Model-free Learning of Regions of Attraction via Recurrent Sets

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***Abstract*— We consider the problem of learning an inner approximation of the region of attraction (ROA) of an asymptotically stable equilibrium point without an explicit model of the dynamics. Rather than leveraging approximate models with bounded uncertainty to find a (robust) invariant set contained in the ROA, we propose to learn sets that satisfy a more relaxed notion of containment known as recurrence. We define a set to be τ-recurrent (resp. k-recurrent) if every trajectory that starts within the set, returns to it after at most τ seconds (resp. k steps). We show that under mild assumptions a τ-recurrent set containing a stable equilibrium must be a subset of its ROA. We then leverage this property to develop algorithms that compute inner approximations of the ROA using counter-examples of recurrence that are obtained by sampling finite-length trajectories. Our algorithms process samples sequentially, which allow them to continue being executed even after an initial offline training stage. We further provide an upper bound on the number of counter-examples used by the algorithm, and almost sure convergence guarantees.**

I. Introduction

The problem of estimating the region of attraction (ROA) of an asymptotically stable equilibrium point has a long standing history in nonlinear control and dynamical systems theory [1]. From a theoretical standpoint, there has been a thorough study of conditions that guarantee several topological properties of such set, e.g., connected, open, dense, smooth [2]. From a practical standpoint, having a representation of such region allows to test the limits of controller designs, that are usually based on (possibly linear) approximations of nonlinear systems [3], and provides a mechanism for safety verification of certain operating conditions [4] [5]. Unfortunately, it is known that finding an analytic form of the region of attraction is difficult and in general impossible [1, p. 122]. As a result, most effort in characterizing the ROA focus on finding inner-approximations by means of invariant sets.

1) Related Work: Several methodologies for computing inner approximations of the ROA have been proposed in the literature. In a broad sense, they can be classified in two groups, depending on whether accurate information of the dynamic model is present or not. Notably, at their core, almost all of the methods rely on finding an invariant set of the system. We briefly review such methods next.

Exact Models: When an exact description of the dynamics is available, it is possible to use this information via two complementary methodologies. Lyapunov methods utilize the fact that Lyapunov functions are certificates of asymptotic stability, and build inner approximations using its level-sets. Methods for finding such Lyapunov functions are surveyed in, e.g., [6]. In particular, [7] and [8] construct Lyapunov functions as solution of Zubov's equation, and [9] searches for piece-wise linear Lyapunov functions that are found via linear programming. Similarly, piece-wise quadratic parameterizations of Lyapunov functions using LMI-based methods are considered in [10]. Finally, recent work [11] leverages the universal approximation property of neural networks to estimate the ROA of general nonlinear dynamical systems.

Alternatively, non-Lyapunov methods focus directly on properties of the ROA. For example, trajectory reversing methods [12] [13] derive the boundary of ROA directly from the stable manifold of the equilibria on the boundary, and the reachable set method [14] generates a grid of sample points and classifies each of them by solving a optimal control problem.

Inexact Models: In the presence of uncertainty, robust ROA approximation methods [15]–[18] generalize Lyapunov approaches by find a common Lyapunov function across the entire uncertainty set. Alternatively, learning-based methods utilize experimental data to estimate the region of attraction. When a Lyapunov function is provided, experimental data expand the Lyapunov function level set through, e.g., Gaussian processes [19], or a simple sampling approach [20]. To address the problem of simultaneouly learning the Lyapunov function and the level set, [21] parameterizes the Lyapunov function as a neural network, and iteratively train it by sampling points that are outside of current Lyapunov level set, but come back in within T steps.

Notably, learning methods play a crucial role in model-free settings. In particular, similar to the Lyapunov methods, [22] uses trajectory data to fit values of a Lyapunov function by leveraging converse Lyapunov results. Perhaps most relevant to our paper is [23], which establishes a non-Lyapunov approach that determine the boundary of ROA directly from a support vector machine, trained from experimental data that is sampled via a hybrid active learning techniques.

2) Contributions: In this paper, we provide a novel approach for learning inner approximations of the region of attraction of an asymptotically stable equilibrium with unknown system model. Rather than focusing on learning invariant sets that require trajectories to always lie within the set, we propose to learn sets that satisfy a more flexible notion of invariance. The contributions of this work are manifold:

- We propose the notion of recurrence as an alternative property that can be used to guarantee a set to be contained in the region of attraction.

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- We show that under mild conditions, a compact set containing an asymptotically stable equilibrium point is a subset of the region of attraction if and only if it is recurrent.
- We leverage this property to develop several algorithms that can learn inner-approximations of the region of attraction using counter-examples of recurrence that are based on finite-length trajectory samples.
- We further provide guarantees on the worst-case number of counter-examples required to compute a recurrent set.

3) Organization: The rest of the paper is organized as follows. In Section II, we formulate the problem we aim to solve, as well as revisit some classical results that will be leveraged in this work. The notion of recurrence to be used in this work is introduced in Section III, together with our first core set of results that show the relationship between recurrence and containment within the region of attraction. The proposed algorithms and the corresponding guarantees are given in Section IV. Numerical examples are provided in Section V and we conclude in Section VI.

II. Problem Formulation

We consider a continuous time dynamical system

$$\dot{x}(t) = f(x(t)), \tag{1}$$

where $x(t) \in \mathbb{R}^d$ is the state at time t, and the map $f : \mathbb{R}^d \to \mathbb{R}^d$ is continuously differentiable and (globally) Lipschitz. Given initial condition $x(0) = x_0$, we use $\phi(t, x_0)$ to denote the solution of (1). Using this notation, the positive orbit of x_0 is given by $\mathcal{O}_+(x_0) = \{y \in \mathbb{R}^d : y = \phi(t, x_0), t \in \mathbb{R}^+\}$.

Definition 1 (ω-limit Set)**.** *Given an initial condition x_0, its ω-limit set $\Omega(x_0)$ is the set of points $y \in \mathbb{R}^d$ for which there exists a sequence t_n indexed by $n \in \mathbb{N}$ satisfying $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \phi(t_n, x_0) = y$.*

We will further use $\Omega(f)$ to denote the ω-limit set of (1), which is the union of ω-limit sets of all $x \in \mathbb{R}^d$.

Note that by definition, if x^* is an equilibrium of (1), then it follows that $x^* \in \Omega(f)$.

A. Region of Attraction

We would like then to learn the set of initial conditions that converge to x^*.

Definition 2 (Region of Attraction)**.** *Given a set $S \subseteq \Omega(f)$, the region of attraction (ROA) of S under (1) is defined as*

$$\mathcal{A}(S) := \left\{ x_0 \in \mathbb{R}^d | \lim_{t\to\infty} \phi(t, x_0) \in S \right\}. \tag{2}$$

When the set S is a singleton that contains exactly one equilibrium point (say x), we abbreviate $\mathcal{A}(S) = \mathcal{A}(\{x\})$ as $\mathcal{A}(x)$.

By definition, $\mathcal{A}(S)$ satisfies the invariant property that every trajectory that starts in the set $\mathcal{A}(S)$ remains in the set for all future times, i.e., $\mathcal{A}(S)$ is a positively invariant set [1].

Definition 3 (Positively Invariant Set)**.** *A set $\mathcal{I} \subseteq \mathbb{R}^d$ is positively invariant w.r.t.* (1) *if and only if:*

$$x_0 \in \mathcal{I} \implies \phi(t, x_0) \in \mathcal{I}, \quad \forall\, t \in \mathbb{R}^+. \tag{3}$$

The notion of positively invariance is fundamental for control. It is used to trap trajectories in compact sets and allows the development of Lyapunov theory. By trapping trajectories on level sets of a function one can guarantee, boundedness of trajectories, stability, and even asymptotic stability via a gradual reduction of the value of the Lyapunov function. Unfortunately, without further assumptions, the set (2) may be a singleton, have measure zero, or be disconnected, making the problem of characterizing (2) from samples, almost impossible. We thus make the following assumption.

