Linear-Convex Optimal Steady-State Control

Liam S. P. Lawrence *Student Member, IEEE*, John W. Simpson-Porco, *Member, IEEE*, and Enrique Mallada *Member, IEEE*

***Abstract*—We consider the problem of designing a feedback controller for a multivariable linear time-invariant system which regulates an arbitrary system output to the solution of a constrained convex optimization problem despite parametric modelling uncertainty and unknown constant exogenous disturbances; we term this the linear-convex optimal steady-state control problem. We introduce the notion of an optimality model, and show that the existence of an optimality model is sufficient to reduce the problem to a stabilization problem; several instances of optimality models are given under various assumptions. This yields a constructive design framework for optimal steady-state control that unifies and extends many existing design methods in the literature. We illustrate our contributions via several numerical examples, including an application to optimal frequency control of power networks, where our methodology recovers centralized and distributed controllers reported in the recent literature.**

***Index Terms*—Reference tracking and disturbance rejection, output regulation, convex optimization, online optimization**

I. INTRODUCTION

Many engineering systems are required to operate at an "optimal" steady-state defined by the solution of a constrained optimization problem that seeks to minimize operational costs while satisfying equipment constraints. Consider, for example, the problem of optimizing the production setpoints of generators in an electric power system while maintaining supply-demand balance and system stability. The current approach involves a time-scale separation between the optimization and control objectives: optimal generation setpoints are computed offline using demand projections and a model of the network, then the operating points are dispatched as reference commands to local controllers at each generation site [1]–[3]. This process is repeated with a fixed update rate: a new optimizer is computed, dispatched, and tracked. If the supply and demand of power changes on a time scale that is slow compared to the update rate, then this method is adequate.

If the optimizer changes rapidly, however, as is the case for power networks with a high penetration of renewable energy sources, the conventional approach is inefficient [4]. Profit is reduced as a result of operating in a sub-optimal regime between optimizer updates. In the rapidly-changing optimizer case, then, it would be advantageous to eliminate the time-scale separation by combining the local generator controllers with an online optimization algorithm, so that the optimal operating condition could be tracked in real time. This is the direction of much recent research in power system control [5]–[13].

The same theme of real-time regulation of system variables to optimal values emerges in diverse areas. Fields of application besides the power network control example mentioned already include network congestion management [14], [15], chemical processing [16], wind turbine power capture [17], and temperature regulation in energy-efficient buildings [18]. The breadth of applications motivates the need for a *general* theory and design procedure for controllers that regulate a plant to a maximally efficient operating point defined by an optimization problem, even as the optimizer changes over time due to changing market prices, disturbances to the plant dynamics, and operating constraints that depend on external variables. We refer to the problem of designing such a controller as the *optimal steady-state (OSS) control problem*.

A number of recent publications have formulated problem statements and solutions for variants of the OSS control problem [19]–[26]. Many of the currently-proposed controllers, however, have limited applicability: some solutions only apply to systems of a special form [23]; some require asymptotic stability of the uncontrolled plant [25], [26]; some attempt to optimize only the steady-state input [21] or output [20], [24], [27], [28] alone; some apply only to equality-constrained [26] or unconstrained optimization problems [29]; and in all cases, the effects of parametric modelling uncertainty are omitted from consideration.

Broadly speaking, these design methodologies consist of modifying an off-the-shelf optimization algorithm to accept system measurements; the algorithm then produces a converging estimate of the optimal steady-state control input, yielding a feedback controller. This procedure, while modular, unnecessarily restricts the design space of dynamic controllers. Moreover, none of the reported approaches adequately consider (i) the effect of system dynamics on achievable steady-states, (ii) the dynamic performance of the closed-loop system, and (iii) the effect of system uncertainty on the optimal steady-state. Our goal in this paper is to present a holistic analysis and design framework which emphasizes these issues.

A. Contributions

The main contribution of this paper is a complete constructive solution to the *linear-convex OSS control problem*, in which the plant is a causal linear time-invariant (LTI) system with constant parametric uncertainty, the optimization problem is convex, and the disturbances are constant in time.

L. S. P. Lawrence and J. W. Simpson-Porco are with the Department of Electrical and Computer Engineering, University of Waterloo, Waterloo ON, N2L 3G1 Canada. Email:{lsplawre, jwsimpson}@uwaterloo.ca.

E. Mallada is with the Department of Electrical and Computer Engineering, The Johns Hopkins University, 3400 N. Charles Street, Baltimore, MD 21218, Email: mallada@jhu.edu

We acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC), [CGS M Program, Discovery Grant RGPIN-2017-04008].

Cette recherche a été financée par le Conseil de recherches en sciences naturelles et en génie du Canada (CRSNG), [Le Programme de BESC M, Le Programme de subventions à la découverte RGPIN-2017-04008].

This work was also supported in part by UW ECE start-up funding.

We show that the linear-convex OSS control problem can be reduced to a stabilization problem via a dynamic filter we call an *optimality model*. We present three optimality models, and discuss associated robustness considerations arising from parametric uncertainty in the plant model. We provide stabilizer existence results when the optimization problem is an equality-constrained quadratic program and discuss several design strategies for the case of a general convex optimization problem. Through a series of examples, we show that the OSS control framework is more general than several frameworks in the literature on feedback optimization of dynamical systems.

B. Notation

The symbol $\bullet$ in $\mathbb{R}^{\bullet\times\bullet}$ indicates that the dimension is unspecified. For a continuously differentiable map $f:\mathbb{R}^n\to\mathbb{R}$, $\nabla f:\mathbb{R}^n\to\mathbb{R}^n$ denotes its gradient. When the arguments of a function $f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ are separated by a semicolon, $\nabla f(x;y)$ refers to the gradient of f with respect to its first argument, evaluated at (x,y). The symbol $\mathbb{0}$ denotes a matrix or vector of zeros whose dimensions can be inferred from context. The symbol $\mathbb{1}_n$ denotes the n-vector of all ones. For scalars or column vectors $\{v_1,v_2,\ldots,v_k\}$, $\mathrm{col}(v_1,v_2,\ldots,v_k)$ is a column vector obtained by vertical concatenation of $v_1,\ldots,v_k$. For vectors α and β, the notation $\alpha\geq\beta$ indicates that every entry of α is greater than or equal to the corresponding entry of β. For symmetric matrices A and B, $A\succ B$ means $A-B$ is positive definite, while $A\succeq B$ means $A-B$ is positive semidefinite.

II. Background on Output Regulation

We will define the OSS control problem in terms of the *output regulation problem*. This section reviews the output regulation problem for an uncertain nonlinear plant subject to constant disturbances, and its solution via *integral control*. We draw upon [30] and [31, Section 12.3]. Consider a nonlinear plant

$$\begin{aligned}\dot{x} &= f(x,u,w,\delta)\,, \qquad x(0)\in\mathbb{R}^n\,,\\ e &= h_e(x,u,w,\delta)\,,\\ y_{\mathrm{m}} &= h_{\mathrm{m}}(x,u,w,\delta)\,,\end{aligned}\tag{1}$$

where $x\in\mathbb{R}^n$ is the state, $u\in\mathbb{R}^m$ is the control input, and $y_{\mathrm{m}}\in\mathbb{R}^{p_{\mathrm{m}}}$ is the vector of available measurements. The *error signal* $e\in\mathbb{R}^p$ consists of variables which should be driven to zero asymptotically using feedback control. The function f is assumed to be locally Lipschitz in x and continuous in u, w, and δ, while h_e and h_{m} are assumed to be continuous.

The vector $w\in W\subseteq\mathbb{R}^{n_w}$ is a constant exogenous input which might include disturbances to the plant dynamics or reference signals, and $\delta\in\boldsymbol{\delta}\subset\mathbb{R}^{n_\delta}$ is a vector representing parametric uncertainty in the plant model. We will often write "for every (w,δ)" as a shortened version of "for every $w\in W$ and $\delta\in\boldsymbol{\delta}$." When $\boldsymbol{\delta}=\{\mathbb{0}\}$ the plant model is precisely known, and we refer to this as the *nominal case*.

A general nonlinear feedback controller for (1) is given by

$$\begin{aligned}\dot{x}_{\mathrm{c}} &= f_{\mathrm{c}}(x_{\mathrm{c}},y_{\mathrm{m}})\,, \qquad x_{\mathrm{c}}(0)\in\mathbb{R}^{n_{\mathrm{c}}}\,,\\ u &= h_{\mathrm{c}}(x_{\mathrm{c}},y_{\mathrm{m}}).\end{aligned}\tag{2}$$

The function f_{c} is assumed to be locally Lipschitz in x_{c} and continuous in y_{m}, while h_{c} is assumed to be continuous.

The dynamics of the closed-loop system consist of (1) and (2). For a given (w,δ), the closed-loop system is said to be *well-posed* if there exists a unique and continuous solution for the state vector $(x(t),x_{\mathrm{c}}(t))$ and control input $u(t)$ for all $t\geq 0$ and for every initial condition $(x(0),x_{\mathrm{c}}(0))\in\mathbb{R}^n\times\mathbb{R}^{n_{\mathrm{c}}}$. The problem of *output regulation* is to design the feedback controller such that the closed-loop system is well-posed, stable, and such that the error signal e is driven to zero.

