, -1)

if $f_2(\mu_{\min}, 0, \alpha(\mu_{\min})) < r$ *and* $f_2(\mu_{\max}, 0, \alpha(\mu_{\max})) < r$ **then**

  | **return** (False, -1)

Definite success, all that is left is to find a good κ

if $f_2(\mu_{\min}, 0, \alpha(\mu_{\min})) \geq r$ **then**

  | $\mu = \mu_{\min}$

else

  | $\mu = \mu$ **s.t.** $f_1(\mu) = f_2(\mu, 0, \alpha(\mu))$

$\kappa = \max\{\kappa \geq 0 \mid f_2(\mu, \kappa, \alpha(\mu)) > r\}$

return (True, κ)

B. Verification of a Region

Having developed an algorithm to verify a ball $B_r(x)$, c.f. Algorithm 1, we are now ready to integrate it into an algorithm that can verify a region of the state space. A critical defining aspect of Algorithm 1 is that the radius r of the ball $B_r(x)$ that can be verified, i.e., (20), depends on the value $V(x)$ through (21a) and (21b). Thus, as x gets closer to (resp. farther away from) x , the radius r will be necessarily smaller (resp. larger).

We therefore focus here on verifying the κ-exponential τ-recurrence of a given function $V(x)$ in a region of the state space given by $D_0 = B_R(x\ )\backslash B_\varepsilon(x\ )$, and choose the grid points to account for the dependence on $V(x)$ of r. For concreteness, in this section we use $V(x) = \|x\| = \|x\|_{\ 1}$, that is, the infinity norm $\|x\| := \max_i |x_i|$.

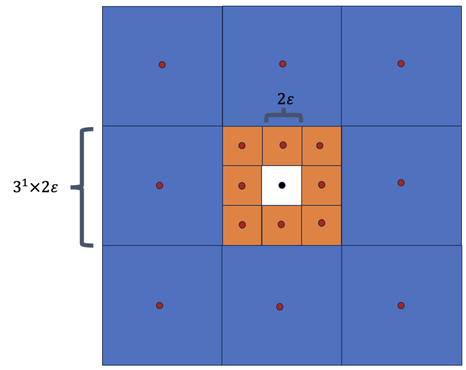

Fig. 1: Illustration of initial grid setup for $R = 3^m \varepsilon$, with $m = 2$ layers. The red dots are the grid points, while the central black dot is x

a) Initial Grid Setup: As mentioned before, given x , ε, and R, we will seek to verify the region of the state space $D_0 := B_R(x\ )\backslash B_\varepsilon(x\ )$. To this end, we cover D_0 with m layers, with each layer containing $3^d - 1$ points, where the radius of the point of the lth layer is given by $r_l := 3^{l\ \ 1}\varepsilon$, $l \in \{1, \ldots m\}$. The number of layers m is chosen so that the entire region D_0 is covered, i.e, so that

$$R \leq \varepsilon + \sum_{l=1}^{m} 2r_l = \varepsilon \left(1 + 2\sum_{l=1}^{m} 3^{l\ \ 1}\right) = 3^m \varepsilon.$$

Note that such an arrangement only requires $\mathcal{O}(3^d m)$ number of initial grid points, which is significantly smaller than a uniform ε-grid that requires $\mathcal{O}((\frac{R}{\varepsilon})^d)$. An illustration of a 2-layer initial grid setup is provided in Figure 1.

b) Estimation of a Consistent Lipschitz Constant: In order to successfully apply Proposition 1, one is required to find a set S^0 that contains all trajectories starting from $D_0 = B_R(x\ )\backslash B_\varepsilon(x\ )$. To that end, we will seek to find a ball $B_{R'}(x\ ) =: S^0$, for sufficiently large R^0. To compute R^0, we build a uniform grid G within the boundary set $\partial B_R(x\ )$ with maximum separation of $\ell > 0$. Starting from each grid point $x \in G$, we simulate a trajectory of length τ, and verify that all grid points return to $B_R(x\ )$ within time τ; else either R or τ must be changed. Then we compute $R_{\max} := \max_{x2G, t2[0,\tau]} \|\phi(t, x) - x\ \|$, and set $R^0 = R_{\max} + \delta$ for some small $\delta > 0$. Finally, we estimate $L := \max_{z \in B_{R'}(x^*)} L_z$ using a fine grid, and verify that $\max_{t \in (0,\tau]} \max_{x \in G} (R^0 - \|\phi(t,x)\|)\, e^{-tL} \geq \ell$.[1] Upon success, this procedure guarantees no trajectory starting within $B_R(x^*)$ leaves $B_{R'}(x^*)$. Otherwise, we halve the grid separation ℓ and repeat until we succeed.

