Outer Approximations of Minkowski Operations on Complex Sets via Sum-of-Squares Optimization

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***Abstract*— We study the problem of finding closed-form outer approximations of Minkowski sums and products of sets in the complex plane. Using polar coordinates, we pose this as an optimization problem in which we find a pair of contours that give lower and upper bounds on the radial distance at a given angle. Through a series of variable transformations we rewrite this as a sum-of-squares optimization problem. Numerical examples are given to demonstrate the performance.**

I. INTRODUCTION

Set operations on complex sets naturally arise in many control applications [1], [2]. The most prominent is robustness analysis in which Nyquist-like criteria is used to assess the stability of a control system. Given a plant $P(s)$ and associated controller $C(s)$, the Nyquist stability criteria involves plotting their product as s travels along a contour of the right half plane [3]. If both plant and controller are known exactly, the numerical evaluation of this criteria at a given s involves a simple product of two points in the complex plane. Uncertainty in the plant and controller leads to these points becoming sets in the complex plane. Evaluation of the stability criteria then involves determining all possible complex products of points drawn from the two sets. Beyond multiplication, forming parallel or feedback connections of uncertain transfer functions leads to addition and division operations applied to sets. Following [4], we refer to these various operations on complex sets as Minkowski operations.

Minkowski operations on complex sets are relevant to other domains including computer-aided design [5] and geometric optics [6]. More recently, the authors of [7] use Minkowski products in analyzing the convergence of optimization algorithms. The authors introduce the *Scaled Relative Graph* which visualizes nonlinear operators as sets in the complex plane. Composition of these operators then involves computing Minkowski products. This can be used to provide formal proofs of convergence with geometric arguments.

Closed-form expressions of the sets resulting from Minkowski operations are not known except for cases involving relatively simple sets. The most widely studied case involves discs in the complex plane which are parameterized by their center and radius. This is sometimes referred to as complex circular arithmetic [8]. The results of [1], [6], [7] are limited to operations involving such disks.

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When exact closed-form expressions are not attainable, one may instead seek to find an outer approximation. When done through manual derivation, this quickly becomes a time-intensive process which requires dedicated efforts for each class of contours considered. For example, in [1], the authors develop an outer approximation for the sum of two complex discs.

As an alternative to manual derivation, an optimization-based approach offers the promise of automating this process. A recent body of literature demonstrates the versatility of sum-of-squares (SOS) optimization for approximating semi-algebraic sets with polynomial functions. Applications include encapsulating 3D point clouds [9], bounding regions of stability for PID controllers [10], and representing unions of sets with a single polynomial [11]. The main contribution of this paper is a method for finding outer approximations of Minkowski operations of addition, multiplication, and division of an arbitrary number of complex sets that belong to a fairly general class.

The rest of the paper is organized as follows. Section II sets up the problem and defines the Minkowski operations considered. Section III develops SOS-based optimization problems for finding outer approximations to the Minkowski operations. Section IV provides examples of the resulting outer approximations. Section V concludes the paper and discusses future directions.

A. Notation

Let $\mathbf{r} = x + iy$ be a complex number with magnitude $r = \sqrt{x^2 + y^2}$ and angle $\theta = \arctan(y/x)$. For $\xi \in \mathbb{R}^n, \mathbb{R}[\xi]$ is the set of polynomials in ξ with real coefficients. The subset $\sum[\xi] = \{p = p_1^2 + p_2^2 + \ldots + p_n^2 : p_1, \ldots, p_n \in \mathbb{R}[\xi]\}$ of $\mathbb{R}[\xi]$ is the set of SOS polynomials in ξ. $\mathbb{Z}\,(\mathbb{Z}_+)$ is the set of non-negative (positive) integers. For convenience, we define the following sets of indices

$$\begin{aligned}
\mathcal{H} &= \{0, 1, \ldots, m\}, \\
\mathcal{J} &= \{1, 2, \ldots, n\}, \\
\mathcal{K} &= \{n+1, n+2, \ldots, m\}.
\end{aligned}$$

We use $x_{[j]}$ to denote element j of vector $x \in \mathbb{R}^n$. Similarly we use $x_{[j:k]}$ to denote the vector $[x_{[j]} \;\; x_{[j+1]} \;\; \ldots \;\; x_{[k]}]^T$ Instead of $\displaystyle\sum_{j=1}^{n} x_{[j]}$, we use $\displaystyle\sum_{j} x_{[j]}$, when the dimension n is implicit from the context.

II. PRELIMINARIES

A. Representation of Complex Sets

Let $\mathcal{R}$ denote the set of points in the complex plane between two polar contours, $r^l(\theta)e^{i\theta}$ and $r^u(\theta)e^{i\theta}$, evaluated over the angle range $\theta \in [\theta^l,\, \theta^u]$, i.e.,

$$\mathcal{R} = \{re^{i\theta} | 0 \leq r^l(\theta) \leq r \leq r^u(\theta), \theta^l \leq \theta \leq \theta^u\}\,. \quad (1)$$

Throughout we use the superscripts l and u to denote lower and upper bounds. We use subscripts where appropriate to distinguish between different sets of this form. Figure 1 provides an example of this notation for the following set:

$$\mathcal{R} = \{re^{i\theta} | 1 + \frac{1}{4}\sin\theta \leq r \leq 1.5 - \frac{1}{4}\cos\theta, 0 \leq \theta \leq \frac{\pi}{3}\}\,. \quad (2)$$