*Problem 2.1 (***Output Regulation***)***:** For the plant (1), design, if possible, a dynamic feedback controller of the form (2) such that for every (w,δ):

(i) the closed-loop system is well-posed;
(ii) the closed-loop system possesses a globally asymptotically stable equilibrium point;
(iii) for every initial condition $(x(0),x_{\mathrm{c}}(0))$ of the closed-loop system, the error signal asymptotically tends to zero, i.e., $\lim_{t\to\infty} e(t)=\mathbb{0}$. △

The output regulation problem has the well-known solution of *integral control* when the error e is measurable, i.e., when e may be computed from y_{m}. The controller takes the form

$$\dot{\eta} = e\,, \tag{3a}$$
$$\dot{x}_{\mathrm{s}} = f_{\mathrm{s}}(x_{\mathrm{s}},\eta,y_{\mathrm{m}},e)\,, \tag{3b}$$
$$u = h_{\mathrm{s}}(x_{\mathrm{s}},\eta,y_{\mathrm{m}},e). \tag{3c}$$

The *integrator* (3a) ensures the error term e is zero in steady-state, while the *stabilizer* (3b)-(3c) is responsible for ensuring closed-loop stability. If the closed loop-system consisting of the plant (1) in feedback with the controller (3) is well-posed and possesses a globally asymptotically stable equilibrium point for every (w,δ), then (3) solves the output regulation problem [31, Section 12.3].

We can give necessary and sufficient conditions for the existence of a stabilizer for a *fixed* $\delta\in\boldsymbol{\delta}$ when the plant (1) is linear and time-invariant. Consider applying integral control to solve the output regulation problem with the plant

$$\begin{aligned}\dot{x} &= A(\delta)x+B(\delta)u+B_w(\delta)w\,, \qquad x(0)\in\mathbb{R}^n\,,\\ e &= C(\delta)x+D(\delta)u+Q(\delta)w\,,\\ y_{\mathrm{m}} &= C_{\mathrm{m}}(\delta)x+D_{\mathrm{m}}(\delta)u+Q_{\mathrm{m}}(\delta)w.\end{aligned}\tag{4}$$

The series connection of the plant (4) and the integrator (3a) is called the *augmented plant*, and is given by

$$\begin{aligned}\dot{x} &= A(\delta)x+B(\delta)u+B_w(\delta)w\,,\\ \dot{\eta} &= C(\delta)x+D(\delta)u+Q(\delta)w\,,\\ y_{\mathrm{a}} &= \mathrm{col}(y_{\mathrm{m}},\eta)\,.\end{aligned}\tag{5}$$

The augmented plant has state $\mathrm{col}(x,\eta)$, control input u, and measured output y_{a}. As is well-known, stabilizability and detectability of (5) are necessary and sufficient for the existence of a dynamic stabilizer, and hence for solution of the output regulation problem. The following classic result [30, Theorem 1] can be established by applying the Popov-Belevitch-Hautus tests for stabilizability and detectability.

Theorem 2.2 (**Stabilizability and Detectability of Augmented Plant):** The augmented plant (5) is stabilizable and detectable for a fixed $\delta \in \boldsymbol{\delta}$ if and only if:

(i) $(C_{\mathrm{m}}(\delta), A(\delta), B(\delta))$ is stabilizable and detectable;

(ii) the matrix

$$\begin{bmatrix} A(\delta) & B(\delta) \\ C(\delta) & D(\delta) \end{bmatrix}$$

has full row rank. △

III. Problem Statement

In the linear-convex optimal steady-state control problem, our objective is to design a feedback controller for a linear time-invariant plant so that a specified output is asymptotically driven to a cost-minimizing steady-state, determined by the solution of a convex optimization problem. We can phrase this problem in the language of output regulation by defining an appropriate error signal.

The plant under consideration is a linear time-invariant system with parametric uncertainty in the matrices:

$$\begin{aligned} \dot{x} &= A(\delta)x + B(\delta)u + B_w(\delta)w\,, \qquad x(0) \in \mathbb{R}^n\,, \\ y &= C(\delta)x + D(\delta)u + Q(\delta)w\,, \\ y_{\mathrm{m}} &= h_{\mathrm{m}}(x, u, w, \delta). \end{aligned} \tag{6}$$

For reasons that will become clear (see Assumption 4.8), the measurements y_{m} are permitted to be general nonlinear functions of state, input, disturbance, and uncertainty. The vector $y \in \mathbb{R}^p$ is the *optimization output*, containing states, tracking errors, and control inputs that should be driven to cost-minimizing values in equilibrium. The optimal value of y is determined by the solution of a convex optimization problem:

$$y^\star(w, \delta) := \operatorname*{argmin}_{y \in \mathbb{R}^p} \{ f_0(y; w) \ \mid y \in \mathcal{C}(w, \delta) \}\,. \tag{7}$$

We will elaborate on the construction of the feasible set $\mathcal{C}(w, \delta)$ shortly. We make a number of assumptions regarding the problem (7).

Assumption 3.1 (**Optimization Problem Assumptions):**

(i) the optimizer $y^\star$ exists and is unique for every (w, δ);

(ii) the objective function $f_0 : \mathbb{R}^p \times W \to \mathbb{R}$ is differentiable and convex in y for each $w \in W$;

(iii) the feasible region $\mathcal{C}(w, \delta)$ is convex and has non-empty relative interior for every (w, δ).

Our objective in linear-convex OSS control is to drive the optimization output y of the plant (6) to the solution $y^\star(w, \delta)$ of the convex optimization problem (7) using a feedback controller, while also ensuring well-posedness and stability of the closed-loop system. We can rephrase these goals in the language of output regulation.

Problem 3.2 (**Linear-Convex Optimal Steady-State Control):** The linear-convex optimal steady-state control problem is the output regulation problem, Problem 2.1, with plant (6) and error signal $e = y - y^\star(w, \delta)$, i.e., with $h_e(x, u, w, \delta) := C(\delta)x + D(\delta)u + Q(\delta)w - y^\star(w, \delta)$. △

For brevity, we drop the phrase "linear-convex" for the remainder of this paper, and simply refer to "OSS control."

Remark 3.3 (**Relation to Optimal Control):** The OSS control problem appears similar to a classical optimal tracking control problem; however, the two are distinct in their assumptions and demands. In the latter, one would minimize a dynamic cost criteria over system trajectories, either on a finite or infinite time horizon, possibly with terminal costs. The exact solution to this problem is computationally intensive to compute and requires a perfect dynamic system model. Moreover, the resulting feedback policy will require measurement of the state and of any exogeneous disturbances. The OSS control problem is less demanding; as a result, its solution requires fewer assumptions. We ask only for optimal behaviour asymptotically, not optimal trajectories. As a result, we do not need to assume the plant state and disturbances are measurable, nor do we require a perfect dynamic plant model. △

We now elaborate on the structure of the feasible region $\mathcal{C}(w, \delta)$ of the optimization problem (7). A necessary condition for solvability of the output regulation problem is the existence of a *forced equilibrium point* $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ for which the error output is zero [31]. We will embed constraints in the optimization problem to *guarantee* that such a forced equilibrium point exists.

Let $\overline{Y}(w, \delta)$ be the set of optimization outputs achievable from a forced equilibrium:

$$\begin{aligned} \overline{Y}(w, \delta) := \{ \bar{y} \in \mathbb{R}^p \mid & \text{ there exists an } (\bar{x}, \bar{u}) \text{ such that} \\ & \mathbb{0} = A(\delta)\bar{x} + B(\delta)\bar{u} + B_w(\delta)w \\ & \bar{y} = C(\delta)\bar{x} + D(\delta)\bar{u} + Q(\delta)w \}\,. \end{aligned} \tag{8}$$

We rewrite $\overline{Y}(w, \delta)$ in algebraic form so that we may include membership in $\overline{Y}(w, \delta)$ as a constraint of the optimization problem in standard equality form. For each (w, δ), the set $\overline{Y}(w, \delta)$ is an affine subspace of $\mathbb{R}^p$. It may therefore be written as the sum of a (non-unique) "offset vector" and a unique subspace, which we denote by $\mathbf{sub}(\overline{Y}(w, \delta))$. In the following lemma, we construct a matrix $G(\delta)$ whose columns span this unique subspace.

Lemma 3.4 (**Construction of $\boldsymbol{G(\delta)}$):** Fix a $\tilde{y}(w, \delta) \in \overline{Y}(w, \delta)$. If $\mathcal{N}(\delta) \in \mathbb{R}^{(n+m)\times \bullet}$ is a matrix such that

$$\mathbf{range}\,\mathcal{N}(\delta) = \mathbf{null}\begin{bmatrix} A(\delta) & B(\delta) \end{bmatrix},$$

then the columns of the matrix

$$G(\delta) := \begin{bmatrix} C(\delta) & D(\delta) \end{bmatrix} \mathcal{N}(\delta) \in \mathbb{R}^{p \times \bullet} \tag{9}$$

span the subspace $\mathbf{sub}(\overline{Y}(w, \delta))$. △

The proof is straightforward and is omitted. Note that when $A(\delta)$ is invertible, one may select

$$\mathcal{N}(\delta) := \begin{bmatrix} -A(\delta)^{-1}B(\delta) \\ I_m \end{bmatrix}$$

which yields $G(\delta) = -C(\delta)A(\delta)^{-1}B(\delta) + D(\delta)$. This is precisely the DC gain matrix of the $u \to y$ channel for the plant (6). One may think of $G(\delta)$ in (9) as a *generalized DC gain matrix* which one can compute regardless of whether or not $A(\delta)$ is invertible. Compared to [25], [26], we do not require $A(\delta)$ to be Hurwitz.