Assumption 1. *The system* (1) *has an asymptotically stable equilibrium at $x^* \in \mathbb{R}^d$.*

Remark 1. *It follows from Assumption 1 that the (positively invariant) ROA $\mathcal{A}(x^*)$ is an open contractible set [24], i.e., the identity map of $\mathcal{A}(x^*)$ to itself is null-homotopic [25].*

As a result, Assumptions 1 provides the necessarily regularity conditions for $\mathcal{A}(x^*)$ to be of practical use. A natural approach under this setting is therefore to search for Lyapunov functions [1] that render its level sets as invariant inner-approximations of $\mathcal{A}(x^*)$. Such methods are particularly justified after the fundamental result by Vladimir Zubov [26] that guarantees the existence of such function:

Theorem 1 (Zubov's Existence Criterion)**.** *A set $\mathcal{A}$ containing x^* in its interior is the region of attraction of x^* under* (1) *if and only if there exist continuous functions V, h such that the following hold:*

- *$V(x^*) = h(x^*) = 0$, $0 < V(x) < 1$ for $x \in \mathcal{A}\backslash\{x^*\}$, $h(x) > 0$ for $x \in \mathbb{R}^d\backslash\{x^*\}$.*
- *For every $\gamma_2 > 0$, there exists $\gamma_1 > 0$, $\alpha_1 > 0$ such that $V(x) > \gamma_1$, $h(x) > \alpha_1$, whenever $\|x\| \geq \gamma_2$.*
- *$V(x_k) \to 1$ for all sequences $\{x_k\}$ such that $x_k \to \partial\mathcal{A}$ or $\|x_k\| \to \infty$.*
- *V and h satisfy*

  $$(\mathcal{L}_f V)(x) = -h(x)(1 - V(x))\sqrt{1 + \|f(x)\|^2}, \tag{4}$$

  where $(\mathcal{L}_f V)(x)$ is the Lie derivative of V under the flow induced by f.

Particularly, given $f(x)$ continuously differentiable, $h(x)$ can always be selected such that V is differentiable, i.e., $(\mathcal{L}_f V)(x) = \nabla V(x)^T f(x)$.

Corollary 1. *Under Assumption 1, there exists a Lyapunov function V with domain on $\mathcal{A}(x^*)$ such that for any $c \in (0, 1)$ the level set $V_{<c} := \{x : V(x) < c\}$ is a contractible invariant subsets of $\mathcal{A}(x^*)$.*

Proof. Let V be the Zubov's function whose existence is guaranteed by Theorem 1. Thus by the definition of V, for $c \in (0, 1)$, $V_{<c} \subseteq \mathcal{A}(x^*)$. Further from (4), it follows that $(\mathcal{L}_f V)(x) < 0$, for $x \in V_{<c} \subset \mathcal{A}(x^*)$. Thus, $V_{<c}$ is positive invariant.

To prove the $V_{<c}$ is contractible, we need to provide a continuous mapping $H : [0,1] \times V_{<c} \to V_{<c}$ such that $H(0,x) = x$ and $H(1,x) = x^*$ for all $x \in V_{<c}$. Similar to [24], we define $H(s,x) := \phi(\frac{s}{1-s}, x)$ for $s < 1$, and $H(1,x) \equiv x^*$. Note that H is continuous in s and x for $s < 1$, as in [1]. We are thus left to prove continuity at each $(1,x)$. To do so, we take any such x and pick any open neighborhood $\mathcal{V}$ of $H(1,x) = x^*$. By Assumption 1 as well as the definition of asymptotic stability, it follows that there exist another open neighborhood $\mathcal{W} \subseteq \mathcal{A}(x^*)$ of x^* for which all trajectories starting in $\mathcal{W}$ remain in $\mathcal{V}$, i.e., $\phi(t,x_0) \in \mathcal{V}$ for all $x_0 \in \mathcal{W}$ and $t > 0$. Given $V_{<c} \subseteq \mathcal{A}(x^*)$, any point $x \in V_{<c}$ satisfies $\phi(T,x) \in \mathcal{W}$ for some $T > 0$. This, together with the continuity of $\phi(T,\cdot)$, implies that there is a neighborhood $\mathcal{V}' \subseteq V_{<c}$ of x such that $\phi(T,y) \in \mathcal{W}$ for all $y \in \mathcal{V}'$, which let us conclude:

$$H(s,y) \in \mathcal{V} \quad \text{whenever } y \in \mathcal{V}' \text{ and } s > 1 - \frac{1}{T+1}$$

and continuity follows since $\mathcal{V}$ could be made arbitrarily small. □

The Zubov's function of Theorem 1 provides a parametric family $\{V_{<c} : c \in (0,1)\}$ of positive invariant sets inside $\mathcal{A}(x^*)$. Further, while Zubov's result provides a constructive method for $V(x)$, by means of solving a partial differential equation, such method becomes impractical in the absence of a descriptive model for (1). Thus, in the absence of an exact model of the dynamics, it is natural then to try to find a set inside $\mathcal{A}(x^*)$ that is positive invariant in a robust sense, in the presence of bounded uncertainty [18], or that is positive invariant with high probability [19].

However, one of the caveats of positive invariant sets is that they need to be specified very carefully, in the sense that even a good approximation of an positively invariant set is not necessarily positively invariant. Particularly, subsets of positively invariant sets need not be positively invariant. This indirectly imposes strict constraints on the complexity of the set that one needs to learn via (3). This motivates the alternative proposed in the next section.

III. Recurrent Sets

We now introduce the relaxed notion of invariance to be used in this paper, which refer here as recurrence. We will then illustrate how recurrent sets constitute a more flexible and general class of objects to study.

Definition 4 (Recurrent Set)**.** *A set $\mathcal{R} \subseteq \mathbb{R}^d$ is recurrent w.r.t.* (1)*, if whenever $x_0 \in \mathcal{R}$, then*

$$\exists\, t' > 0 \quad s.t. \quad \phi(t', x_0) \in \mathcal{R}.$$

The following Lemma shows how recurrent sets, while not invariant, guarantee that solutions that start in this set, will visit it back infinitely often.

Lemma 1. *Consider a compact recurrent set $\mathcal{R}$. Then for any point $x_0 \in \mathcal{R}$ and time $\tau > 0$, there exist a $\tau' > \tau$, such that $\phi(\tau', x_0) \in \mathcal{R}$.*

Proof. Given a recurrent set $\mathcal{R}$, any point $x_0 \in \mathcal{R}$ and any $\tau > 0$, we denote $t_1 \in [0,\tau]$ as the last time before τ that the solution $x_1 := \phi(t_1, x_0)$ lies within $\mathcal{R}$, i.e., $\phi(t_1,x_0) \in \mathcal{R}$ and $\phi(t,x_0) \notin \mathcal{R}$ for all $t \in (t_1,\tau]$. Note that such a t_1 always exist since $\phi(0,x_0) = x_0 \in \mathcal{R}$ and $\mathcal{R}$ is compact. We then prove the result by contradiction. Precisely, assume that Lemma 1 is not true, that is, $\phi(t,x_0) \notin \mathcal{R}$ for all $t > \tau$. Then it follws that $x_1 \in \mathcal{R}$ and $\forall t > 0\ \phi(t,x_1) \notin \mathcal{R}$, which contradicts our assumption that $\mathcal{R}$ is recurrent (Definition 4). □

By Definition 3, every positive invariant set $\mathcal{I}$ is recurrent. Thus, Definition 4 generalizes the notion of positive invariance by allowing the solution $\phi(t,x_0)$ to step outside the set $\mathcal{R}$ for some finite time. One concern may be however that by allowing $\phi(t,x_0)$ to leave the set $\mathcal{R}$, this will lead to trajectories that diverge, thus leading to unstable behavior. The following theorem shows that under mild assumptions this should not be a source of concern.