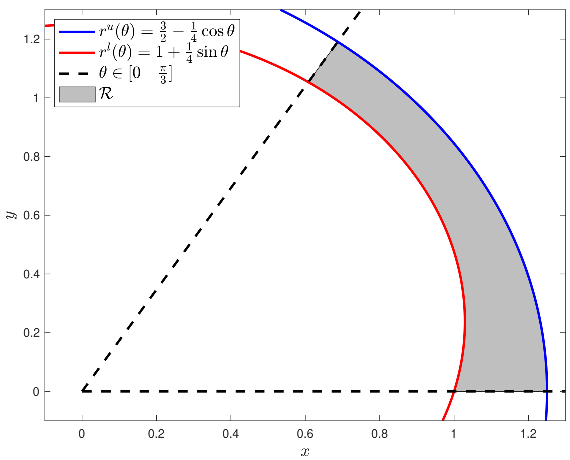

Fig. 1. Complex Set of the Form (1)

B. Minkowski Operations on Complex Sets

Consider a family of n sets of the form (1) and let $\mathcal{S}_{\bigotimes}$ denote the set obtained by forming all possible complex products. Following [6] we refer to this as the Minkowski product

$$\mathcal{S}_{\bigotimes} = \{\prod_{j \in \mathcal{J}} \mathbf{r}_j | \mathbf{r}_j \in \mathcal{R}_j, j \in \mathcal{J}\}\,. \quad (3)$$

Similarly, we define Minkowski division as the set obtained by forming all possible pair-wise complex divisions between two sets:

$$\mathcal{S}_{\div} = \{\mathbf{r}_1\mathbf{r}_2^{-1} | \mathbf{r}_1 \in \mathcal{R}_1, \mathbf{r}_2 \in \mathcal{R}_2\}. \quad (4)$$

The Minkowski sum is defined as follows:

$$\mathcal{S}_{\bigoplus} = \{\sum_{j \in \mathcal{J}} \mathbf{r}_j | \mathbf{r}_j \in \mathcal{R}_j, j \in \mathcal{J}\} \quad (5)$$

In this work we focus on two operations that often arise in control applications. The first operation contains multiplication and division as special cases:

$$\mathcal{S}_{\frac{\bigotimes}{\bigotimes}} = \{\prod_{j \in \mathcal{J}} \mathbf{r}_j \prod_{k \in \mathcal{K}} \mathbf{r}_k^{-1}, \mathbf{r}_j \in \mathcal{R}_j, \mathbf{r}_k \in \mathcal{R}_k, j \in \mathcal{J},\, k \in \mathcal{K}\} \quad (6)$$

The second operation extends the Minkowski sum to allow inversion of some sets.

$$\mathcal{S}_{\bigoplus + \bigoplus^{-1}} = \{\sum_{j \in \mathcal{J}} \mathbf{r}_j + \sum_{k \in \mathcal{K}} \mathbf{r}_k^{-1} | \mathbf{r}_j \in \mathcal{R}_j, \mathbf{r}_k \in \mathcal{R}_k, j \in \mathcal{J},\, k \in \mathcal{K}\} \quad (7)$$

C. Problem Setup

In general, closed-form expressions do not exist for the sets $\mathcal{S}_\bullet$ resulting from the Minkowski operation denoted by $\bullet$. Here we focus on finding a set $\mathcal{R}_\bullet$ of the form (1) that provides an outer approximation of $\mathcal{S}_\bullet$. A natural objective is to minimize the volume (area) of $\mathcal{R}_\bullet$ subject to the set-containment condition $\mathcal{S}_\bullet \subseteq \mathcal{R}_\bullet$. This can be posed as an optimization problem:

$$\begin{aligned} \min_{\alpha^l, \alpha^u} \quad & \int_{\theta^l}^{\theta^u} r^u(\theta, \alpha^u) - r^l(\theta, \alpha^l) d\theta \\ \text{s.t.} \quad & \mathcal{S}_\bullet \subseteq \mathcal{R}_\bullet \end{aligned} \quad (8)$$

where

$$\mathcal{R}_\bullet = \{re^{i\theta} | 0 \leq r^l(\theta, \alpha^l) \leq r \leq r^u(\theta, \alpha^u), \theta^l \leq \theta \leq \theta^u\}. \quad (9)$$

Assumption 1. *We assume that each contour $r(\theta, \alpha)$ is a function of $\cos(\theta)$ and $\sin(\theta)$ with associated real coefficient vector α, i.e.,*

$$\begin{aligned} r(\theta, \alpha) &= \alpha_{[1]} + \alpha_{[2]}\cos(\theta) + \alpha_{[3]}\sin(\theta) + \alpha_{[4]}\cos(\theta)^2 + \ldots \\ &= \sum_j \alpha_{[j]}(\cos\theta)^{u_j}(\sin\theta)^{v_j}\,, \quad \alpha_{[j]} \in \mathbb{R}, u_j, v_j \in \mathbb{N}. \end{aligned}$$

We will sometimes refer to this parameterization as a polynomial of $\cos\theta$ and $\sin\theta$, as introducing independent variables for each would yield a polynomial expression. This parameterization readily admits an upper bound which we will utilize.

Lemma 1. *Let $r(\theta, \alpha)$ be a polynomial function of $\cos(\theta)$ and $\sin(\theta)$ with associated real coefficient vector α. The following inequality holds:*

$$r(\theta, \alpha) \leq \bar{r} \quad (10)$$

where:

$$\bar{r} = \sum_j \left|\alpha_{[j]}\right| \quad (11)$$

Proof. Note the following inequality:

$$\left|\alpha_{[j]}(\cos\theta)^m(\sin\theta)^n\right| \leq \left|\alpha_{[j]}\right| \quad \forall \theta \in \mathbb{R}, m, n \in \mathbb{N} \quad (12)$$

The inequality for the polynomial follows immediately. $\square$

Assumption 2. *We assume that any set which is inverted has a known, positive lower bound for $r^l(\theta)$ which we denote $\underline{r}^l$.*

$$r^l(\theta) \geq \underline{r}^l > 0 \quad \forall\, \theta \in \mathbb{R} \quad (13)$$

Assumption 2 ensures the set does not contain the origin and therefore its inverse is bounded. The sets resulting from the introduced Minkowski operations are then bounded as well. This is important as seeking an outer approximation of an unbounded set would be trivially infeasible. Knowledge of the constant $\underline{r}^l$ allows us to calculate an upper bound as given by the following lemma.