c) Splitting Failed Points: Once a common Lipschitz constant L has been computed for all trajectories that lie within $B_{R'}(x^*)$, one is ready to apply Algorithm 1. However, it is possible that for a given grid point pair (x, r) Algorithm 1 fails. Upon failure, we will seek to refine the local section of the grid by splitting the ball $B_r(x)$ into 3^d balls. We refer the reader to Algorithm 2 for details and to Figure 2 for an illustration for $d = 2$.

Algorithm 2: Splitting a Ball

Input $x \in \mathbb{R}^n, r > 0$
$(x_1, ..., x_n) = x$
points = $(x_1 \pm (2/3)r, ..., x_n), ..., (x_1, ..., x_n \pm (2/3)r)$
radii = 3^n copies of $r/3$
return (points, radii)

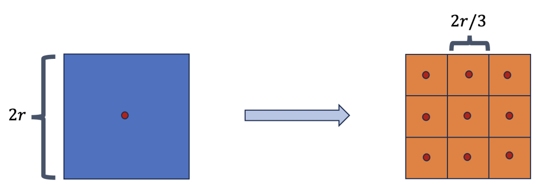

Fig. 2: Splitting a Ball according to Algorithm 2

d) Algorithm Summary: The combination of the above-mentioned steps provides us with an algorithm that allows us to verify a region $B_R(x^*) \backslash B_\varepsilon(x^*)$. The proposed algorithm, which can verify many points in parallel on a GPU, is summarized in Algorithm 3. In short, Algorithm 3 establish the initial grid and verify each ball, recursively splitting the failures a number of times until each ball is either verified or we reach the maximum number of splits we are willing to perform. On success, we take the minimum κ from across all balls and can assert that V is κ-exponentially τ-decreasing over $B_R(x^*) \backslash B_\varepsilon(x^*)$.

C. Numerical validation

We end this section by providing a preliminary validation of the proposed algorithm. To investigate the efficiency of our proposed method, we consider the following systems:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B_1 \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}; \qquad (24)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0.5 & -1 & 0 \\ 0.5 & 0.5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + B_2 \begin{bmatrix} x_1^2 \\ \dots \\ x_3^2 \end{bmatrix}, \qquad (25)$$

[1]This condition is sufficient to guarantee that, not only the simulated trajectories stay within $B_{R'}(x^*)$, but also the trajectories within an l-neighborhood.

Algorithm 3: Verification of a Region

Input $x^* \in \mathbb{R}^n, R > 0, \varepsilon \in (0, R), \tau, L > 0, V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}, \alpha_2 \geq \alpha_1 \geq 0, max\_splits > 0$
$\kappa = \infty$
$splits = 0$
points, radii = Grid of balls (x, r) covering $B_R(x^*) \backslash B_\varepsilon(x^*)$
while $splits < max\_splits$ **do**
  (verified, kappas) = VerifyBall(points,radii, τ,L,V,α_1, α_2)
  $\kappa = \min\{\kappa$, kappalist[verified == True]$\}$
  failed_points, failed_radii = points[verified == False], radius[verified == False]
  if *failed_points is empty* **then**
    return (True, κ)
  points, radii = Split(failed_points, failed_radii)
  $splits += 1$

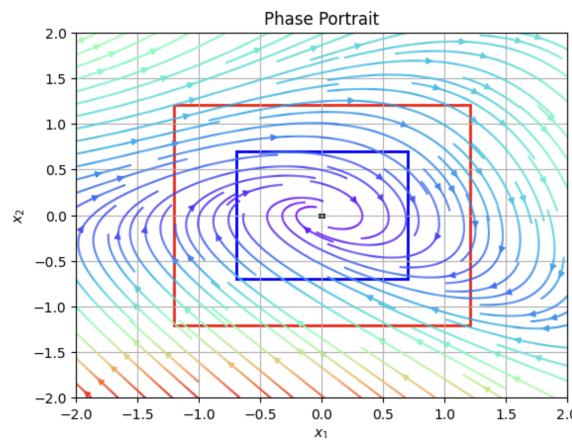

Fig. 3: Phase portrait of system (24), wherein the black box surrounds the region which we do not verify ($B_\varepsilon(x^*)$), the blue box represents surrounds the region which we verify in Algorithm 3 ($B_R(x^*)$), and the red box surrounds the region which trajectories that begin in the blue box do not leave ($B_{R'}(x^*)$).