Theorem 2. *Let $\mathcal{R} \subset \mathbb{R}^d$ be a compact set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$. Then $\mathcal{R}$ is recurrent if and only if $\Omega(f) \cap \mathcal{R} \neq \emptyset$ and $\mathcal{R} \subset \mathcal{A}(\Omega(f) \cap \mathcal{R})$.*

Proof. ($\implies$): If $\mathcal{R}$ is recurrent, then for any $x_0 \in \mathcal{R}$, we can construct an infinite sequence $\{x_n\}_{n=0}^\infty$ that lies within $\mathcal{R}$, i.e., $\{x_n\}_{n=0}^\infty \subset \mathcal{R}$. Precisely, we start from $t_0 = 0$ that gives solution $x_0 := \phi(0,x_0) \in \mathcal{R}$. Then, given $x_n := \phi(t_n,x_0) \in \mathcal{R}$ and some fixed time interval $\tau > 0$, we defined t_{n+1} as the first time since $t_n + \tau$ that the solution $x_{n+1} := \phi(t_{n+1},x_0)$ lies within $\mathcal{R}$, i.e., $\phi(t_{n+1},x_0) \in \mathcal{R}$ and $\phi(t,x_0) \notin \mathcal{R}$ for all $t \in [t_n+\tau, t_{n+1})$. Note that Lemma 1 ensures there exist such a t_{n+1} and x_{n+1}.

Now since $\mathcal{R}$ is compact, by Bolzano-Weierstrass theorem, $\{x_n\}_{n=0}^\infty$ must have a sub-sequence $\{x_{n_i}\}_{i=1}^\infty$ that converges to an accumulation point $\bar{x} \in \mathcal{R}$. It follow then by the definition of ω-limit sets (Definition 1) that $\bar{x} = \lim_{i\to\infty} x_{n_i} \in \Omega(f) \cap \mathcal{R} \neq \emptyset$. Thus, we have that $x_0 \in \Omega(f) \cap \mathcal{R}$. Finally, since x_0 was chosen arbitrarily within $\mathcal{R}$, it follows that $\mathcal{R} \subset \mathcal{A}(\Omega(f) \cap \mathcal{R})$.

($\Longleftarrow$): By assumption $\Omega(f) \cap \mathcal{R} \subset \operatorname{int}\mathcal{R}$ and $\mathcal{R} \subset \mathcal{A}(\Omega(f) \cap \mathcal{R})$. Therefore, if $x_0 \in \mathcal{R}$, then $x_0 \in \mathcal{A}(\Omega(f) \cap \mathcal{R})$ and it follows that $\phi(t,x_0)$ converges to $\Omega(f) \cap \mathcal{R} \subset \mathcal{R}$. Therefore for all $x_0 \in \mathcal{R}$, since $\Omega(f) \cap \mathcal{R} \subset \operatorname{int}\mathcal{R}$, it follows from the continuity of ϕ that there always exists some time $t > 0$ such that $\phi(t,x_0) \in \mathcal{R}$. Thus $\mathcal{R}$ is recurrent. □

Theorem 2 illustrates that recurrence necessarily implies containment on the region of attraction of $\Omega(f) \cap \mathcal{R}$. As a result, by imposing mild conditions on $\Omega(f)$, one leads to the following quite useful result.

Assumption 2. *The ω-limit set $\Omega(f)$ of* (1) *is composed by hyperbolic equilibrium points, with only one of them, say x^*, being asymptotically stable.*

Corollary 2. *Let assumptions 1 and 2 hold. Further, let $\mathcal{R}$ be a compact set satisfying $\partial\mathcal{R} \cap \Omega(f) = \emptyset$. Then the set $\mathcal{R}$ is recurrent if and only if $\Omega(f) \cap \mathcal{R} = \{x^*\}$ and $\mathcal{R} \subset \mathcal{A}(x^*)$.*

Proof. ($\Longrightarrow$): By assumption $\mathcal{R}$ is compact, $\partial\mathcal{R}\cap\Omega(f)=\emptyset$, Theorem 2 implies that if $\mathcal{R}$ is recurrent then $\Omega(f)\cap\mathcal{R}\neq\emptyset$ and $\mathcal{R}\subset\mathcal{A}(\Omega(f)\cap\mathcal{R})$. Since all the equilibrium points inside $\Omega(f)$ are hyperbolic, the regions of attraction of the unstable ones are not full dimensional [13], and therefore $\mathcal{A}(\Omega(f)\cap\mathcal{R})\backslash\mathcal{A}(x^*)$ is not full dimensional. It follows that the set $\mathcal{R}\backslash\mathcal{A}(x^*)\subset\mathcal{A}(\Omega(f)\cap\mathcal{R})\backslash\mathcal{A}(x^*)$ is also not full dimensional. Together with the fact that ROA $\mathcal{A}(x^*)$ is an open contractible set, one can conclude that $\mathcal{R}\backslash\mathcal{A}(x^*)\subseteq\partial\mathcal{R}$. Otherwise, there exists a point $x\in\mathcal{R}\backslash\mathcal{A}(x^*)$ satisfying $x\in\operatorname{int}(\mathcal{R})$, which contradict with $\mathcal{R}\backslash\mathcal{A}(x^*)$ not full dimensional.

Now since $(\Omega(f)\cap\mathcal{R})\backslash x^*\subseteq\mathcal{R}\backslash\mathcal{A}(x^*)\subseteq\partial\mathcal{R}$ contradict with $\partial\mathcal{R}\cap\Omega(f)=\emptyset$ if $(\Omega(f)\cap\mathcal{R})\backslash x^*$ is non-empty, we can further conclude that $\Omega(f)\cap\mathcal{R}=\{x^*\}$ given $\Omega(f)\cap\mathcal{R}\neq\emptyset$. And the other conclusion follows from $\mathcal{R}\subset\mathcal{A}(\Omega(f)\cap\mathcal{R})=\mathcal{A}(x^*)$.

($\Longleftarrow$): This direction is trivial given Theorem 2. ☐

From a practical standpoint, Corollary 2 implies that whenever $\mathcal{R}$ is recurrent virtually all points within $\mathcal{R}$ lie within the region of attraction of the stable equilibrium. This suggests that we may use recurrence as a mechanism for practically finding inner approximations for $\mathcal{A}(x^*)$. However, one limitation of the above results is that although $\mathcal{R}$ is recurrent, we do not know a priori how long it may take for a trajectory to come back to $\mathcal{R}$ after it leaves it. This motivates the following stricter notion of recurrence.

Definition 5 (τ-Recurrent Set)**.** *A set $\mathcal{R}\subseteq\mathbb{R}^d$ is τ-recurrent w.r.t.* (1)*, if whenever $x_0\in\mathcal{R}$, then*

$$\exists\, t'\in(0,\tau]\quad s.t.\quad \phi(t',x_0)\in\mathcal{R}.$$

Theorem 3. *Let Assumption 1 hold, and consider a compact set $\mathcal{R}\subseteq\mathcal{A}(x^*)$ satisfying $x^*\in\operatorname{int}\mathcal{R}$ and $\mathcal{R}\cap\partial\mathcal{A}(x^*)=\emptyset$. Then there exists positive constants $\underline{c}$, $\overline{c}$, and a, depending on $\mathcal{R}$, such that for all*

$$\tau\geq\bar{\tau}:=\frac{\overline{c}-\underline{c}}{a},$$

the set $\mathcal{R}$ is τ-recurrent. Further, starting from any point $x\in\mathcal{R}$, the solution $\phi(t,x)\in\mathcal{R}$ for all $t\geq\bar{\tau}$.