Lemma 2. *Let* $\mathbf{r}$ *be a point in* $\mathcal{S}_{\bigoplus+\bigoplus^{-1}}$ *as defined by (7). Let Assumptions 1 and 2 hold. Then the following inequality holds:*

$$|\mathbf{r}| \leq \sum_{j\in\mathcal{J}} (\bar{r}_j^u) + \sum_{k\in\mathcal{K}} (\underline{r}_k^l)^{-1}, \quad \forall \mathbf{r} \in \mathcal{S}_{\bigoplus+\bigoplus^{-1}}. \tag{14}$$

Proof. Given that $\mathbf{r} \in \mathcal{S}_{\bigoplus+\bigoplus^{-1}}$, there exists points $\mathbf{r}_j \in \mathcal{R}_j, \mathbf{r}_k \in \mathcal{R}_k, j \in \mathcal{J}$, $k \in \mathcal{K}$ such that the following equality holds:

$$\begin{aligned} |\mathbf{r}| &= \left|\sum_{j\in\mathcal{J}} \mathbf{r}_j + \sum_{k\in\mathcal{K}} \mathbf{r}_k^{-1}\right| \\ &\leq \sum_{j\in\mathcal{J}} |\mathbf{r}_j| + \sum_{k\in\mathcal{K}} \left|\mathbf{r}_k^{-1}\right| \\ &\leq \sum_{j\in\mathcal{J}} (\bar{r}_j^u) + \sum_{k\in\mathcal{K}} (\underline{r}_k^l)^{-1} \quad \text{Lem. 1, Asm. 2} \end{aligned} \tag{15}$$

□

Assumption 3. *Let* Θ *denote the set of angles in* $\mathcal{S}_\bullet$*:*

$$\Theta = \{\arctan(\mathbf{r}) | \mathbf{r} \in \mathcal{S}_\bullet\} \tag{16}$$

We assume that we know Θ *exactly so that we can specify the lower and upper bounds* θ^l, θ^u *in our objective.*

The range of possible angles is easy to calculate for the product and division of complex sets as angles simply add and subtract. For Minkowski sums of complex sets the set of possible angles is not easily calculated. We discuss methods for doing so in section III-C.

D. Generalized $\mathcal{S}$-Procedure and SOS Optimization

In the development that follows, we will be interested in solving optimization problems of the following form:

$$\begin{aligned} \min_{\alpha^h} \ & \sum_{h=1}^{j} c_h^T \alpha^h \\ \text{s.t. } & g_1(\xi_1, \alpha^1) d_1(\xi_1) - f_1(\xi_1) \geq 0 \quad \forall \xi_1 \in \mathcal{X}_1 \\ & g_2(\xi_2, \alpha^2) d_2(\xi_2) - f_2(\xi_2) \geq 0 \quad \forall \xi_2 \in \mathcal{X}_2 \\ & \vdots \\ & g_j(\xi_j, \alpha^j) d_j(\xi_j) - f_j(\xi_j) \geq 0 \quad \forall \xi_j \in \mathcal{X}_j \end{aligned} \tag{17}$$

where

$$\mathcal{X}_h = \{\xi_h | h_{h,k}(\xi_h) \geq 0, k = 1, \dots, n_h\}. \tag{18}$$

In each constraint, $\xi_j \in \mathbb{R}^{n_j}$ is a vector of free variables and $g_j(\xi_j, \alpha^j), d_j(\xi_j), f_j(\xi_j), h_{j,k}(\xi_j) \in \mathbb{R}[\xi_j]$ are polynomials of these variables. The coefficients α^j of $g_j(\xi_j, \alpha^j)$ are explicitly listed to highlight that they are decision variables. The objective is linear with each c_j being a given weighting of the decision variable vector α^j. The constraints consist of non-negativity conditions that must hold for all ξ_j in the semi-algebraic set $\mathcal{X}_j$ which is described by polynomial inequalities of ξ_j. This can be seen as a set-containment condition.

The generalized $\mathcal{S}$-procedure provides an inequality that is sufficient for the set-containment condition to hold [12]. Introducing non-negative multipliers $s_{j,k}(\xi_k, \beta^{j,k})$ we can replace the set-containment condition with a simple non-negativity condition.