With Lemma 3.4, we have that

$$\bar{y} \in \overline{Y}(w,\delta) \quad \Longleftrightarrow \quad \bar{y} = \tilde{y}(w,\delta) + G(\delta)v \tag{10}$$

for some vector $v \in \mathbb{R}^{\bullet}$. Now let $G_{\perp}(\delta) \in \mathbb{R}^{\bullet \times p}$ be any full-row-rank matrix satisfying $\mathbf{null}\, G_{\perp}(\delta) = \mathbf{range}\, G(\delta)$. Then from (10), one finds that

$$\overline{Y}(w,\delta) = \{ y \in \mathbb{R}^p \mid G_{\perp}(\delta) y = b(w,\delta) \} .$$

where $b(w,\delta) := G_{\perp}(\delta)\tilde{y}(w,\delta)$. We will see shortly that, for our controller design, the matrix $G_{\perp}(\delta)$ is important and the vector $b(w,\delta)$ is unimportant. We can now write the optimization problem of (7) explicitly in standard form:

$$\begin{aligned}
&\underset{y \in \mathbb{R}^p}{\text{minimize}} && f_0(y; w) && \text{(11a)}\\
&\text{subject to} && G_{\perp}(\delta) y = b(w,\delta) && \text{(11b)}\\
& && Hy = Lw && \text{(11c)}\\
& && f_i(y; w) \leq 0\,, \quad i \in \{1, \ldots, n_{\mathrm{ic}}\}. && \text{(11d)}
\end{aligned}$$

The constraint (11b) is the equilibrium constraint just discussed. The constraints (11c) represent n_{ec} engineering equality constraints determined by the matrices $H \in \mathbb{R}^{n_{\mathrm{ec}} \times p}$ and $L \in \mathbb{R}^{n_{\mathrm{ec}} \times n_w}$. The constraints (11d) are n_{ic} engineering inequality constraints which should be satisfied in the desired steady-state. To ensure the optimization problem is convex, the functions $f_i : \mathbb{R}^p \times W \to \mathbb{R}$ must be convex in y for each $w \in W$ for all $i \in \{1, \ldots, n_{\mathrm{ic}}\}$. The matrices H, L and the functions f_i are part of the design specification, and are therefore not subject to parametric uncertainty.

*Remark 3.5 (***Necessity of Equilibrium Constraints***):* The steady-state constraints (11b) ensure *compatibility* between the plant and the specified optimization problem. Failing to include these constraints can result in an instance of the OSS control problem in which $y^\star(w,\delta) \notin \overline{Y}(w,\delta)$ for some (w,δ), i.e., in which the optimizer of (11) might be *inconsistent* with steady-state operation of the plant (6). If this is the case, Problem 3.2 is insolvable. For an example of what occurs when we fail to include the equilibrium constraints, see Section V-B. △

By Assumption 3.1, the feasible region of (11) has non-empty relative interior for all (w,δ) and thus the *Karush-Kuhn-Tucker (KKT) conditions* are necessary and sufficient for optimality [32, Sections 5.2.3 and 5.5.3]. For each (w,δ), the optimal solution $y^\star \in \mathbb{R}^p$ is characterized as the unique vector such that $y^\star$ is feasible for (11) and there exist $\lambda^\star \in \mathbb{R}^r$, $\mu^\star \in \mathbb{R}^{n_{\mathrm{ec}}}$, and $\nu^\star \in \mathbb{R}^{n_{\mathrm{ic}}}$ such that $(y^\star, \lambda^\star, \mu^\star, \nu^\star)$ satisfy

$$\mathbb{0} = \nabla f_0(y^\star; w) + G_{\perp}(\delta)^{\mathsf{T}} \lambda^\star + H^{\mathsf{T}} \mu^\star + \sum_{i=1}^{n_{\mathrm{ic}}} \nu_i^\star \nabla f_i(y^\star; w) \tag{12a}$$

$$0 = \nu_i^\star f_i(y^\star; w)\,, \quad \nu_i^\star \geq 0\,, \quad i \in \{1, \ldots, n_{\mathrm{ic}}\}. \tag{12b}$$

IV. Constructive Solutions for Optimal Steady-State Control

Since the OSS control problem is an output regulation problem, integral control appears to be a natural solution. The barrier to this approach is that the error signal of the OSS control problem, $e = y - y^\star(w,\delta)$, is unmeasurable since (w,δ) are unknown and the mapping from (w,δ) to the optimizer $y^\star$ may be unknown. Integral control requires a measurable error signal — see (3a). Our design framework uses a dynamic filter called an *optimality model* to convert the OSS control problem to a related output regulation problem with measurable error. One then solves this output regulation problem using an integral controller. An optimality model therefore reduces the OSS control problem to a stabilization problem.

A. Optimality Models

An optimality model is a filter applied to the measured output of the plant that produces a signal ϵ that acts as a proxy for the optimality error $e = y - y^\star(w,\delta)$. To make this idea precise, consider a filter (φ, h_ϵ) with state $\xi \in \mathbb{R}^{n_\xi}$, input y_{m}, output $\epsilon \in \mathbb{R}^{n_\epsilon}$, and dynamics

$$\begin{aligned}
\dot{\xi} &= \varphi(\xi, y_{\mathrm{m}})\,,\\
\epsilon &= h_\epsilon(\xi, y_{\mathrm{m}}).
\end{aligned} \tag{13}$$

*Definition 4.1 (***Optimality Model***):* The filter (13) is said to be an *optimality model* (for the OSS control problem, Problem 3.2) if the following implication holds: if the triple $(\bar{x}, \bar{\xi}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^m$ satisfies

$$\begin{aligned}
\mathbb{0} &= A(\delta)\bar{x} + B(\delta)\bar{u} + B_w(\delta) w\\
\mathbb{0} &= \varphi(\bar{\xi}, h_{\mathrm{m}}(\bar{x}, \bar{u}, w, \delta))\\
\mathbb{0} &= h_\epsilon(\bar{\xi}, h_{\mathrm{m}}(\bar{x}, \bar{u}, w, \delta))
\end{aligned} \tag{14}$$

then the pair $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfies

$$y^\star(w,\delta) = C(\delta)\bar{x} + D(\delta)\bar{u} + Q(\delta) w. \quad \triangle$$

In the OSS control framework, we place the plant and optimality model in series and attempt to solve the output regulation problem with ϵ as the (measurable) error signal. Doing so with an integral controller converts the OSS control problem to the problem of stabilizing the *augmented plant*

$$\dot{x} = A(\delta)x + B(\delta)u + B_w(\delta)w\,, \tag{15a}$$

$$\dot{\xi} = \varphi(\xi, h_{\mathrm{m}}(x, u, w, \delta))\,, \tag{15b}$$

$$\dot{\eta} = \epsilon := h_\epsilon(\xi, h_{\mathrm{m}}(x, u, w, \delta)) \tag{15c}$$

using a *stabilizer*

$$\dot{x}_{\mathrm{s}} = f_{\mathrm{s}}(x_{\mathrm{s}}, \eta, \xi, y_{\mathrm{m}}, \epsilon)\,, \tag{16a}$$

$$u = h_{\mathrm{s}}(x_{\mathrm{s}}, \eta, \xi, y_{\mathrm{m}}, \epsilon). \tag{16b}$$

This design framework, depicted in Figure 1, is justified by the following theorem, a proof of which may be found in the appendix.

*Theorem 4.2 (***Reduction of OSS to Stabilization***):* Suppose that (φ, h_ϵ) is an optimality model. If the stabilizer $(f_{\mathrm{s}}, h_{\mathrm{s}})$ is designed such that the closed-loop system (15)–(16) is well-posed and possesses a globally asymptotically stable equilibrium point for every (w,δ), then the controller (15b), (15c), (16a), (16b) solves the OSS control problem. △

Solving the OSS control problem therefore amounts to (i) constructing an optimality model and (ii) designing (if possible) a stabilizer for the augmented plant. We explore optimality model and stabilizer designs in Sections IV-B and IV-C, respectively.

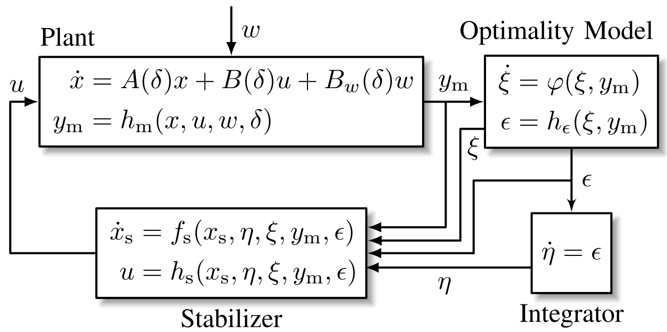


Fig. 1: The OSS control framework. The optimality model produces a proxy error signal ϵ for the tracking error $y - y^\star(w, \delta)$, the integrator enforces $\epsilon = \mathbb{0}$ in equilibrium, and the stabilizer ensures well-posedness and stability of the closed-loop system. Feedback from the stabilizer to the optimality model can also be included, but is omitted for simplicity.

B. Optimality Model Design

1) The Gradient Condition: According to Definition 4.1, an optimality model encodes sufficient conditions for optimality when it is in equilibrium with the plant and its output ϵ is held at zero. We can incorporate the KKT conditions — which are sufficient for optimality under our assumptions — in an optimality model for this purpose.

Note that the gradient condition (12a) involves the dual variable $\lambda^\star$ associated with the equilibrium constraints. Dual variables associated with equality constraints are typically calculated using an integrator on the constraint violation: see [20, Equation (4a)], or [23, Equation (8f)], for example. Integrating the equilibrium constraint violation $G_\perp(\delta)y - b(w, \delta)$ is impossible since (w, δ) are unknown. Luckily, doing so is unnecessary, as the plant already enforces this constraint in steady-state.

We now describe how to incorporate the gradient condition (12a) into an optimality model without calculating $\lambda^\star$. In the nominal case (when $\boldsymbol{\delta} = \{\mathbb{0}\}$) the following constructions may be directly applied. If we additionally consider uncertainty, then we must be more careful; we will shortly introduce robustness conditions to ensure that an optimality model can be constructed without knowledge of δ.