where $B_1 \in \mathbb{R}^{2 \times 3}$ and $B_2 \in \mathbb{R}^{3 \times 9}$ are drawn independently from a Gaussian distribution, i.e., $[B_1]_{ij}, [B_2]_{ij} \sim \mathcal{N}(0, \sigma)$. We will increase the standard deviation σ as a means to increase the complexity of the dynamics. In our experiments, we choose the ℓ_1 norm as our choice of $V(x)$ and as the norm used to measure distances between trajectories. Thus, $\alpha_1 = \alpha_2 = 1$. Sample trajectories for the system (24) with $\sigma = 0.3$ are shown in Figure 3, where we also illustrate the ball of radius R (blue) selected, the computed ball of radius R^0 (red), and the small region around the origin (x^*) not certified (black). We also show in Figure 4 the verified region and a coloring scheme illustrating the different ball sizes used at different parts of $B_R(x^*) \backslash B_\varepsilon(x^*)$.

In all of our experiments we use $R = 0.7$, $\varepsilon = 0.01$, and $\tau = 1.5$. Table I and Table II summarize the results obtained by running Algorithm 3 together with a comparison with SOSTOOLS. When running our algorithm, we use

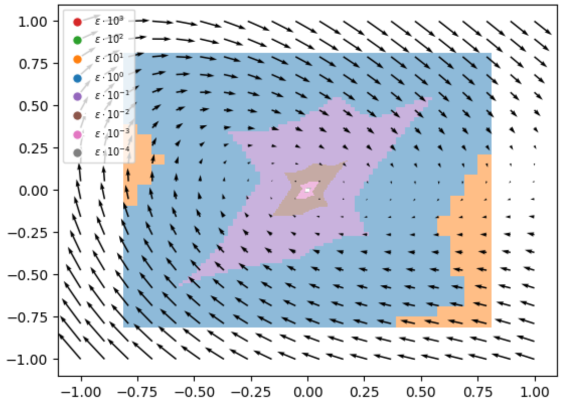

Fig. 4: Sizes of blocks resulting from applying Algorithm 3 to system (24).

the Torchode toolbox [22] to compute system trajectories in parallel. Note that our algorithm not only outputs the domain $D_0 = B_R(x\ )\backslash B_\varepsilon(x\ )$ wherein the function V is κ-exponentially τ-decreasing, but also the parameter κ, for which this is satisfied. When using SOSTOOLS [23], we use both R and κ as inputs and select the minimum polynomial order sufficient to guarantee κ-exponential stability in the region $B_R(x\ )$. Note that this speeds up the computation of SOSTOOLS as one does not need to search for them. It can be seen that as σ grows, i.e., the system becomes more nonlinear, our algorithm outperforms SOSTOOLS.

2D system (24)	$\sigma = 0,$	0.1,	0.3
R^0:	1.176	1.193	1.206
L:	2.00	2.53	3.24
κ:	1.6e-3	9.7e-7	1.11e-7
Alg. 3 Time:	2.29s	2.40s	2.40s
SOS Time:	0.35	1.96s	3.3s

TABLE I: Parameter values and performance comparison between our algorithm and the SOSTOOLS for system (24)

3D system (25)	$\sigma = 0,$	0.1,	0.3
R^0:	1.322	1.487	1.560
L:	2.81	3.23	4.07
κ:	1.63e-5	1.95e-7	4.22e-8
Alg. 3 Time:	14.15s	14.38s	14.79s
SOS Time:	0.73s	9.82s	30.63s

TABLE II: Parameter values and performance comparison between our algorithm and the SOSTOOLS for system (25)

VII. Conclusions

In this paper, we seek to relax the notion of set invariance, a fundamental tool in the analysis of dynamical systems. We thus propose and use the notion of set recurrence and show that under mild conditions, recurrence can be used to guarantee stability, asymptotic stability, and exponential stability of an equilibrium point. On the back of this theory, we have constructed an algorithm that lets us verify that a set, other than a ball arbitrarily close to the equilibrium, is κ-exponentially τ-decreasing. This algorithm is entirely deterministic and can be run in parallel on GPUs, resulting in time improvements over the state-of-the-art Sum of Squares method.

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