$$\begin{aligned} \min_{\alpha^h, \beta^{j,k}} \ & \sum_{h=1}^{j} c_h^T \alpha^h \\ \text{s.t. } & g_1(\xi_1, \alpha^1) d_1(\xi_1) - f_1(\xi_1) \\ & \quad - \sum_k s_{1,k}(\xi_1, \beta^{1,k}) h_{1,k}(\xi_1) \geq 0 \quad \forall \xi_1 \in \mathbb{R}^{n_1} \\ & \quad s_{1,k}(\xi_1) \geq 0 \quad \forall\, \xi_1 \in \mathbb{R}^{n_1}, k \in 1, \dots, n_1 \\ & g_2(\xi_2, \alpha^2) d_2(\xi_2) - f_2(\xi_2) \\ & \quad - \sum_k s_{2,k}(\xi_2, \beta^{2,k}) h_{2,k}(\xi_2) \geq 0 \quad \forall \xi_2 \in \mathbb{R}^{n_2} \\ & \quad s_{2,k}(\xi_2) \geq 0 \quad \forall\, \xi_2 \in \mathbb{R}^{n_2}, k \in 1, \dots, n_2 \\ & \vdots \\ & g_j(\xi_j, \alpha^j) d_j(\xi_j) - f_j(\xi_j) \\ & \quad - \sum_k s_{j,k}(\xi_j, \beta^{j,k}) h_{j,k}(\xi_j) \geq 0 \quad \forall \xi_j \in \mathbb{R}^{n_j} \\ & \quad s_{j,k}(\xi_j) \geq 0 \quad \forall\, \xi_j \in \mathbb{R}^{n_j}, k \in 1, \dots, n_j \end{aligned} \tag{19}$$

The left hand side of each inequality j describes a polynomial of free variables ξ_j with decision variables α^j and $\beta^{j,k}$ entering linearly. We can replace each non-negativity constraint with the more restrictive condition that the polynomial be a SOS polynomial. The resulting semidefinite program can be solved readily.

Although we only show inequality constraints above, any equality constraint $h(\xi) = 0$ can be represented by two constraints $h(\xi) \geq 0, h(\xi) \leq 0$. In the development that follows we focus on transforming problems of interest into the form of (17). Once in this form, the subsequent application of the $\mathcal{S}$-procedure and SOS conditions is straight-forward and due to page limits we do not explicitly include this step.

III. OUTER APPROXIMATIONS OF MINKOWSKI OPERATIONS ON COMPLEX SETS

We first develop a method for finding outer approximations of the set $\mathcal{S}_{\bigotimes \bigotimes}$, a generalization of the Minkowski product and division for complex sets. Through a series of variable transformations we pose this as a polynomial optimization problem with set-containment constraints. The $\mathcal{S}$-procedure and SOS-based sufficient conditions for non-negativity are then used to obtain a convex optimization problem. A similar approach is followed to develop a method for outer approximating the set $\mathcal{S}_{\bigoplus+\bigoplus^{-1}}$ which generalizes the Minkowski sum of complex sets.

A. Minkowski Product and Division of Complex Sets

We seek to minimize an outer approximation of $\mathcal{S}_{\underset{\otimes}{\otimes}}$. This can be posed as follows:

$$\min_{\alpha^u,\alpha^l} \int_{\theta^l}^{\theta^u} r^u(\theta,\alpha^u) - r^l(\theta,\alpha^l)d\theta$$
$$\text{s.t. } r^l(\theta_0,\alpha^l) \le \left| \frac{\prod_{j\in\mathcal{J}} r_j e^{i\theta_j}}{\prod_{k\in\mathcal{K}} r_k e^{i\theta_k}} \right| \le r^u(\theta_0,\alpha^u) \tag{20}$$
$$\forall(\theta_{[0:m]}, r_{[1:m]}) \in \mathcal{X}$$

where $\mathcal{X}$ is the semi-algebraic set:

$$\begin{aligned}\mathcal{X} =\{(\theta_{[0:m]}, r_{[1:m]}) \ &: \theta_0 = \sum_{j\in\mathcal{J}}\theta_j - \sum_{k\in\mathcal{K}}\theta_k,\\ r_j^l(\theta_j) \le r_j \le r_j^u(\theta_j), &\theta_j^l \le \theta_j \le \theta_j^u, j\in\mathcal{J}\\ r_k^l(\theta_k) \le r_k \le r_k^u(\theta_k), &\theta_k^l \le \theta_k \le \theta_k^u, k\in\mathcal{K}\}\end{aligned} \tag{21}$$

Given that we know the bounds θ_l, θ_u, we can evaluate the integral within our objective to eliminate the dependency on θ. This yields a linear objective in terms of the coefficients.

$$\int_{\theta^l}^{\theta^u} r^u(\theta,\alpha^u) - r^l(\theta,\alpha^l)d\theta = c_l^T\alpha^l + c_u^T\alpha^u$$

We introduce intermediate variables ϕ_j such that the sum of angles defining θ_0 can be written as the sum of two angles.

$$\phi_j = \sum_{h=j}^{m} c_h\theta_h \tag{22}$$

where

$$c_h = \begin{cases} +1, & \text{if } h\in\mathcal{J} \\ -1, & \text{if } h\in\mathcal{K}\end{cases} \tag{23}$$

The angle summation can then be replaced with the following semi-algebraic set:

$$\begin{aligned}\mathcal{Z} = \{(\theta_{[0:m]}, \phi_{[2:m-1]}) | \theta_0 &= c_1\theta_1 + \phi_2,\\ \phi_2 &= c_2\theta_2 + \phi_3,\\ &\dots\\ \phi_{m-2} &= c_{m-2}\theta_{m-2} + \phi_{m-1},\\ \phi_{m-1} &= c_{m-1}\theta_{m-1} + c_m\theta_m\}\end{aligned} \tag{24}$$

We then obtain a superset of $\mathcal{Z}$ by replacing each equality constraint with two constraints involving cos and sin.