Let $G(\delta)$ be the matrix of Lemma 3.4. Recall that $\mathbf{null}\, G_\perp(\delta) = \mathbf{range}\, G(\delta)$; by taking the orthogonal complement of both sides, it follows that $\mathbf{range}\, G_\perp(\delta)^\mathsf{T} = \mathbf{null}\, G(\delta)^\mathsf{T}$. Therefore, the existence of a $(y^\star, \lambda^\star, \mu^\star, \nu^\star)$ satisfying (12a) is equivalent to the existence of a $(y^\star, \mu^\star, \nu^\star)$ satisfying

$$G(\delta)^\mathsf{T}\left(\nabla f_0(y^\star; w) + H^\mathsf{T}\mu^\star + \sum_{i=1}^{n_{\mathrm{ic}}} \nu_i^\star \nabla f_i(y; w)\right) = \mathbb{0}. \tag{17}$$

The left-hand side of this equation is a natural choice for inclusion in the proxy error signal ϵ, since driving ϵ to zero will then enforce the gradient KKT condition. The only major obstruction is the presence of the uncertain parameters δ. The uncertain parameters do not pose an issue when it is possible to construct the matrix $G(\delta)$ *without knowledge of* δ. We will elaborate on this idea shortly.

First however, note that we could also rewrite the gradient condition in a second, equivalent, manner. Define a matrix $T(\delta)$ such that

$$\mathbf{range}\, T(\delta) = \mathbf{null} \begin{bmatrix} G_\perp(\delta) \\ H \end{bmatrix}. \tag{18}$$

The existence of a $(y^\star, \lambda^\star, \mu^\star, \nu^\star)$ satisfying (12a) is equivalent to the existence of a $(y^\star, \nu^\star)$ satisfying

$$T(\delta)^\mathsf{T}\left(\nabla f_0(y^\star; w) + \sum_{i=1}^{n_{\mathrm{ic}}} \nu_i^\star \nabla f_i(y^\star; w)\right) = \mathbb{0}. \tag{19}$$

This procedure can also be generalized by including only some rows of H in the construction of $T(\delta)$, leading to a hybrid between (17) and (19); the details are omitted. As we did with (17), we can make the expression on the left-hand side of (19) one of the components of an optimality model's error output; the only remaining barrier is to understand when an appropriate matrix $T(\delta)$ can be constructed without knowledge of δ.

2) Robust Subspaces: We just saw that we can rewrite the gradient condition (12a) as either (17) or (19), the left-hand sides of which are suitable for use in the error signal ϵ of an optimality model. We now explore conditions under which we can construct the matrices $G(\delta)$ or $T(\delta)$ without knowledge of δ.

*Definition 4.3 (***Robust Feasible Subspace (RFS)***):* The optimization problem (11) is said to satisfy the *robust feasible subspace (RFS) property* when there exists a fixed matrix T_0 such that

$$\mathbf{range}\, T_0 = \mathbf{null} \begin{bmatrix} G_\perp(\delta) \\ H \end{bmatrix} \quad \text{for all } \delta \in \boldsymbol{\delta}. \tag{20}$$

An even stronger property is the following.

*Definition 4.4 (***Robust Output Subspace (ROS)***):* Let $G(\delta)$ be the matrix of Lemma 3.4. The uncertain LTI plant (6) is said to satisfy the *robust output subspace (ROS) property* when there exists a fixed matrix G_0 such that

$$\mathbf{range}\, G_0 = \mathbf{range}\, G(\delta) \quad \text{for all } \delta \in \boldsymbol{\delta}. \tag{21}$$

*Remark 4.5 (***Relationship Between ROS and RFS***):* The ROS property implies the RFS property, but not conversely (see Section V-F for an example). Recalling that $\mathbf{null}\, G_\perp(\delta) = \mathbf{range}\, G(\delta)$ by definition, the existence of a G_0 satisfying (21) implies $\mathbf{null}\, G_\perp(\delta) = \mathbf{range}\, G_0$ for all $\delta \in \boldsymbol{\delta}$. Hence, any matrix T_0 with the property $\mathbf{range}\, T_0 = \mathbf{range}\, G_0 \cap \mathbf{null}\, H$ will satisfy (20). In the special case when the optimization problem (11) has no engineering equality constraints, the matrix T_0 of (20) satisfies $\mathbf{range}\, T_0 = \mathbf{range}\, G(\delta)$ for all $\delta \in \boldsymbol{\delta}$. In this case, the RFS property and the ROS property are equivalent, and one make take $T_0 = G_0$. △

*Remark 4.6 (***Nominal Case***):* In the nominal case (with $\boldsymbol{\delta} = \{\mathbb{0}\}$) the robust output subspace and robust feasible subspace properties automatically hold. △

A sufficient condition for the ROS property is as follows.

*Proposition 4.7 (***Robust Full Rank $\implies$ ROS***):* The LTI system (6) satisfies the ROS property with $G_0 := I_p$ if

$$\mathbf{rank} \begin{bmatrix} A(\delta) & B(\delta) \\ C(\delta) & D(\delta) \end{bmatrix} = n + p, \qquad \text{for all } \delta \in \boldsymbol{\delta}. \tag{22}$$

Proof: If (22) holds, then by (8) we have $\overline{Y}(w,\delta) = \mathbb{R}^p$ for all (w,δ). Hence $\mathbf{sub}(\overline{Y}(w,\delta)) = \mathbf{range}\, I_p = \mathbf{range}\, G(\delta)$ for all $\delta \in \boldsymbol{\delta}$. □

We emphasize that (22) is only a sufficient condition for the ROS property, and can hold only when the number of outputs to be optimized is less than or equal to the number of control inputs. From Theorem 2.2, we see that the rank condition (22) is a necessary assumption for solvability of the linear output regulation problem with error signal $e = C(\delta)x + D(\delta)u + Q(\delta)w$; however, the optimality models of Section IV-B3 will produce error signals that make the rank condition (22) *not necessary* for solvability of the OSS control problem. See Propositions 4.14, 4.15, and 4.16 for details.

3) Optimality Models: We are now ready to construct optimality models for OSS control. The options available to us depend on which of the two previously-defined subspace robustness properties hold. We shall require our measurement vector to contain some key information about the optimization problem (11).

*Assumption 4.8 (***Measurement Assumptions***):* The measurement vector y_{m} contains, at minimum, the constraint violations, $Hy - Lw$ and $f_i(y;w)$, and the gradients $\nabla f_0(y;w)$ and $\nabla f_i(y;w)$.

Incorporating the inequality constraints and associated dual variable conditions relies on a function $\varphi_\nu : \mathbb{R}^{n_{\mathrm{ic}}} \times \mathbb{R}^{n_{\mathrm{ic}}} \to \mathbb{R}^{n_{\mathrm{ic}}}$ that satisfies the following implication:

$$\varphi_\nu(\alpha,\beta) = \mathbb{0} \implies \alpha \geq 0\,,\ \beta \leq 0\,,\ \alpha^{\mathsf{T}}\beta = 0. \tag{23}$$

For example, let $\mathbf{max}(\cdot,\mathbb{0})$ take the elementwise maximum between its first argument and $\mathbb{0}$. The function $\varphi_\nu(\alpha,\beta) \coloneqq \mathbf{max}(\alpha+\beta,\mathbb{0}) - \alpha$ from [20, Equation (8)] satisfies (23), as does the function

$$\varphi_\nu(\alpha,\beta)_i = \begin{cases} \beta_i & \text{if } \alpha_i > 0 \\ \max\{0,\beta_i\} & \text{if } \alpha_i = 0 \end{cases}$$

which appears in saddle-point dynamics [33]. So that we may write the inequality constraints in a compact manner, define $\mathsf{f}(y;w) \coloneqq \mathrm{col}(f_1(y;w), f_2(y;w), \ldots, f_{n_{\mathrm{ic}}}(y;w))$.

*Proposition 4.9 (***Robust Feasible Subspace Optimality Model (RFS-OM)***):* Suppose the optimization problem (11) satisfies the robust feasible subspace property with the matrix T_0, and let φ_ν satisfy the implication (23). Then

$$\begin{aligned} \dot{\nu} &= \varphi_\nu(\nu, \mathsf{f}(y;w)) \\ \epsilon &= \begin{bmatrix} Hy - Lw \\ T_0^{\mathsf{T}}\left(\nabla f_0(y;w) + \sum_{i=1}^{n_{\mathrm{ic}}} \nu_i \nabla f_i(y;w)\right) \end{bmatrix} \end{aligned} \tag{24}$$

is an optimality model for the OSS control problem.

Proof: For each (w,δ), consider the solutions $(\bar{x},\bar{\nu},\bar{u})$ to:

$$\mathbb{0} = A(\delta)\bar{x} + B(\delta)\bar{u} + B_w(\delta)w \tag{25a}$$
$$\bar{y} = C(\delta)\bar{x} + D(\delta)\bar{u} + Q(\delta)w \tag{25b}$$
$$\mathbb{0} = \varphi_\nu(\bar{\nu}, \mathsf{f}(\bar{y};w)) \tag{25c}$$
$$\mathbb{0} = H\bar{y} - Lw \tag{25d}$$
$$\mathbb{0} = T_0^{\mathsf{T}}\left(\nabla f_0(\bar{y};w) + \sum_{i=1}^{n_{\mathrm{ic}}} \bar{\nu}_i \nabla f_i(\bar{y};w)\right). \tag{25e}$$

The equations (25) correspond to the equations (14) in the definition of an optimality model. We show that (25) imply the KKT conditions. The first two equations (25a) and (25b) imply $\bar{y} \in \overline{Y}(w,\delta)$, which is equivalent to the first set of equality constraints, (11b). The equation (25d) is the engineering equality constraint, (11c). The engineering inequality constraints (11d) and associated dual variable conditions (12b) are implied by (25c). Finally, because the robust feasible subspace property holds, (25e) implies the gradient condition (19) for any δ. Since the KKT conditions are sufficient for optimality, the following implication holds for all (w,δ): if $(\bar{x},\bar{\mu},\bar{\nu},\bar{u})$ satisfy (25), then $(\bar{x},\bar{u})$ satisfy

$$y^\star(w,\delta) = C(\delta)\bar{x} + D(\delta)\bar{u} + Q(\delta)w.$$

The filter (24) satisfies the criterion of Definition 4.1, and is therefore an optimality model. □

The above optimality model may be employed whenever the RFS property holds. If, furthermore, the stronger ROS property holds, then we have a second option. The proof of the following is essentially identical to the proof of Proposition 4.9.