Proof. The proof of the theorem relies on Zubov's existence criterion stated in Theorem 1. Given $\mathcal{R}$, let us now define

$$\underline{c}:=\min_{x\in\partial\mathcal{R}}V(x),\quad \overline{c}:=\max_{x\in\partial\mathcal{R}}V(x),$$
$$\text{and}\quad a:=\max_{x\in C}\nabla V(x)^Tf(x),$$

where $C=\{x\in\mathbb{R}^d:\underline{c}\leq V(x)\leq\overline{c}\}$ is compact.

We first argue that $V_{\leq\underline{c}}:=\{x:V(x)\leq\underline{c}\}\subseteq\mathcal{R}$. Let $\underline{x}$ be the point in $\partial\mathcal{R}$ that achieves the minimum, i.e, $V(\underline{x})=\underline{c}$, and let $\mathcal{R}'$ be the connected component of $\mathcal{R}$ containing $\underline{x}$. Note that $x^*\in\operatorname{int}\mathcal{R}$ must be contained in $\mathcal{R}'$, since otherwise, the trajectory $\phi(t,\underline{x})$, which strictly decreases V must eventually find a point $x'\in\partial R$ with $V(x')<\underline{c}$; which contradicts the definition of $\underline{c}$. Thus, $x^*\in\mathcal{R}'\subseteq\mathcal{R}$.

Suppose then that $V_{\leq\underline{c}}\not\subseteq\mathcal{R}'\subseteq\mathcal{R}$, for any point $\tilde{x}\in V_{\leq\underline{c}}\backslash\mathcal{R}'$, $V(\phi(t,\tilde{x}))<\underline{c}$, for $t>0$, and $\lim_{t\to\infty}\phi(t,\tilde{x})=x^*$. Thus there exists $\tilde{t}>0$ s.t. $V(\phi(t,\tilde{x}))<\underline{c}$ and $\tilde{x}\in\partial\mathcal{R}$; contradiction. It follow then that $V_{\leq\underline{c}}\subseteq\mathcal{R}'\subseteq\mathcal{R}$.

Similarly, since $V_{\leq\overline{c}}$ contains every point in the boundary of $\mathcal{R}$, there cannot be any point in $x\in\mathcal{R}$ with $V(x)>\overline{c}$. We therefore get that the following inclusions must hold:

$$V_{\leq\underline{c}}\subseteq\mathcal{R}\subseteq V_{\leq\overline{c}}. \tag{5}$$

Finally, by (5), for any point $x\in\mathcal{R}$ we must have $V(x)\leq\overline{c}$. Since the time derivative of $V(x)$ is at most $a<0$, it follows that after $t\geq\bar{\tau}:=\frac{\underline{c}-\overline{c}}{a}$ the Lyapunov value $V(\phi(t,x))\leq\underline{c}$, which implies that $\phi(t,x)\in\mathcal{R}$ and result follows. ☐

Note that the lower bound on τ in Theorem 3 implicitly depends on the set $\mathcal{R}$. This makes the process of learning a recurrent set difficult as τ would change and the set is updated. To eliminate this dependence, one is required to introduce conservativeness. To that end we consider the set

$$\mathcal{A}_\delta:=\mathcal{A}(x^*)\backslash\left(\{\partial\mathcal{A}(x^*)+\operatorname{int}\mathcal{B}_\delta\}\cup\{\operatorname{int}\mathcal{B}_\delta+x^*\}\right), \tag{6}$$

where $\partial\mathcal{A}(x^*)$ is the boundary of $\mathcal{A}(x^*)$, '+' stands for the Minkowski sum, $\mathcal{B}_\delta$ is a closed δ ball centered at the origin, i.e., $\mathcal{B}_\delta=\{x|\,\|x\|_2\leq\delta\}$ and $\delta>0$ is chosen to be small enough such that $\mathcal{B}_\delta+x^*\subseteq\mathcal{A}(x^*)\backslash\{\partial\mathcal{A}(x^*)+\mathcal{B}_\delta\}$.

Then, by denoting $\overline{c}(\delta)$ and $\underline{c}(\delta)$ as the max and min Lyapunov function values in $\mathcal{A}_\delta$ respectively, i.e.,

$$\overline{c}(\delta):=\max_{x\in\mathcal{A}_\delta}V(x),\quad \underline{c}(\delta):=\min_{x\in\mathcal{A}_\delta}V(x),$$
$$\text{and}\quad a(\delta):=\max_{x\in C_\delta}\nabla V(x)^Tf(x),$$

where $C_\delta=\{x\in\mathbb{R}^d:\underline{c}(\delta)\leq V(x)\leq\overline{c}(\delta)\}$, we obtain a lower bound on τ that is mostly independent of $\mathcal{R}$.

Theorem 4. *Under Assumption 1, any compact set $\mathcal{R}$ satisfying:*

$$\mathcal{B}_\delta+x^*\subseteq\mathcal{R}\subseteq\mathcal{A}(x^*)\backslash\{\partial\mathcal{A}(x^*)+\operatorname{int}\mathcal{B}_\delta\}$$

is τ-recurrent for any $\tau\geq\bar{\tau}(\delta):=(\underline{c}(\delta)-\overline{c}(\delta))/a(\delta)$. Moreover, when $t\geq\bar{\tau}(\delta)$, $\phi(t,x)\in\mathcal{R}$ for any point $x\in\mathcal{R}$.

The proof of Theorem 4 is analogous to Theorem 3 and omitted due to space constraints.

IV. Learning recurrent sets

Having laid down the basic theory underlying recurrent sets, we now propose a method to compute inner-approximations of the region of attraction $\mathcal{A}(x^*)$ based on checking the recurrence property on finite-length trajectory samples. For concreteness, we consider the following type of sample trajectories for system (1):

$$x_n=\phi(n\tau_s,x_0),\qquad x_0\in\mathbb{R}^d,\quad n\in\mathbb{N}, \tag{7}$$

where $\tau_s>0$ is the sampling period.

In this setting, we define the notion of discrete recurrence w.r.t. a length k trajectory:

Definition 6 (k-Recurrent Set)**.** *A set $\mathcal{R} \subseteq \mathbb{R}^d$ is k-steps recurrent (k-recurrent for short) w.r.t.* (7)*, if whenever $x_0 \in \mathcal{R}$, then*

$$\exists\, n \in \{1, ..., k\} \quad s.t. \quad x_n \in \mathcal{R}.$$

Remark 2. *Note that a set $\mathcal{R}$ being k-recurrent implies that $\mathcal{R}$ is τ-recurrent with $\tau = k\tau_s$. One can then conclude that $\mathcal{R} \subset \mathcal{A}(x^*)$ under the assumptions of Corollary 2. However, the converse is not necessarily true.*

To ensure one can find such a k-recurrent set, we consider again the specific set $\mathcal{A}_\delta$ defined in (6) that gives the following sufficient conditions for a set $\mathcal{R}$ to be k-recurrent.