$$\begin{aligned}\mathcal{Y} = \{(\theta_{[0:m]}, \phi_{[2:m-1]}) |&\\ \cos\theta_0 &= \cos(c_1\theta_1 + \phi_2)\,,\\ \sin\theta_0 &= \sin(c_1\theta_1 + \phi_2)\,,\\ \cos\phi_2 &= \cos(c_2\theta_2 + \phi_3)\,,\\ \sin\phi_2 &= \sin(c_2\theta_2 + \phi_3)\,,\\ &\dots\\ \cos\phi_{m-2} &= \cos(c_{m-2}\theta_{m-2} + \phi_{m-1})\,,\\ \sin\phi_{m-2} &= \sin(c_{m-2}\theta_{m-2} + \phi_{m-1})\,,\\ \cos\phi_{m-1} &= \cos(c_{m-1}\theta_{m-1} + c_m\theta_m)\,,\\ \sin\phi_{m-1} &= \sin(c_{m-1}\theta_{m-1} + c_m\theta_m)\}\end{aligned} \tag{25}$$

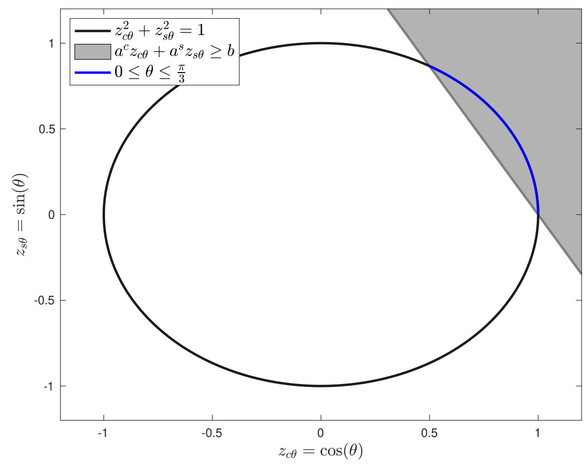

Fig. 2. Constraints for Angle Interval

Remark 1. $\mathcal{Y}$ is a superset of $\mathcal{Z}$ as the trigonometric identities still hold when angles have multiples of 2π added. Given we are working with periodic functions (Assumption 1) this is a subtlety of no consequence.

Recall the following trigonometric identities involving angles a and b with signs $c_a, c_b \in \{-1, 1\}$:

$$\cos(c_a a + c_b b) = \cos a\cos b - c_a c_b \sin a \sin b\,, \tag{26}$$
$$\sin(c_a a + c_b b) = c_a \sin a\cos b + c_b \cos a \sin b\,. \tag{27}$$

Applying these identities we can write the constraints defining $\mathcal{Y}$ in terms of $\cos\theta_h, \sin\theta_h, \cos\phi_l, \sin\phi_l$. We then eliminate the trigonometric terms by introducing new variables along with a quadratic equality constraint.

$$\begin{aligned}z_{c\theta_h} = \cos(\theta_h), z_{s\theta_h} = \sin(\theta_h), z_{c\theta_h}^2 + z_{s\theta_h}^2 &= 1\\ \forall\, h \in 0\cup\mathcal{J}\cup\mathcal{K}&\\ z_{c\phi_l} = \cos(\phi_l), z_{s\phi_l} = \sin(\phi_l), z_{c\phi_l}^2 + z_{s\phi_l}^2 &= 1\\ \forall\, l = 2,\dots,m-1&\end{aligned}$$

Next, we rewrite the angle constraints $\theta_h^l \le \theta_h \le \theta_h^u$ in terms of $z_{c\theta_h}, z_{s\theta_h}$. In the new variables, the points satisfying the angle interval constraint can be represented by the intersection of the quadratic equality constraint and a halfplane that passes through the points $(\cos\theta_h^l, \sin\theta_h^l)$ and $(\cos\theta_h^u, \sin\theta_h^u)$. Figure 2 visualizes this for $\theta^l = 0, \theta^u = \frac{\pi}{3}$. Defining the midpoint angle $\theta_h^m = \frac{1}{2}(\theta_h^l + \theta_h^u)$, it can be shown that the halfplane is the set of points $(z_{c\theta_h}, z_{s\theta_h})$ satisfying:

$$a_h^c z_{c\theta_h} + a_h^s z_{s\theta_h} \ge b_h \tag{28}$$

where

$$a_h^c = \cos\theta_h^m, a_h^s = \sin\theta_h^m, b_h = \cos\theta_h^m\cos\theta_h^u + \sin\theta_h^m\sin\theta_h^m \tag{29}$$

With this change of variables, the optimization problem is rewritten as follows:

$$
\begin{aligned}
\min_{\alpha^l,\alpha^u} \; & c_l^T\alpha^l + c_u^T\alpha^u \\
\text{s.t.} \quad & r^l(z_{c\theta_0}, z_{s\theta_0}, \alpha^l) \prod_{k\in\mathcal{K}} r_k \le \prod_{j\in\mathcal{J}} r_j \\
& r^u(z_{c\theta_0}, z_{s\theta_0}, \alpha^u) \prod_{k\in\mathcal{K}} r_k \ge \prod_{j\in\mathcal{J}} r_j \\
& \forall (z_{c\theta_{[0:m]}}, z_{s\theta_{[0:m]}}, z_{c\phi_{[2:m-1]}}, z_{s\phi_{[2:m-1]}}, r_{[1:m]}) \in \mathcal{W}
\end{aligned} \tag{30}
$$

where:

$$
\begin{aligned}
\mathcal{W} = \{ & (z_{c\theta_{[0:m]}}, z_{s\theta_{[0:m]}}, z_{c\phi_{[2:m-1]}}, z_{s\phi_{[2:m-1]}}, r_{[1:m]}) : \\
& z_{c\theta_0} = z_{c\theta_1} z_{c\phi_2} - c_1 z_{s\theta_1} z_{s\phi_2} \\
& z_{s\theta_0} = c_1 z_{s\theta_1} z_{c\phi_2} + z_{c\theta_1} z_{s\phi_2} \\
& z_{c\phi_l} = z_{c\theta_l} z_{c\phi_{l+1}} - c_l z_{s\theta_l} z_{s\phi_{l+1}}, \quad l \in 2,\ldots,m-2 \\
& z_{s\phi_l} = c_l z_{s\theta_l} z_{c\phi_{l+1}} + z_{c\theta_l} z_{s\phi_{l+1}}, \quad l \in 2,\ldots,m-2 \\
& z_{c\phi_{m-1}} = z_{c\theta_{m-1}} z_{c\theta_m} - c_{m-1} c_m z_{s\theta_{m-1}} z_{s\theta_m} \\
& z_{s\phi_{m-1}} = c_{m-1} z_{s\theta_{m-1}} z_{c\theta_m} + c_m z_{c\theta_{m-1}} z_{s\theta_m} \\
& z_{c\theta_h}^2 + z_{s\theta_h}^2 = 1 \quad h \in 0,\ldots,m \\
& z_{c\phi_l}^2 + z_{s\phi_l}^2 = 1 \quad l \in 2,\ldots,m-1 \\
& r_h^l(z_{c\theta_h}, z_{s\theta_h}) \le r_h \le r_h^u(z_{c\theta_h}, z_{s\theta_h}) \quad h \in 1,\ldots,m \\
& a_h^c z_{c\theta_h} + a_h^s z_{s\theta_h} \ge b_h \quad h \in 1,\ldots,m \}
\end{aligned} \tag{31}
$$

This is a polynomial optimization problem with set-containment constraints of the form (17). Following the process outlined in section II-D it is easily turned into a semidefinite program involving SOS conditions. Due to page limits we do not list the SOS program.

B. Minkowski Sum of Complex Sets

Calculating the Minkowski sum of complex sets is more involved as we must convert between polar and Euclidean coordinates. We develop an outer approximation of (7) in which points belonging to sets $\mathcal{R}_k$, $k \in \mathcal{K}$, are inverted. The resulting Euclidean coordinates (x, y) are given by:

$$
\begin{aligned}
x_j = r_j \cos\theta_j, y_j = r_j \sin\theta_j, \quad & \forall r_j \in \mathcal{R}_j, j \in \mathcal{J}, \\
x_k = \cos\theta_k / r_k, y_k = -\sin\theta_k / r_k, \quad & \forall r_k \in \mathcal{R}_k, k \in \mathcal{K}.
\end{aligned}
$$

We sum the Euclidean coordinates to obtain the point $(x_0 + iy_0) \in \mathcal{S}_{\bigoplus + \bigoplus^{-1}}$. We then must determine the angle θ_0 and non-negative radial distance of this point. This is achieved with the following equations:

$$
\begin{aligned}
x_0 = \sum_{h\in\mathcal{J}\cup\mathcal{K}} x_h, \quad y_0 = \sum_{h\in\mathcal{J}\cup\mathcal{K}} y_h \\
x_0 = r_0\cos\theta_0, \quad x_0 = y_0\cos\theta_0, \quad r_0 \ge 0
\end{aligned}
$$

The optimization problem is then:

$$
\begin{aligned}
\min_{\alpha^l,\alpha^u} & \int_{\theta^l}^{\theta^u} r^u(\theta,\alpha^u) - r^l(\theta,\alpha^l) d\theta \\
\text{s.t. } & r^l(\theta_0,\alpha^l) \le r_0 \le r^u(\theta_0,\alpha^u), \\
& \forall (\theta_{[0:m]}, r_{[0:m]}, x_{[0:m]}, y_{[0:m]}) \in \mathcal{X}
\end{aligned} \tag{32}
$$

where $\mathcal{X}$ is the semi-algebraic set

$$
\begin{aligned}
\mathcal{X} = \{ & (\theta_{[0:m]}, r_{[0:m]}, x_{[0:m]}, y_{[0:m]}) : \\
& r_0 \ge 0, r_0\cos\theta_0 = x_0, r_0\sin\theta_0 = y_0 \\
& x_0 = \sum_{h\in\mathcal{J}\cup\mathcal{K}} x_h, y_0 = \sum_{h\in\mathcal{J}\cup\mathcal{K}} y_h \\
& r_j\cos\theta_j = x_j, r_j\sin\theta_j = y_j \forall j \in \mathcal{J} \\
& r_k x_k = \cos\theta_k, r_k y_k = -\sin\theta_k \forall k \in \mathcal{K} \\
& r_j^l(\theta_j) \le r_j \le r_j^u(\theta_j), \theta_j^l \le \theta_j \le \theta_j^u, \forall j \in \mathcal{J} \\
& r_k^l(\theta_k) \le r_k \le r_k^u(\theta_k), \theta_k^l \le \theta_k \le \theta_k^u, \forall k \in \mathcal{K} \}
\end{aligned} \tag{33}
$$

Following a similar procedure as before, we first integrate the objective to eliminate the dependence on θ. We then introduce new variables for the trigonometric terms:

$$
\begin{aligned}
z_{c\theta_h} = \cos\theta_h, z_{s\theta_h} = \sin\theta_h \\
z_{c\theta_h}^2 + z_{s\theta_h}^2 = 1 \qquad \forall h \in 0 \cup \mathcal{J} \cup \mathcal{K}
\end{aligned}
$$