*Proposition 4.10 (***Robust Output Subspace Optimality Model (ROS-OM)***):* Suppose the plant (6) satisfies the robust output subspace property with the matrix G_0. Then

$$\begin{aligned} \dot{\mu} &= Hy - Lw \\ \dot{\nu} &= \varphi_\nu(\nu, \mathsf{f}(y;w)) \\ \epsilon &= G_0^{\mathsf{T}}\left(\nabla f_0(y;w) + H^{\mathsf{T}}\mu + \sum_{i=1}^{n_{\mathrm{ic}}} \nu_i \nabla f_i(y;w)\right) \end{aligned} \tag{26}$$

is an optimality model for the OSS control problem. △

In special circumstances, one can modify the RFS-OM of Proposition 4.9 to obtain an optimality model with an error signal of reduced dimension; this reduces the number of integrators required. A proof of the following may be found in the appendix.

*Proposition 4.11 (***Reduced-Error RFS-OM***):* Suppose the optimization problem (11) satisfies the robust feasible subspace property with the matrix $T_0 \in \mathbb{R}^{p \times n_{\mathrm{ec}}}$, and let φ_ν satisfy the implication (23). Let $G(\delta)$ be the matrix defined in Lemma 3.4. The filter

$$\begin{aligned} \dot{\nu} &= \varphi_\nu(\nu, \mathsf{f}(y;w)) \\ \epsilon &= Hy - Lw + T_0^{\mathsf{T}}\left(\nabla f_0(y;w) + \sum_{i=1}^{n_{\mathrm{ic}}} \nu_i \nabla f_i(y;w)\right) \end{aligned} \tag{27}$$

is an optimality model for the linear-convex OSS control problem if $\mathbf{range}\, HG(\delta) \cap \mathbf{range}\, T_0^{\mathsf{T}} = \{\mathbb{0}\}$ for all $\delta \in \boldsymbol{\delta}$. △

*Remark 4.12 (***Uncertain Equality Constraints***):* It is possible to apply the RFS optimality models even when the engineering equality constraint matrices are uncertain — that is, when (11c) reads $H(\delta)y = L(\delta)w$ — by redefining the RFS property as the existence of a matrix T_0 satisfying

$$\mathbf{range}\, T_0 = \mathbf{null} \begin{bmatrix} G_\perp(\delta) \\ H(\delta) \end{bmatrix} \ \text{ for all } \delta \in \boldsymbol{\delta}\,. \tag{28}$$

Under Assumption 4.8 and using a T_0 satisfying (28), Propositions 4.9 and 4.11 still hold with $Hy - Lw$ replaced by $H(\delta)y - L(\delta)w$. △

C. Stabilizer Design

In this section we discuss the design of a stabilizer for the augmented plant (15) when we employ one of the three optimality models of Section IV-B3.

1) Stabilizer Design for Quadratic Programs: We first consider the case when the optimization problem (11) is an equality-constrained convex quadratic program — that is, when the objective function is quadratic in y and the only constraints of the problem are affine equality constraints. In this case the closed-loop system becomes linear and time-invariant, and we can obtain strong and explicit results. To be precise, suppose the optimization problem (11) is of the form

$$\begin{aligned} \underset{y\in\mathbb{R}^p}{\text{minimize}} \quad & \tfrac{1}{2}y^\mathsf{T}My - y^\mathsf{T}Nw \\ \text{subject to} \quad & G_\perp(\delta)y = b(w,\delta) \\ & Hy = Lw, \end{aligned} \tag{29}$$

where $M \succeq \mathbb{0}$.[1]

*Definition 4.13 (***Nonredundant Constraints***):* The problem (29) is said to have *nonredundant constraints* when the matrix

$$\begin{bmatrix} G_\perp(\delta) \\ H \end{bmatrix} \tag{30}$$

is full row rank. △

If the matrix (30) is not full row rank, then either the feasible region is empty or one may eliminate constraints without changing the geometry of the feasible region [34].

Under mild assumptions on the optimization problem and plant, we can ensure that the augmented plant arising from from the RFS-OM, ROS-OM, or reduced-error RFS-OM is both stabilizable and detectable for a given $\delta \in \boldsymbol{\delta}$. When these conditions hold in the nominal case (when $\boldsymbol{\delta} = \{\mathbb{0}\}$), the OSS control problem is solvable via standard LTI design tools, e.g., [35, Section 16.7].

The following propositions mention the measurement matrix C_m; Assumption 4.8 in the present context implies that we can take the available measurements y_m as a linear function of (x,u,w), i.e., $y_\mathrm{m} = C_\mathrm{m}(\delta)x + D_\mathrm{m}(\delta)u + Q_\mathrm{m}(\delta)w$.

The proofs of the following three propositions may be found in the appendix.

*Proposition 4.14 (***Stabilizability and Detectability Using RFS-OM***):* Consider the augmented plant (15) using the RFS-OM (24) as the optimality model (φ, h_ϵ) for the OSS control problem with optimization problem (29). This augmented plant is stabilizable and detectable for a given $\delta \in \boldsymbol{\delta}$ if and only if

(i) $(C_\mathrm{m}(\delta), A(\delta), B(\delta))$ is stabilizable and detectable;
(ii) the problem (29) has nonredundant constraints;
(iii) the problem (29) has a unique optimizer;
(iv) T_0 is full column rank. △

*Proposition 4.15 (***Stabilizability and Detectability Using ROS-OM***):* Consider the augmented plant (15) using the ROS-OM (26) as the optimality model (φ, h_ϵ) for the OSS control problem with optimization problem (29). This augmented plant is stabilizable and detectable for a given $\delta \in \boldsymbol{\delta}$ if and only if

(i) $(C_\mathrm{m}(\delta), A(\delta), B(\delta))$ is stabilizable and detectable;
(ii) the problem (29) has nonredundant constraints;
(iii) the problem (29) has a unique optimizer;
(iv) G_0 is full column rank. △

*Proposition 4.16 (***Stabilizability and Detectability Using Reduced-Error RFS-OM***):* Consider the augmented plant (15) using the reduced-error ROS-OM (27) as the optimality model (φ, h_ϵ) for the OSS control problem with optimization problem (29). This augmented plant is stabilizable and detectable for a given $\delta \in \boldsymbol{\delta}$ if and only if

(i) $(C_\mathrm{m}(\delta), A(\delta), B(\delta))$ is stabilizable and detectable;
(ii) the problem (29) has a unique optimizer;
(iii) $(\mathbf{range}\, HG(\delta))^\perp \cap (\mathbf{range}\, T_0^\mathsf{T})^\perp = \{\mathbb{0}\}$. △

These results provide explicit conditions which guarantee solvability of the OSS control problem, at least in the nominal case. Unfortunately, even if the augmented plant is stabilizable and detectable for every $\delta \in \boldsymbol{\delta}$, this does not guarantee the existence of a *fixed* stabilizer that stabilizes the closed-loop system for every $\delta \in \boldsymbol{\delta}$. The conditions of Propositions 4.14/4.15/4.16 being met for every $\delta \in \boldsymbol{\delta}$ are necessary *but not sufficient* for the existence of such a robust stabilizer. If the structure of the uncertainty set $\boldsymbol{\delta}$ is known, one may attempt robust stabilizer design using the robust control techniques described in the following section.

2) Stabilizer Design in the General Case: If the optimization problem has a non-quadratic convex objective function, or if inequality constraints are present, or if the uncertainty set $\boldsymbol{\delta}$ is non-trivial, then we can turn to tools from robust control theory.

Supposing that the inequality constraints (11d) of the optimization problem are linear, and that we employ one of the optimality models of Section IV-B3 with $\varphi_\nu(\alpha,\beta) := \mathbf{max}(\alpha+\beta,\mathbb{0}) - \alpha$, the only nonlinearities in the closed-loop system are the static, slope-restricted nonlinearities $\max(\cdot,0)$ and $\nabla f_0(\cdot,w)$. The function $\max(\cdot,0)$ satisfies a $[0,1]$ slope restriction, while $y \mapsto \nabla f_0(y;w)$ satisfies a $[\kappa, L]$ slope restriction if $y \mapsto f_0(y;w)$ is κ-strongly convex and $y \mapsto \nabla f_0(y;w)$ is L-Lipschitz continuous. As a result, we can manipulate the closed-loop system into the standard configuration of the robust control synthesis problem depicted in Figure 2a (see [36, Chapter 9] for background). Stabilizer design can then be accomplished by describing the Δ block with *integral quadratic constraints* (IQCs) and applying standard *linear matrix inequality* (LMI) tools. See [37] for a detailed exposition on IQC analysis.