Theorem 5. *Under Assumption 1, any compact set $\mathcal{R}$ satisfying $\mathcal{B}_\delta + x^* \subseteq \mathcal{R} \subseteq \mathcal{A}(x^*)\backslash\{\partial\mathcal{A}(x^*) + \operatorname{int}\mathcal{B}_\delta\}$ or, equivalently $\partial\mathcal{R} \subset \operatorname{int}\mathcal{A}_\delta$, is k-recurrent for $k > \bar{k} := \bar{\tau}(\delta)/\tau_s$, where $\bar{\tau}(\delta)$ is defined in Theorem 4.*

Proof. Given Theorem 4, this result follows directly from $\phi(t, x) \in \mathcal{R}$ for all $x \in \mathcal{R}$ when $t \geq \bar{\tau}(\delta)$. ☐

In the rest of the paper, we assume for simplicity that the ω-limit set $\Omega(f)$ of (1) is composed by hyperbolic equilibrium points with a unique asymptotically stable one (Assumption 2). We further assume w.l.o.g. that the asymptotically stable equilibrium is at the origin, i.e., $x^* = 0$.

We will restrict our search to a compact initial approximation $\hat{\mathcal{S}}^{(0)} \subset \mathbb{R}^d$ of the ROA satisfying $\hat{\mathcal{S}}^{(0)} \supseteq \mathcal{B}_\delta$. Precisely, we will seek to find an subset of the ROA within $\mathcal{A}(x^*) \cap \hat{\mathcal{S}}^{(0)}$ by computing k-recurrent sets $\mathcal{R}$ that seek to satisfy the properties of Theorem 5. In this approach, starting from $\hat{\mathcal{S}}^{(0)}$, we sequentially generate a sequence of approximations $\hat{\mathcal{S}}^{(i)}$. For each $\hat{\mathcal{S}}^{(i)}$, we sample points $p_{ij} \in \hat{\mathcal{S}}^{(i)}$ and check whether a trajectory that starts at p_{ij} is k-recurrent for each $j = 0, 1, 2, ...$ Once a counter-example is found, we update approximation $\hat{\mathcal{S}}^{(i)}$ to $\hat{\mathcal{S}}^{(i+1)}$ and restart the sampling process. This method is summarized in Algorithm 1.

The rest of this section provides a detailed explanation of each step of the algorithm, as well as a rigorous justification of the proposed methodology.

Algorithm 1: Learning a k-recurrent set

```
Initialize Ŝ_0 according to (9) or (12)
for Iteration i = 0, 1, ... do
    for Iteration j = 0, 1, ... do
        Generate random sample p_ij ∈ Ŝ^(i) uniformly
        if p_ij is a counter-example w.r.t Ŝ^(i) then
            Update Ŝ^(i) according to (10) or (13)
            break
        end
    end
end
```

A. Classification of sample points

We say that a sample point p_{ij} is valid k-recurrent point w.r.t current approximation $\hat{\mathcal{S}}^{(i)}$ if starting from $x_0 = p_{ij}$,

$$\exists\, n \in \{1, ..., k\},\ s.t.\ x_n \in \hat{\mathcal{S}}^{(i)}. \tag{8}$$

If (8) does not hold, we say p_{ij} is a *counter-example*. We will use such counter-examples to update our current set approximation $\hat{\mathcal{S}}^{(i)}$.

B. Construction of set approximations

In order to gradually update the sets $\hat{\mathcal{S}}^{(i)}$, we consider two parametric families of set approximations.

1) Sphere approximation: To construct a sphere approximation, we start by choosing a radius $c > 0$ large enough such that the set

$$\hat{\mathcal{S}}^{(0)} := \{x|\, \|x\|_2 \leq b^{(0)} := c\} \ \supseteq\ \mathcal{B}_\delta. \tag{9}$$

The sphere approximation for iteration i is then defined as $\hat{\mathcal{S}}^{(i)} := \{x|\, \|x\|_2 \leq b^{(i)}\}$. Finally, given a sample point $p_{ij} \in \hat{\mathcal{S}}^{(i)}$ we update $\hat{\mathcal{S}}^{(i)}$ based on the following criterion:

$$\textbf{If } p_{ij} \textbf{ is a counter-example} \quad b^{(i+1)} = \|p_{ij}\|_2 - \varepsilon, \tag{10}$$

where $\varepsilon > 0$ is an algorithm parameter expressing the level of conservativeness in our update.

If the process reaches a value of $b^{(i)} < 0$, we declare the search a failure. At such point, one may choose to either reduce the value of ε or increase the length of the trajectories sampled.

2) Polyhedron approximation: One could also choose to construct $\hat{\mathcal{S}}^{(i)}$ using polyhedron approximations. To that end, we first construct a matrix $A :== [a_1, ..., a_n]^T \in \mathbb{R}^{n\times d}$, where each row vector a_n is a *normalized* ($\|a_l\| = 1\ \forall l$) exploration direction indexed by $l \in \{1, ..., n\}$. Precisely, we generate A randomly with row dimension n large enough such that for any arbitrary direction $v \in \mathbb{R}^d$, there exist an exploration direction a_l with an angle $\theta_{v,l}$ between v and a_l satisfying

$$\theta_{v,l} := \arccos\left(\frac{v^T a_l}{\|v\|\|a_l\|}\right) \leq \frac{2}{3}\pi. \tag{11}$$

Note that the aforementioned process is analog to constructing an ε-net over a unit Euclidean hyper-sphere, for which several algorithms exist [27], [28]. Upper bounds on the size of n can also be found in the literature, see, e.g., [29].

The polyhedron approximation at iteration i is defined as $\hat{\mathcal{S}}^{(i)} := \{x|Ax \leq b^{(i)}\}$, consisting on n inequalities aimed at approximating a k-recurrent set via counter examples. In this paper, keep A fixed for every iteration, and update the constraint coefficients $b^{(i)} = [b_1^{(i)}, ..., b_n^{(i)}]^T \in \mathbb{R}^n$.

Similarly to the sphere approximation, we initialize $b^{(0)} := c1_n$, such that

$$\hat{\mathcal{S}}^{(0)} = \{x|Ax \leq b^{(0)} := c\mathbb{1}_n\} \supseteq \mathcal{B}_\delta, \tag{12}$$

which can be done for $c > 0$ large enough; here $\mathbb{1}_n \in \mathbb{R}^n$ is the vector of all ones.

Afterwards $b^{(i)}$ is updated by sampling points $p_{ij} \in \hat{\mathcal{S}}^{(i)}$ and checking the following criterion:

$$\begin{aligned} &\textbf{if } p_{ij} \textbf{ is a counter-example} \\ &\begin{cases} b_{l^*}^{(i+1)} = a_{l^*} p_{ij} - \varepsilon \\ b_l^{(i+1)} = b_l^{(i)}, \quad \forall\, l = \{1, .., n\}\backslash l^*, \end{cases} \end{aligned} \tag{13}$$

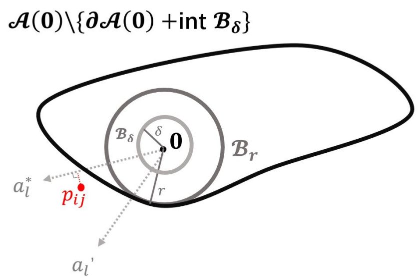

Fig. 1: An illustration of the proof of Theorem 16. In particular, given arbitrary point $p_{ij} \notin \mathcal{A}(0)\backslash\{\partial\mathcal{A}(0) + \mathrm{int}\, \mathcal{B}_\delta\}$, in the sphere case, it follows that $\|p_{ij}\|_2 \geq r$. And in the polyhedron case, the closest projection $\max_{l\in\{1,\dots,n\}} a_l^T p_{ij} \geq r/2$ under Assumption (11) that every pair of exploration directions are close enough.

where $\varepsilon > 0$ is fixed and $l^* = \arg\max_{l\in\{1,\dots,n\}} \frac{a_l^T p_{ij}}{\|a_l\|\|p_{ij}\|}$ is the index of exploration direction that minimizes the angle between p_{ij} and a_l. If l^* consists of more then one index, we simply choose one at random. As before, we declare the search a failure whenever $b_l^{(i)} < 0$ for some l, since this implies that the equilibrium $x^* = 0$ is outside the set $\hat{\mathcal{S}}^{(i)}$.