With this change of variables the optimization problems is rewritten as:

$$
\begin{aligned}
\min_{\alpha^l,\alpha^u} \; & c_l^T\alpha^l + c_u^T\alpha^u \\
\text{s.t.} \quad & r^l(z_{c\theta_0}, z_{s\theta_0}, \alpha^l) \le r_0 \\
& r^u(z_{c\theta_0}, z_{s\theta_0}, \alpha^u) \ge r_0 \\
& \forall (z_{c\theta_{[0:m]}}, z_{s\theta_{[0:m]}}, r_{[0:m]}, x_{[0:m]}, y_{[0:m]}) \in \mathcal{W}
\end{aligned} \tag{34}
$$

where $\mathcal{W}$ is the semialgebraic set:

$$
\begin{aligned}
\mathcal{W} = \{ & (z_{c\theta_{[0:m]}}, z_{s\theta_{[0:m]}}, r_{[0:m]}, x_{[0:m]}, y_{[0:m]}) : \\
& r_0 \ge 0, x_0 = \sum_{h\in\mathcal{J}\cup\mathcal{K}} x_h, \quad y_0 = \sum_{h\in\mathcal{J}\cup\mathcal{K}} y_h \\
& r_0 z_{c\theta_0} = x_0, r_0 z_{s\theta_0} = y_0, \\
& r_j z_{c\theta_j} = x_j, r_j z_{s\theta_j} = y_j, \quad \forall j \in \mathcal{J} \\
& r_k x_k = z_{c\theta_k}, r_k z_{s\theta_k} = -y_k, \quad \forall k \in \mathcal{K} \\
& z_{c\theta_h}^2 + z_{s\theta_h}^2 = 1 \quad h \in 0,\ldots,m \\
& r_h^l(z_{c\theta_h}, z_{s\theta_h}) \le r_h \le r_h^u(z_{c\theta_h}, z_{s\theta_h}) \quad h \in 1,\ldots,m \\
& a_h^c z_{c\theta_h} + a_h^s z_{s\theta_h} \ge b_h \quad h \in 1,\ldots,m \}
\end{aligned} \tag{35}
$$

As before, applying the $\mathcal{S}$-procedure followed by replacing the non-negativity conditions with SOS constraints yields a semidefinite optimization problem.

C. Determining Angle Interval

As stated in Assumption 2, we assume that we know the exact set of angles Θ contained in the set $\mathcal{S}_\bullet$. For the Minkowski sum this is not readily calculated. Here we outline an iterative approach for conservatively bounding Θ within an interval $\tilde{\Theta} = [\tilde{\theta}^l, \tilde{\theta}^u]$ such that $\Theta \subseteq \tilde{\Theta}$.

We initialize our estimate to $\tilde{\Theta} = [0, 2\pi]$. If Θ is a strict subset of this interval, then there exists an angle ψ such that $\psi \in \tilde{\Theta} \setminus \Theta$. Along this angle, there is no element of $\mathcal{S}_\bullet$ constraining $r^l(\psi, \alpha^u)$ and $r^u(\psi, \alpha^u)$. Thus our objective which minimizes r^u and maximizes r^l would be unbounded. To resolve this, we add known upper and lower bounds, $\bar{r}^l$ and $\underline{r}_u$, on $r^l(\cdot, \alpha^l)$ and $r^u(\cdot, \alpha^u)$, respectively. For $r^u(\theta, \alpha^u)$ we use the trivial lower bound of zero. For $r^l(\theta, \alpha^l)$ we make use of the bound provided by Lemma 2. To enforce these bounds, we augment problem (32) with the following conditions:

$$r^l(z_{c\theta_0}, z_{s\theta_0}, \alpha^l) \leq \sum_{j\in\mathcal{J}} (\bar{r}_j^u) + \sum_{k\in\mathcal{K}} (\underline{r}_k^l)^{-1} \,\forall\, (z_{c\theta_0}, z_{s\theta_0}) \in \mathcal{V}$$
$$r^u(\theta, \alpha^u) \geq 0 \quad \forall\, (z_{c\theta_0}, z_{s\theta_0}) \in \mathcal{V} \tag{36}$$

where

$$\mathcal{V} = \{(z_{c\theta_0}, z_{s\theta_0}) | z_{c\theta_0}^2 + z_{s\theta_0}^2 = 1\}. \tag{37}$$

We solve this augmented problem and then examine the bounding contours $r^l(\theta, \alpha^l, r^u(\theta, \alpha^l$. For any angles ψ at which the lower bound exceeds the upper bound ($r^l(\psi) > r^u(\psi)$), we can conclude that $\psi \notin \Theta$ and update our angle interval $\tilde{\Theta}$ appropriately. We then repeat this process, solving the augmented problem with the tighter approximation of Θ, examining the resulting bounds to further tighten the interval $\tilde{\Theta}$ and repeating. We stop once the returned bounds satisfy ($r^l(\theta) \leq r^u(\theta) \,\forall\, \theta \in \tilde{\Theta}$).

As an aside we note that determining the range of angles in $\mathcal{S}_{\bigoplus + \bigoplus^{-1}}$ can also be solved via global optimization methods using branch-and-bound techniques. Our initial experience with this approach yielded solutions in under a second for the examples considered herein.

IV. Examples

A. Minkowski Product

Consider the following set formed from Minkowski products and division:

$$\mathcal{S} = \mathcal{R}_1 \bigotimes \mathcal{R}_2 \bigotimes (\mathcal{R}_3 \bigotimes \mathcal{R}_4)^{-1} \tag{38}$$

where each set $\mathcal{R}_j$ is as shown in Figure 1.

$$\mathcal{R}_j = \{re^{i\theta} | 1 + \frac{1}{4}\sin\theta \leq r \leq 1.5 - \frac{1}{4}\cos\theta, 0 \leq \theta \leq \frac{\pi}{3}\} \tag{39}$$
$$j = 1, 2, 3, 4$$

By inspection, the possible angles of $\mathcal{S}$ are $\Theta \in [-\frac{2\pi}{3}, \frac{2\pi}{3}]$. Limiting ourselves to 4th-order contours we solve the SOS form of (30). Figure 3 plots the resulting contour along with points sampled from $\mathcal{S}$. Empirically the outer approximation is close to the true contour suggested by the sampled points.