The robust control framework presents two distinct stabilizer design options. One can propose a stabilizer structure, such as a proportional-integral controller, and tune the parameters of the stabilizer until closed-loop stability can be verified using an analysis LMI. Alternatively, one can employ a synthesis LMI to generate a dynamic stabilizer whose order is, in general, the same as that of the augmented plant. The latter method often results in a more complex controller structure than the former, but may be useful when stabilizing gains cannot be found "by hand." For an example of each approach, see our previous work [19].

[1] Any constant term of the form $y^\mathsf{T}c$ with $c \in \mathbb{R}^p$ may be included in the term $y^\mathsf{T}Nw$ by appropriate redefinition of N and w.

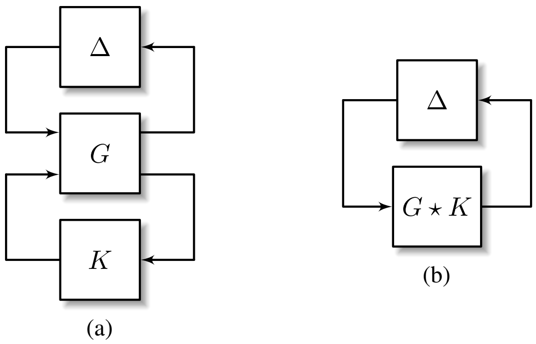


Fig. 2: A robust control framework for stabilizer design and analysis. System uncertainty $\boldsymbol{\delta}$ and optimality model nonlinearities ∇f_0 and $\max(\cdot, 0)$ are extracted into Δ. The LTI system K is the stabilizer, while all certain LTI dynamics are contained in G. (a) Robust controller synthesis, where the stabilizer K is designed for stability/performance using, for example, $\mathcal{H}_\infty/\mathcal{H}_2$ synthesis techniques. (b) Robust stability/performance analysis, which can be used to assess the performance of a specified stabilizer design K upon closing the lower loop between G and K.

*Remark 4.17 (***Inequality Constraints***):* While, in principle, our controller design is capable of enforcing hard inequality constraints, the φ_ν function often complicates the stability analysis. For example, when $\varphi_\nu(\alpha, \beta) = \mathbf{max}(\alpha + \beta, \mathbb{0}) - \alpha$, the lower bound of zero on the $[0, 1]$ slope restriction of $\max(\cdot, 0)$ can lead to inconclusive stability results when using tools from robust control theory. This difficulty is described in detail in [38, Section 5.6.3]. In practice it may be more useful to enforce inequality constraints by including either barrier or penalty terms in the objective function; see [39, Section 3.1] and [40, Section VI-B] for examples of this approach. △

*Remark 4.18 (***Unique Optimizer Assumption***):* Assumption 3.1 (i), which requires the optimizer $y^\star(w, \delta)$ of (7) to be unique, is necessary for the closed-loop system corresponding to one of the previously-described optimality models to possess a unique equilibrium point. This assumption is therefore necessary for one of our controllers to solve Problem 3.2, which demands a globally asymptotically stable equilibrium point for the closed-loop dynamics. Relaxing the problem statement to require only a stable equilibrium set would allow one to drop the assumption of optimizer uniqueness, but would complicate the stability analysis. One would have to invoke, for example, the invariance theorems of LaSalle. See [38, Section 5.6] for more details. △

V. ILLUSTRATIVE EXAMPLES

In this final section, we illustrate the ideas of the preceding sections through several example problems. We compare application of the OSS control framework and previous frameworks from the literature to highlight our contributions.

A. Reference Tracking

Consider now an *asymptotic reference tracking* problem, in which it is desired that the output of a plant $y \in \mathbb{R}^p$ asymptotically approaches a vector of prescribed values $r \in \mathbb{R}^p$. The reference tracking problem can be formulated as a case of the output regulation problem of Section II by defining the error signal as $e := y - r$ (the reference signal r is a component of the exogenous input w). In this section, we show that formulating the reference tracking problem in the OSS control framework yields a variety of tracking problems with different steady-state objectives. Additionally, we show that application of the ROS-OM to one of these tracking problems recovers and generalizes the control scheme of [41, Section 4] for reference tracking in underactuated systems.

Supposing that we wish the optimization output y of the plant (6) to track the reference signal r, we formulate the optimization problem (11) as follows:

$$\begin{aligned} \min_{y \in \mathbb{R}^p} \quad & n(y - r) \\ \text{subject to} \quad & G_\perp(\delta) y = b(w, \delta)\,, \end{aligned} \tag{31}$$

where $n : \mathbb{R}^p \to \mathbb{R}$ is a smooth and convex norm. An OSS controller will guide y to the solution $y^\star(w, \delta)$ of (31). If r is an achievable equilibrium output of the plant — that is, if $G_\perp(\delta) r = b(w, \delta)$ — then $y^\star(w, \delta) = r$, and an OSS controller will guide y to r exactly. If r is not an achievable equilibrium output, then it is impossible for y to equal r in steady-state; however, $y^\star(w, \delta)$ will be the nearest point in the equilibrium set $\overline{Y}(w, \delta) = \{y \in \mathbb{R}^p \mid G_\perp(\delta) y = b(w, \delta)\}$ to the vector r as measured by the norm n. An OSS controller will therefore minimize the discrepancy between y and r to the greatest extent possible in steady-state.

Formulating the reference tracking problem in the OSS control framework allows us to specify different "types" of tracking through our choice of norm. For example:

RMS Tracking With n equal to the ℓ_2-norm, we seek to minimize the root-mean-square error between y and r.

Sparse Tracking With n equal to the ℓ_1-norm, we seek to make the vector $y - r$ as sparse as possible.

MinMax Tracking With n equal to the ℓ_∞-norm, we seek to minimize the maximum entry of $y - r$.

Using other norms yields other tracking interpretations.

Note that for use in the OSS control framework, we must replace ℓ_1 and ℓ_∞ norms with differentiable approximations. Differentiable approximations include the Moreau envelope [42, Section 3.1] and the functions in [43], among others.

Consider, for example, the optimization problem (31) with output $y = \mathrm{col}(y_\mathrm{m}, u)$, reference $r = \mathrm{col}(r_\mathrm{m}, \mathbb{0})$, and norm $n : \mathbb{R}^{p_\mathrm{m}} \times \mathbb{R}^m \to \mathbb{R}$ defined as

$$n(y) = n(y_\mathrm{m}, u) := \|y_\mathrm{m}\|_2 + \theta \|u\|_1$$

for some $\theta > 0$. By seeking to minimize $n(y_\mathrm{m} - r_\mathrm{m}, u)$ in equilibrium, we are attempting to minimize the root-mean-square error between the measured output y_m and the reference r_m using the least number of control inputs possible (we interpret the 1-norm as penalizing nonzero entries of its argument). The strength of our desire for u to be sparse is expressed through the magnitude of the parameter θ. We simulated the feedback interconnection of a random, stable LTI plant generated using the `rss` function in MATLAB with 4 states, 2 control inputs, 1 disturbance input, and 3 measured outputs and an OSS controller using the ROS-OM and pure integral feedback with a gain of 10. We made the smoothing approximation $\|u\|_1 \approx \frac{1}{20}\sum_{i=1}^{2}\log(\cosh(20u))$. In Figure 3, we compare the behaviour of the closed-loop system for $\theta = 0.05$ (we want u to be sparse) and $\theta = 10^{-9}$ (we do not care about sparsity of u).

As a final comment on solving tracking problems using the OSS control framework, we note that the control strategy of [41, Section 4] is a special case of the setup described in this section. Assuming that the plant (6) satisfies the ROS property, we apply the ROS-OM to yield the augmented plant

$$\begin{aligned} \dot{x} &= A(\delta)x + B(\delta)u + B_w(\delta)w \\ \dot{\eta} &= \epsilon = G_0^{\mathsf{T}}\nabla n(y-r). \end{aligned} \tag{32}$$

We set $n(\cdot) := \frac{1}{2}\|\cdot\|_2$, assume that $A(\delta)$ is invertible, and take $G_0 := -C(\delta)A(\delta)^{-1}B(\delta)+D(\delta)$ to recover the scheme of [41, Section 4]. Observe that we have generalized the framework of [41] in three ways: we introduce flexibility in the choice of norm, remove the assumption of invertibility of $A(\delta)$, and do not require that G_0 be the DC gain of the plant.

B. Equilibrium Constraints

Recall the purpose of including the equilibrium constraints in the optimization problem: by doing so, we guarantee the existence of a forced equilibrium point $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ such that the output $\bar{y}$ is optimal. If we omit the equilibrium constraints, we have no assurance that such a forced equilibrium exists. The following example demonstrates the consequences.

Consider the problem of regulating the stable plant

$$\begin{aligned} \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}-1 & 0\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ -1\end{bmatrix}u + \begin{bmatrix}1\\ 1\end{bmatrix}w \\ y &= \mathrm{col}(x_1, u) \\ y_{\mathrm{m}} &= y \end{aligned} \tag{33}$$

to an equilibrium point such that y equals the optimizer

$$y^\star := \operatorname*{argmin}_{y\in\mathbb{R}^2}\left(f_0(y) := \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2\right) = \begin{bmatrix}0\\ 0\end{bmatrix}. \tag{34}$$

The set of forced equilibria $(\bar{x}_1, \bar{x}_2, \bar{u})$ yielding the optimal output is the set of solutions to

$$\left[\begin{array}{cc|c} -1 & 0 & 1\\ 1 & -1 & -1\\ \hline 1 & 0 & 0\\ 0 & 0 & 1\end{array}\right]\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \bar{u}\end{bmatrix} = \begin{bmatrix}-w\\ -w\\ 0\\ 0\end{bmatrix}.$$

For $w \neq 0$, these equations fail to have a solution, and this problem is not solvable as a result.