C. Bound on the number of updates

As mentioned before, the aforementioned search for approximations will fail, whenever one of the parameters $b^{(i)}$ (sphere) or $b_l^{(i)}$ (polyhedron) become negative. We will show next that, provided that k and ε are chosen appropriately, will never meet our failure condition. In other words, will fail to find counter-examples after a finite number of set updates.

Given $k \geq \bar{k}$, and an arbitrary approximation $\hat{\mathcal{S}}^{(i)}$ satisfying $\mathcal{B}_\delta \subseteq \hat{\mathcal{S}}^{(i)} \subseteq \mathcal{A}(0)\backslash\{\partial\mathcal{A}(0) + \mathrm{int}\, \mathcal{B}_\delta\}$, then Theorem 5 guarantees that any sample $p_{ij} \in \hat{\mathcal{S}}^{(i)}$ will lead to a k-recurrent trajecotry, i.e., condition (8). As a result, the algorithm will stop updating at this point since we cannot find further counter-examples within $\hat{\mathcal{S}}^{(i)}$.

This means that, if it is possible for $\hat{\mathcal{S}}^{(i)}$ to become a subset of $\mathcal{A}(0)\backslash\{\partial\mathcal{A}(0) + \mathrm{int}\, \mathcal{B}_\delta\}$, without violating the condition $\mathcal{B}_\delta \subseteq \hat{\mathcal{S}}^{(i)}$, then the algorithm will stop updating and never fail. The following theorem shows that this is indeed possible, whenever ε and k are properly chosen.

Theorem 6. *Let the initial approximation $\hat{\mathcal{S}}^{(0)}$ satisfy $\mathcal{B}_\delta \subseteq \hat{\mathcal{S}}^{(0)}$ and trajectory length $k > \bar{k}$, for $\bar{k}$ as defined in Theorem 5. Then, given a counter-example p_{ij}, the resulting updated set satisfies $\mathcal{B}_\delta \subseteq \hat{\mathcal{S}}^{(i+1)}$ whenever*

$$\varepsilon \leq \begin{cases} r - \delta & \textbf{\textit{with sphere approximation}} \\ \frac{r}{2} - \delta & \textbf{\textit{with polyhedron approximation}}, \end{cases} \tag{14}$$

where r is the smallest distance between the origin (equilibrium) and the boundary $\partial\left(\mathcal{A}(0)\backslash\{\partial\mathcal{A}(0) + \mathrm{int}\, \mathcal{B}_\delta\}\right)$.

Proof. Given an arbitrary counter-example p_{ij} w.r.t $\hat{\mathcal{S}}^{(i)} \supseteq \mathcal{B}_\delta$, it follows that $p_{ij} \notin \mathcal{A}(0)\backslash\{\partial\mathcal{A}(0) + \mathrm{int}\, \mathcal{B}_\delta\}$ by Theorem 5; since otherwise, p_{ij} would generate a k-recurrent trajectory. Then, as illustrated in Figure 1, $\|p_{ij}\|_2 \geq r$. Further, let $\mathcal{B}_r := \{x | \, \|x\|_2 \leq r\} \subseteq \mathcal{A}(0)\backslash\{\partial\mathcal{A}(0) + \mathrm{int}\, \mathcal{B}_\delta\}$.

We now reason differently depending on the type of approximation.

(Sphere case): It then follows from $\|p_{ij}\|_2 \geq r$ that whenever $\varepsilon \leq r - \delta$, the update leads to $b^{(i+1)} = \|p_{ij}\|_2 - \varepsilon \geq r - \varepsilon \geq \delta$.

(Polyhedron case): It follows from (11), that for any point $p' \notin \mathcal{B}_r$, we have $\max_{l\in\{1,\dots,n\}} a_l^T p' \geq \|p'\| \cos\left(\frac{2}{3}\pi\right) \geq \frac{r}{2}$. Therefore, since by definition of $\mathcal{B}_r$, $p_{ij} \notin \mathcal{B}_r$ we conclude then that $b_{l^*}^{(i+1)} = a_{l^*}^T p_{ij} - \varepsilon \geq \frac{r}{2} - \varepsilon \geq \delta$.

Together with the fact that $\hat{\mathcal{S}}^{(0)} \supseteq \mathcal{B}_\delta$, result follows. □

Theorem 6 establishes that one can choose parameters k and ε so that the sequence of sets $\hat{S}^{(i)}$ never leads to $b^{(i)}$ or $b_l^{(i)}$ negative, i.e., the algorithm never fails. However, this requires prior knowledge of $\bar{k}$, r, and δ. We argue that local information on the dynamics can be sufficient to find conservative bounds for r and δ, and thus ε. However, $\bar{k}$ depends in a highly non-trivial way w.r.t. δ. We solve this issue by, doubling the side of k, i.e. $k^+ = 2k$, every time the failure conditions is met, and re-initializing the sets to back to $\hat{\mathcal{S}}^0$.

In what follows, we use $\mathcal{F}_c$ to denote the parametric family of closed balls (resp. polytopes) defined by $\{x : ||x||_2 \leq b\}$ (resp.$\{x : Ax \leq b\}$), for $b \in [0, c]$ (resp. $b \in [0, c]^n$). This leads to the following total bound on the number of iterations.

Theorem 7. *Given the initial approximation $\hat{\mathcal{S}}^{(0)} \in \mathcal{F}_c$ and initial constant c defined in* (9) *or* (12)*, the total number of counter-examples encountered in Algorithm 1, with k-doubling after each failure, is bounded by $\frac{c}{\varepsilon}\log_2 \bar{k}$ in the sphere case and $n\frac{c}{\varepsilon}\log_2 \bar{k}$ in the polyhedron case.*

Proof. Note that once a counter-example is encountered, we decrease the radius constraint (sphere case) or on one of the exploration directions (polyhedron case) by at least ε. Therefore, $\hat{\mathcal{S}}^{(i)} \in \mathcal{F}_c$ for all $i \in \{1, 2, \dots\}$. And for any fixed k, our method can find at most $\frac{c}{\varepsilon}$ counter-examples with the sphere approximation and $n\frac{c}{\varepsilon}$ counter-examples with the polyhedron approximation without failing. Since it takes at most $\log_2 \bar{k}$ updates on k to find some $k \geq \bar{k}$ using the doubling method, result follows. □

Our results provide an upper bound on the number of updates the set approximation may experience by ensuring that $\hat{\mathcal{S}}^{(i)}$ always contains an δ-ball around the equilibrium point. However, this is not sufficient to guarantee that $\hat{\mathcal{S}}^{(i)}$ is k-recurrent, which is required to guarantee that $\hat{\mathcal{S}}^{(i)} \subseteq \mathcal{A}(0)\backslash\{\partial\mathcal{A}(0) + \mathrm{int}\, \mathcal{B}_\delta\}$. This is issue is addressed next.

D. Convergence guarantee

By Definition 6, a set $\mathcal{S}$ is k-recurrent if every point $p \in \mathcal{S}$ satisfies (8). As shown before, certifying this property will enable us to guarantee that $\mathcal{S} \subset \mathcal{A}(0)$. However, it is infeasible to enforce condition (8) for every point in $\mathcal{S}$. Instead, we will show that under mild conditions, our algorithm converges to a $\hat{\mathcal{S}}^*$ satisfying $\mathrm{int}\, \hat{\mathcal{S}}^* \subseteq \mathcal{A}(0)$ with probability one.