B. Minkowski Sum

Using the same sets as in the previous example, we now find an outer approximation for the following Minkowski sum

$$\mathcal{S} = \mathcal{R}_1 \bigoplus \mathcal{R}_2 \bigoplus (\mathcal{R}_3)^{-1} \bigoplus (\mathcal{R}_4)^{-1} \tag{40}$$

We do not know the possible range Θ of $\mathcal{S}$ so we use the iterative approach previously outlined. For the given set operation, it is straight-forward to obtain an upper bound on r of $2 \times 1.75 + 2 \times (0.75)^{-1} = 6.1667$. We impose the conditions $r^l(\theta) \leq 6.1667$ and $r^u(\theta) \geq 0$. We then solve the SOS form of (34) conservatively assuming $\theta^l = 0, \theta^u = 2\pi$ and augmenting the problem with the bounds of (36). Figure 4 plots the resulting bounds as a function of θ. Examining the plot it is seen that $r^l(\theta) \leq r^u(\theta)$ for $\theta \in [-27.1°, 40.6°]$.

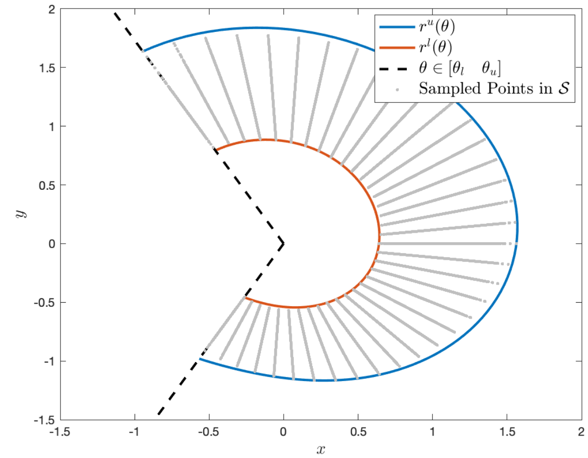

Fig. 3. Outer Bound of Minkowski Product and Ratio (38)

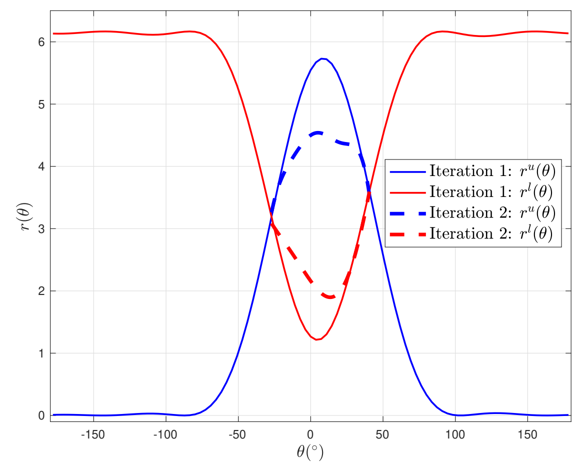

Fig. 4. Iterative Bounds of Minkowski Sum

Outside of this interval, $r^l(\theta)$ approaches its upper bound of 6.1667 and $r^u(\theta)$ approaches its lower bound of zero. We again solve the problem now with $\theta^l = -27.1°, \theta^u = 40.6°$ and obtain the dashed lines in Figure 4. With the new bounds, $r^l(\theta) \leq r^u(\theta)$ for $\theta \in [-27.1°, 40.4°]$. We again solve the problem with our slightly tightened angle interval. The resulting bounds have $r^l(\theta) \leq r^u(\theta)$ for all $\theta \in [-27.1°, 40.4°]$. At this point we can no longer improve our estimate of Θ so we stop. Figure 5 plots the resulting contour along with points sampled from $\mathcal{S}$. Empirically the outer approximation is close to the true contour suggested by the sampled points.

C. Implementation Details

All examples were solved using MOSEK [13] in conjunction with the SOS module of YALMIP [14].

V. Conclusions

In this work we developed optimization-based methods for finding outer approximations of Minkowski sums and

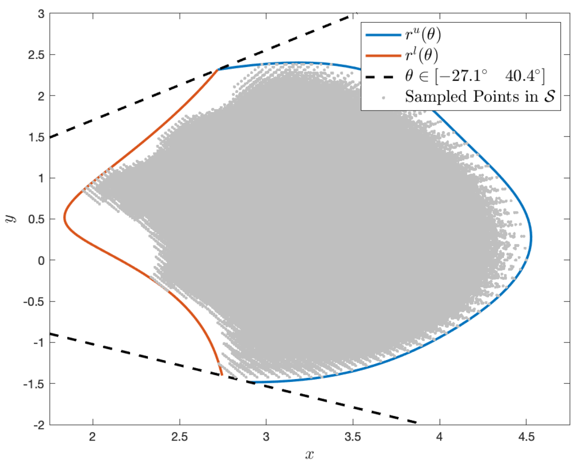

Fig. 5. Outer Bound of Minkowski Sum (40)

products of complex sets. We make few assumptions about the shape and number of these sets. Thus our method is quite general. By introducing some variable transformations, we posed this problem as a sum-of-squares optimization problem which is readily solved by off-the-shelf solvers. Examples provided empirical evidence that the resulting approximations are good. In future work we plan to consider cases in which complex sets are more naturally described using Euclidean coordinates. In addition we plan to leverage these results to develop new methods for certifying the robust stability of networked dynamic systems using Nyquist-like criteria [15].

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