In the OSS control framework, we *always* include the equilibrium constraints and therefore never formulate such insolvable problems. Compare with previous work, such as [20], in which one *assumes* the existence of a forced equilibrium point yielding the optimal output (see [20, Assumption II.3]). When this assumption fails, applying their *KKT controller* may yield an unstable closed-loop system. The KKT controller proposed in [20] for the above example is

$$\dot{\eta} = \nabla f_0(y), \quad u = K\eta.$$

One attempts to select the gain matrix K to stabilize the closed-loop system. However, the augmented plant

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{\eta}_1\\ \dot{\eta}_2\end{bmatrix} = \begin{bmatrix}-1 & 0 & 0 & 0\\ 1 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \eta_1\\ \eta_2\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ 1\end{bmatrix}u$$

consisting of the plant (33) in series with a bank of integrators on $\nabla f_0(y)$ is not stabilizable. No dynamic stabilizing controller exists, let alone a stabilizing integral gain K. Such pathologies are the result of failing to include the equilibrium constraints in the optimization problem.

Consider instead applying an OSS controller to the above example, using the ROS-OM with pure integral feedback:

$$\begin{aligned} \dot{\eta} &= G_0^{\mathsf{T}}\nabla f_0(y) \\ u &= -\eta\,, \end{aligned} \tag{35}$$

with $G_0 = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$. The closed-loop system is stable and the optimization output tracks the optimizer — see Figure 4a for simulation results.

C. Robust Feasible Subspace Property

The OSS control framework explicitly accounts for model uncertainty through the vector of uncertain parameters δ. As stressed in Section IV-B, additional conditions are generally required to ensure that robust regulation to an optimal equilibrium point is possible in the presence of uncertainty; such considerations are omitted from previous work. To illustrate, consider the OSS control problem with a perturbed version of the plant (33)

$$\begin{aligned} \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}-1-\delta & 0\\ 1+\delta & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ -1\end{bmatrix}u + \begin{bmatrix}1\\ 1\end{bmatrix}w \\ y &= \mathrm{col}(x_1, u) \\ y_{\mathrm{m}} &= y \end{aligned} \tag{36}$$

and the OSS optimization problem

$$y^\star(w,\delta) := \operatorname*{argmin}_{y\in\overline{Y}(w,\delta)}\left(f_0(y) := \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2\right). \tag{37}$$

The plant dynamics depend on an uncertain parameter $\delta \in \boldsymbol{\delta} := [-0.5, 0.5]$. One can show that the matrix $G(\delta)$ of Lemma 3.4 is given by $G(\delta) = \mathrm{col}(1, \delta)$. The robust feasible subspace property therefore fails.[2] Our analysis framework allow us to conclude that this problem cannot be solved exactly.

Suppose, however, that we ignore this fact and attempt to construct an OSS controller by assuming a nominal value for the uncertain parameter. This is analogous to the procedure one would follow to construct the controllers [26, Equation (5)] or [25, Equation (9)], for instance. We assume $\delta = 0$ and use the ROS-OM with pure integral feedback to yield the controller (35). The behaviour of the resulting closed-loop system when $\delta = 0.5$ is shown in Figure 4b. The plant fails to track the optimizer.

[2]The RFS and ROS properties are equivalent in this example since no engineering equality constraints are present; see Remark 4.5.

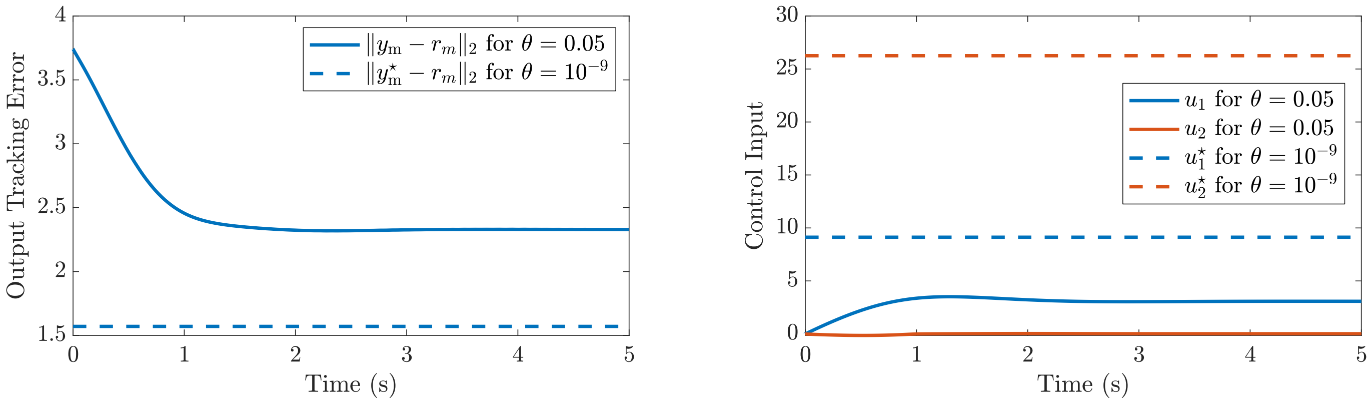
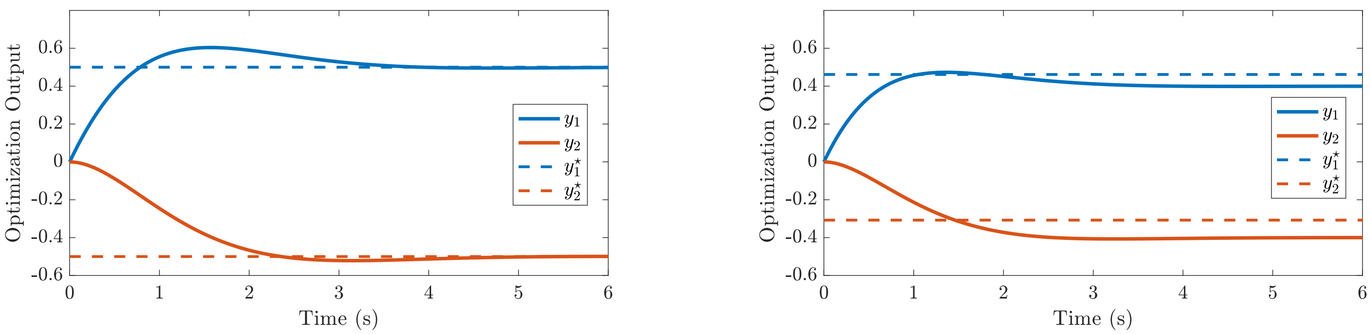


Fig. 3: Simulation plots for the example of Section V-A. The solid curve in the left-hand plot is the root-mean-square error between the measured output y_{m} and the reference r_{m} as a function of time for $\theta = 0.05$; the solid curves in the right-hand plot are the two control inputs as a function of time for $\theta = 0.05$. For comparison are the optimal values of the output tracking error and the control inputs for $\theta = 10^{-9}$ as dashed lines. We observe that by increasing θ, we make a small sacrifice in output tracking performance but save substantially on control effort — one of our control inputs is almost zero, in fact.



(a) The optimization output as a function of time for the plant (33) in feedback with the controller (35).

(b) The optimization output as a function of time for the plant (36) in feedback with the controller (35).

Fig. 4: Simulation plots for the examples of Sections V-B and V-C.

D. No Hurwitz Assumption

One benefit of the OSS control framework over other formulations in the feedback-based optimization literature is that we do not assume the dynamics matrix of the plant A is Hurwitz or even invertible — compare with [26] or [25], both of which rely on *invertibility* and *stability* of A. As a consequence, one may construct OSS controllers for a wider variety of plants than previously considered. The dynamics matrix is allowed to be singular, unstable, or both.

Consider the OSS control problem with the plant

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix} u + \begin{bmatrix}0\\ 1\\ 0\end{bmatrix} w$$

$$\begin{aligned} y &= x\\ y_{\mathrm{m}} &= y \end{aligned}$$

and the OSS optimization problem

$$y^\star(w) := \operatorname*{argmin}_{y\in\overline{Y}(w)} \frac{1}{2} y^{\mathsf{T}} M y,$$

where $M \succ \mathbb{0}$. The eigenvalues of A are $\{0, 0, 1\}$, hence A is neither invertible nor stable. The controllers of [26] or [25] do not apply to this problem. In contrast, a ROS-OM for this problem is

$$\epsilon = \underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}}_{G_0^{\mathsf{T}}} My\,,$$

and the corresponding augmented plant (15) is stabilizable and detectable since the requirements of Proposition 4.15 are satisfied (existence and uniqueness of the optimizer are guaranteed by positive definiteness of M). A stabilizer exists, and therefore a complete OSS controller for this problem exists also. For example, with $M = I_3$, the OSS controller

$$\begin{aligned} \dot{\eta}_1 &= x_1 & u_1 &= 18.5x_1 + 7.5x_2 + 15\eta_1\\ \dot{\eta}_2 &= -x_3 & u_2 &= 3.5x_3 - 1.5\eta_2 \end{aligned}$$

yields a closed-loop stable system with the dynamics matrix having eigenvalues $\{-1, -1.5, -2, -2.5, -3\}$.

E. Designing for Improved Performance

A common strategy for controller design in the feedback-based optimization literature is to employ a standard continuous-time optimization algorithm to compute a steady-state control input $u^\star$ that yields the optimal steady-state optimization output $y^\star$. Unmeasurable terms in the algorithm are replaced with measurements of y and the intermediate values of u are applied as control input while the algorithm converges. Hence, one obtains a feedback controller by making a minor modification to an off-the-shelf algorithm. See, for example, [9], [25], [26].