In our algorithm, we generate samples p_j uniformly within some set $\mathcal{S}$, i.e., $p_j \overset{iid}{\sim} U(\mathcal{S})$ for all $j \in \{0, 1, 2, ...\}$. We use $\mathcal{S}_{\text{counter}}$ to denote the set that contains all the counter-examples that certify $\mathcal{S}$ being not k-recurrent. Given a random sample p_j we define the Bernoulli random variable X_j with $X_j = 1$ if $p_j \in \mathcal{S}_{\text{counter}}$ and $X_j = 0$ otherwise.

Lemma 2. *For any set $\mathcal{S} \in \mathcal{F}_c$ if $\operatorname{int}\mathcal{S} \not\subseteq \mathcal{A}(0)$, then the volume of its counter-example part is positive, i.e., $\operatorname{vol}(\mathcal{S}_{\text{counter}}) > 0$.*

Proof. We will proof this statement by contrapositive, i.e., we will show $\operatorname{vol}(\mathcal{S}_{counter}) = 0$ implies $\operatorname{int}\mathcal{S} \subseteq \mathcal{A}(0)$.

We first argue that if $\operatorname{vol}(\mathcal{S}_{counter}) = 0$ then $\mathcal{S}\backslash\mathcal{A}(0) \subseteq \partial\mathcal{S}$. To see this, we can form a contradiction. Assume $\mathcal{S}\backslash\mathcal{A}(0) \not\subseteq \partial\mathcal{S}$ and recall $\mathcal{A}(0)$ is an open contractible set. It follows then that there exist a full dimensional set $\mathcal{V} \subseteq \mathcal{S}\backslash\mathcal{A}(0)$. Now note that the regions of attraction of the unstable hyperbolic equilibrium points are not full dimensional. There is then a point $p \in \operatorname{int}\mathcal{V}$ satisfying $p \in \mathcal{S}_{counter}$. By the continuity of $\phi(\tau, \cdot)$, we further have an open neighborhood $\mathcal{V}'$ of p satisfying $\operatorname{vol}(\mathcal{V}') > 0$ and $\mathcal{V}' \subseteq \mathcal{S}_{counter}$, which contradict with $\operatorname{vol}(\mathcal{S}_{counter}) = 0$.

In summary, we have $\mathcal{S}\backslash\mathcal{A}(0) \subseteq \partial\mathcal{S}$, which imples that $\{\mathcal{S}\backslash\mathcal{A}(0)\}\backslash\{\partial\mathcal{S}\} = \emptyset$. Therefore, we have $\operatorname{int}\mathcal{S} \subseteq \mathcal{A}(0)$ and result follows. $\square$

Lemma 3. *For any set $\mathcal{S} \in \mathcal{F}_c$ satisfying $\operatorname{int}\mathcal{S} \not\subseteq \mathcal{A}(0)$, we have $\lim_{m\to\infty} \mathbb{P}(X_1 = \cdots = X_m = 1) = 0$. That is, a counter-example is eventually sampled almost surely.*

Proof. Note that we have $\mathcal{S}_{\text{counter}} \subseteq \mathcal{S}$ and $\operatorname{vol}(\mathcal{S}_{\text{counter}}) > 0$ by Lemma 2. Then, denoting the counter-example ratio as $\rho := \operatorname{vol}(\mathcal{S}_{\text{counter}})/\operatorname{vol}(\mathcal{S})$, one can conclude $0 < \rho \leq 1$ and

$$\lim_{m\to\infty} \mathbb{P}(X_0 = ... = X_m = 1) = \lim_{m\to\infty} (1-\rho)^m = 0.$$

$\square$

We now leverage the results in Lemma 2 and Lemma 3 to obtain the following termination guarantee.

Theorem 8. *Consider $\hat{\mathcal{S}}^{(0)} \in \mathcal{F}_c$ with $\hat{\mathcal{S}}^{(0)} \supseteq \mathcal{B}_\delta$. Then, after a finite number of iterations, the updates on set $\hat{\mathcal{S}}^{(i)}$ terminate at some $\hat{\mathcal{S}}^*$ whose interior is a non-empty subset of $\mathcal{A}(0)$ whenever $k > \bar{k}$ and* (14) *holds.*

Proof. Suppose that at any given iteration i the set $\operatorname{int}\hat{\mathcal{S}}^{(i)} \not\subseteq \mathcal{A}(0)$. Then it follows from Lemma 3 that a counter-example is eventually found almost surely, and a new set $\hat{\mathcal{S}}^{(i+1)}$ is obtained. Also Theorem 7 implies the total number of such transitions is finite, since $\hat{\mathcal{S}}^{(0)} \in \mathcal{F}_c$ and $\mathcal{B}_\delta \subseteq \hat{\mathcal{S}}^{(0)}$.

Now let $\hat{\mathcal{S}}^*$ denote the last updated approximation. Note that since there are not further updates to $\hat{\mathcal{S}}^*$ with probability one, this implies that $\operatorname{vol}(\hat{\mathcal{S}}^*_{\text{counter}}) = 0$. We argue then that $\operatorname{int}\hat{\mathcal{S}}^* \subseteq \mathcal{A}(0)$, since otherwise $\operatorname{vol}(\hat{\mathcal{S}}^*_{\text{counter}}) > 0$, which contradicts the fact that $\hat{\mathcal{S}}^*$ is the last iteration. Finally, $\operatorname{int}\hat{\mathcal{S}}^*$ is non-empty since Theorem 6 implies $\hat{\mathcal{S}}^* \supseteq \mathcal{B}_\delta$.

$\square$

E. Multiple center point approximation

When the ROA $\mathcal{A}(0)$ is distorted or non-convex, Algorithm 1 may significantly underestimate $\mathcal{A}(0)$, meaning that the volume of resulted approximation $\operatorname{vol}(\hat{\mathcal{S}}^{(i)}) \ll \operatorname{vol}(\mathcal{A}(0))$. To address this problem, we can refine Algorithm 1 by generating additional approximations similar to $\hat{\mathcal{S}}^{(i)}$ but centered differently from the equilibrium $x^* = 0$.

In particular, we consider $h \in \mathbb{N}^+$ center points x_q indexed by $q \in \{1, 2, ..., h\}$. In this manner, we set the first center point as $x_1 = x^* = 0$. Then other centers, i.e., x_2,...,x_h, can be chosen uniformly within some region of interest or selected to be in some preferred place. At each center point x_q the sphere approximation is defined by $\hat{\mathcal{S}}_q^{(i)} := \{x | \, \|x - x_q\|_2 \leq b_q^{(i)}\}$, where $b_q^{(i)}$ represents the radius to be updated in the presence of counter-examples. As before we initialize $b_q^{(0)} = c$. In the case of polyhedral approximations, we similarly define $b_q^{(i)} = [b_{q,1}^{(i)}, ..., b_{q,n}^{(i)}]^T \in \mathbb{R}^n$, with $b_q^{(0)} = c\mathbb{1}^N$ and let $\hat{\mathcal{S}}_q^{(i)} := \{x | A(x - x_q) \leq b_q^{(i)}\}$. Then, the multi-center ROA approximation $\hat{\mathcal{S}}_{\text{multi}}^{(i)}$ at iteration i is the union of all approximations, i.e., $\hat{\mathcal{S}}_{\text{multi}}^{(i)} := \cup_{q=1}^h \hat{\mathcal{S}}_q^{(i)}$. Note that $\hat{\mathcal{S}}_1^{(i)}$ is equivalent to the original approximation $\hat{\mathcal{S}}^{(i)}$ of previous sections, and $\hat{\mathcal{S}}_2^{(i)}$ to $\hat{\mathcal{S}}_h^{(i)}$ are additional enhancements.

Similar to Algorithm 1, sample points p_{ij} are generated uniformly within $\hat{\mathcal{S}}_{\text{multi}}^{(i)}$ in each sub-iteration $j = 1, 2, ...$. In this multi-center case, p_{ij} is classified as a counter-example if starting from $x_0 = p_{ij}$, $x_n \notin \hat{\mathcal{S}}_{\text{multi}}^{(i)}$ for all $n \in \{1, ..., k\}$. Once encountered a counter-example, we update $\hat{\mathcal{S}}_{\text{multi}}^{(i)}$ and restart sampling iteration j.

In particular, given counter-example $p_{ij} \in \hat{\mathcal{S}}_{\text{multi}}^{(i)}$, every approximations $\hat{\mathcal{S}}_q^{(i)}$ (sphere or polyhedron) satisfying $p_{ij} \in \hat{\mathcal{S}}_q^{(i)}$ are subjected to update respectively via the following criterion:

$$\textbf{(sphere)} \qquad b_q^{(i+1)} = \|p_{ij} - x_q\|_2 - \varepsilon$$

$$\textbf{(polyhedron)} \begin{cases} b_{q,l^*}^{(i+1)} = a_{l^*}(p_{ij} - x_q) - \varepsilon \\ b_{q,l}^{(i+1)} = b_{q,l}^{(i)}, \forall\, l \in \{1, .., n\}\backslash l^*, \end{cases}$$

where $l^* = \arg\max_{l\in\{1,...,n\}} \frac{a_l^T(p_{ij} - x_q)}{\|a_l\|\|p_{ij} - x_q\|}$. Then, those approximations not containing p_{ij} are updated as $\hat{\mathcal{S}}_q^{(i+1)} = \hat{\mathcal{S}}_q^{(i)}$. Note that the conservation parameter ε is strictly positive. Thus, for all center points $x_q \notin \mathcal{A}(0)$, the corresponding constraint parameters $b_q^{(i)}$ could decrease to negative values and result in $\hat{\mathcal{S}}_q^{(i)} = \emptyset$ without affecting our results.

In this multi-center setting, we use $\mathcal{F}_c^h$ to denote the parametric family of h closed balls (resp. polytopes) defined by $\cup_{q=1}^h \mathcal{S}_q$, where $\mathcal{S}_q = \{x : \|x - x_q\|_2 \leq b_q\}$ (resp. $\mathcal{S}_q = \{x : A(x - x_q) \leq b_q\}$), for $b_q \in [0, c]$ (resp. $b_q \in [0, c]^n$) and $x_q \in \mathbb{R}^d$ indexed by $q = \{1, ..., h\}$.

Theorem 9. *For any iteration $i \in \mathbb{N}^+$, the multi-center approximation $\hat{\mathcal{S}}_{\text{multi}}^{(i)}$ is non-vanishing, i.e., $\hat{\mathcal{S}}_{\text{multi}}^{(i)} \supseteq \mathcal{B}_\delta$, if condition* (14) *is satisfied. The total number of counter-examples encountered in Algorithm 1, with k-doubling after each failure, is bounded by $h\frac{c}{\varepsilon}\log_2 \bar{k}$ and $nh\frac{c}{\varepsilon}\log_2 \bar{k}$*

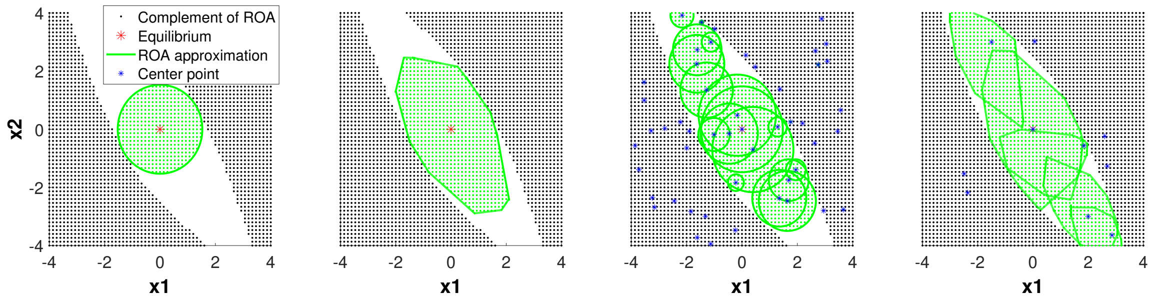

Fig. 2: The region of attraction polyhedron approximation for one center point (left two) and multiple random center points (right two) in the sphere and polyhedron case, respectively.

Approximate method	# of counter examples	# of samples	# of steps simulated	Average # of steps per sample
1-center sphere approximation	14	7024	7935	1.39
1-center polyhedron approximation	94	23130	28127	1.22
50-center sphere approximations	191	17481	53756	3.07
10-center polyhedron approximations	370	46819	66399	1.41

TABLE I: Performance statistics for different configurations of our algorithm.

in the sphere and polyhedron case respectively. Moreover, the last updated multi-center approximation $\hat{\mathcal{S}}^*_{\text{multi}}$ *satisfies* $\operatorname{int}\hat{\mathcal{S}}^*_{\text{multi}} \neq \emptyset$ *and* $\operatorname{int}\hat{\mathcal{S}}^*_{\text{multi}} \subseteq \mathcal{A}(0)$.

Proof. By definition $\hat{\mathcal{S}}^{(i)}_{\text{multi}} \supseteq \hat{\mathcal{S}}^{(i)}_1$ for all $i \in \mathbb{N}^+$, Theorem 6 therefore implies $\hat{\mathcal{S}}^{(i)}_{\text{multi}} \supseteq \hat{\mathcal{S}}^{(i)}_1 \supseteq \mathcal{B}_\delta$ under (14). The bound on the total number of counter examples follows from Theorem 7. Since every additional approximation $\hat{S}^{(i)}_q \in \mathcal{F}_c$ for all $q \in \{1, ..., h\}$ and iteration $i \in \{1, ...\}$. Finally, by generalizing Lemma 2, Lemma 3 and Theorem 8 to the scope of $\mathcal{F}^h_c$, the last statement follows. □

V. Experiments

We illustrate the accuracy of the proposed methodology by approximating the region of attraction of the following autonomous dynamical system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 + \frac{1}{3}x_1^3 - x_2 \end{bmatrix}.$$

The black dotted area in Figure 2 represents the complement of ROA of the origin, which is computed by testing a mesh grid of points. A point is marked black if it does not converge to the equilibrium after $t = 30$. In our algorithm we set $\epsilon = 0.1$, $k = 50$ and $\tau_s = 0.5$. To estimate the number of iterations until convergence, we stop our algorithm when all black are dots excluded from our current approximation.

The outcomes of our approximation are marked in green. In particular, Figure 2 (left two panels) shows the outcome of applying Algorithm 1 using a sphere and a $n = 200$ directions polyhedron approximation. To address the problem of under-estimation, as shown in the right two panels of Figure 2, we can generate random center points. 50 spheres or 10 polyhedrons approximates with random center points give a good approximation. Detailed statistics of our algorithms for the aforementioned scenarios are provided in Table I. Notably, the number of counter-examples and the steps simulated per sample is small, which illustrates the efficiency of our algorithm.

VI. Conclusions and Future Work

We consider the problem of learning the region of attraction of a stable equilibrium. To that end, we propose the use of a more flexible notion of invariance known as recurrence. We provide necessary and sufficient conditions for a recurrent set to be an inner-approximation of the ROA. Our algorithms are sequential, and only incur on a limited number of counter-examples. Future work includes extension of our framework to other family of approximations and to control design.

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