This suggests an "optimization algorithm" design framework that stands in contrast to OSS controller design: first, synthesize a feedback controller to stabilize the plant and achieve good tracking performance from the control input u to the optimization output y; second, interconnect this plant with an optimization algorithm. While this procedure is a logical design strategy, the OSS controller architecture can yield improved performance because of the flexibility available through the choice of optimality model and stabilizer.

An example will illustrate this fact. Consider the plant (6) with no parametric uncertainty ($\boldsymbol{\delta} = \{\mathbb{0}\}$) and a Hurwitz dynamics matrix A. For a fixed control input $\bar{u}$, the steady-state optimization output $\bar{y}$ is given by $\bar{y} = G\bar{u} + G_w w$, where $G \coloneqq -CA^{-1}B + D$ and $G_w \coloneqq -CA^{-1}B_w + Q$. Our objective is to guide the optimization output y to the optimizer of an equality-constrained problem,

$$\begin{aligned} y^\star(w) \coloneqq \quad & \underset{y\in\mathbb{R}^p,\, u\in\mathbb{R}^m}{\operatorname{argmin}} && f_0(y;w) \\ & \text{subject to} && y = Gu + G_w w \\ & && Hy = Lw\,, \end{aligned} \tag{38}$$

where $f_0(y;w)$ is convex in y for all w. The first set of equality constraints in the problem of (38) is equivalent to the equilibrium constraint (11b) of the generic convex optimization problem (11).

In the "optimization algorithm" framework for controller design, we eliminate y from (38), dualize the remaining constraint, apply a primal-descent-dual-ascent algorithm to compute u, and then replace instances of $Gu + G_w w$ with real-time measurements of y. This yields the *primal-dual controller*

$$\dot{\lambda} = Hy - Lw\,, \quad \dot{u} = -k_i G^\mathsf{T}(\nabla f_0(y;w) + H^\mathsf{T}\lambda)\,, \tag{39}$$

as in [26]. The feedback gain $k_i > 0$ determines stability and performance.

The controller (39) is a special case of the generic OSS controller. We may rewrite (39) as

$$\begin{aligned} \dot{\lambda} &= Hy - Lw & \dot{\eta} &= \epsilon \\ \epsilon &= G^\mathsf{T}(\nabla f_0(y;w) + H^\mathsf{T}\lambda) & u &= -k_i\eta\,. \end{aligned} \tag{40}$$

In (40), the controller (39) has been decomposed into an optimality model with state λ and output ϵ, an integrator η, and a constant-gain stabilizer. Whatever the performance of the primal-dual controller, the OSS framework may be applied to design a controller that achieves better performance through modification of the optimality model, stabilizer, or both. For instance, one can add a proportional term to the stabilizer, yielding an OSS controller of the form (40) with modified input equation $u = -k_i\eta - k_p\epsilon$, where k_p is the proportional gain. The additional degree of design freedom allows us to improve the behaviour of the closed-loop system.

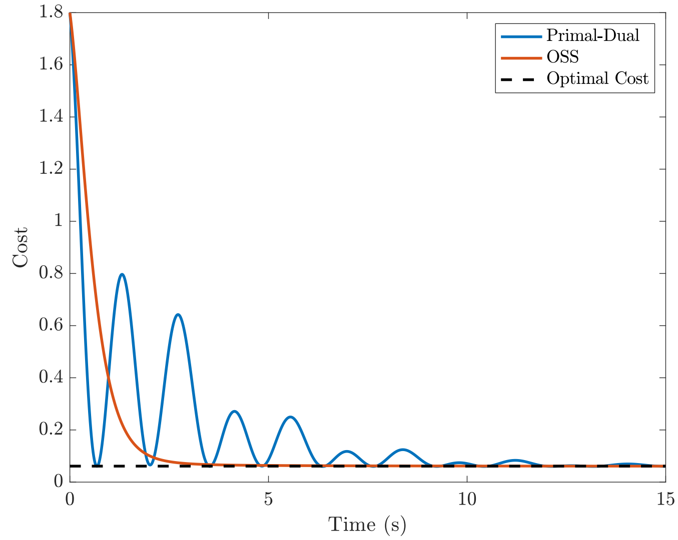


Fig. 5: Simulation plot for the example of Section V-E of the cost $\frac{1}{2}y(t)^\mathsf{T}My(t)$ for the primal-dual controller and the OSS controller when $w(t)$ is the unit step function.

The following numerical example illustrates the difference between the preceding primal-dual controller and OSS controller. Using the OSS control architecture, we can modify the stability and tracking performance of the plant through the stabilizer, while this is not the case for the primal-dual controller (39); hence, for the purpose of making a fair comparison, suppose that we have already designed and implemented an effective tracking controller so that we may approximate the plant by its DC gain, i.e. $y(t) = Gu(t) + G_w w(t)$. Assume the optimization problem (38) has objective function $f_0(y;w) \coloneqq \frac{1}{2}y^\mathsf{T}My$ and matrices

$$\begin{aligned} M &\coloneqq \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \quad G \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad G_w \coloneqq \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \\ H &\coloneqq \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \qquad\qquad\qquad\qquad L \coloneqq 0\,. \end{aligned}$$

Setting $\{k_i = 1\}$ in the primal-dual controller (39) and $\{k_i = 1, k_p = 1\}$ in the OSS controller (40) with $u = -k_i\eta - k_p\epsilon$) yields the behaviour shown in Figure 5 when $w(t)$ is the unit step function. The primal-dual controller causes the cost function to oscillate many times before settling to its optimal value, while the OSS controller smoothly guides the system to the optimizer.

F. Optimal Frequency Regulation in Power Systems

This final section illustrates the application of our theory to a power system control problem. Our main objective is to work through the constructions presented in Section IV, and to simultaneously illustrate the many sources of design flexibility within our proposed framework. In particular, we will show that several centralized and distributed frequency controllers proposed in the literature are recoverable as special cases of our framework. The following example also demonstrates that the robust feasible subspace property may hold even when the robust output subspace property fails.

The dynamics of synchronous generators in a connected AC power network with n buses and n_{t} transmission lines is modelled in a reduced-network framework by the *swing equations*. The vectors of angular frequency (deviations from nominal) $\omega \in \mathbb{R}^n$ and real power flows $p \in \mathbb{R}^{n_{\mathrm{t}}}$ along the transmission lines obey the dynamic equations

$$\begin{aligned} M(\delta)\dot{\omega} &= P^\star - D(\delta)\omega - \mathcal{A}p + u \\ \dot{p} &= \mathcal{B}(\delta)\mathcal{A}^\mathsf{T}\omega, \end{aligned} \tag{41}$$

in which $M(\delta) \succ \mathbb{0}$ is the (diagonal) inertia matrix, $D(\delta) \succ \mathbb{0}$ is the (diagonal) damping matrix, $\mathcal{A} \in \{0, 1, -1\}^{n\times n_{\mathrm{t}}}$ is the signed node-edge incidence matrix of the network, $\mathcal{B}(\delta) \succ \mathbb{0}$ is the diagonal matrix of transmission line susceptances, $P^\star \in \mathbb{R}^n$ is the vector of uncontrolled power injections (generation minus demand) at the buses, and $u \in \mathbb{R}^n$ is the controllable reserve power produced by the generators. The incidence matrix satisfies $\mathbf{null}\,\mathcal{A}^\mathsf{T} = \mathrm{span}(\mathbb{1}_n)$, and strictly for simplicity we assume that the network is acyclic, in which case $n_{\mathrm{t}} = n-1$ and $\mathbf{null}\,\mathcal{A} = \{\mathbb{0}\}$. The vector δ lists the diagonal elements of the inertia, damping, and branch susceptance matrices; the elements of δ are uncertain positive real numbers. See [44, Section VII] for a first-principles derivation of this model.

We consider the *optimal frequency regulation problem* (OFRP), wherein we minimize the total cost $\sum_i J_i(u_i)$ of reserve power production in the system subject to system equilibrium and zero steady-state frequency deviations:

$$\begin{aligned} \underset{u\in\mathbb{R}^n,\omega\in\mathbb{R}^n}{\text{minimize}} \quad & J(u) := \textstyle\sum_{i=1}^n J_i(u_i) \\ \text{subject to} \quad & G_\perp(\delta)\,\mathrm{col}(u,\omega) = b(w,\delta) \\ & F\omega = \mathbb{0}. \end{aligned} \tag{42}$$

We shall compute the matrix $G_\perp(\delta)$ of the equilibrium constraints shortly; the vector $b(w,\delta)$ is unimportant for controller design. The matrix F encodes the steady-state frequency constraint. We will specify the requirements on F later in this section.

With state vector $x := \mathrm{col}(\omega, p)$, the dynamics (41) can be put into the standard LTI form (6) with matrices

$$\begin{aligned} A(\delta) &:= \begin{bmatrix} -M(\delta)^{-1}D(\delta) & -M(\delta)^{-1}\mathcal{A} \\ \mathcal{B}(\delta)\mathcal{A}^\mathsf{T} & \mathbb{0} \end{bmatrix} \\ B(\delta) &:= \begin{bmatrix} M(\delta)^{-1} \\ \mathbb{0} \end{bmatrix} \quad B_w(\delta) := \begin{bmatrix} M(\delta)^{-1} \\ \mathbb{0} \end{bmatrix}. \end{aligned}$$

We identify the optimization output as $y := \mathrm{col}(u, \omega)$. Therefore

$$C := \begin{bmatrix} \mathbb{0} & \mathbb{0} \\ I_n & \mathbb{0} \end{bmatrix} \quad D := \begin{bmatrix} I_n \\ \mathbb{0} \end{bmatrix}. \tag{43}$$

We assume the measured output is given by $y_{\mathrm{m}} = \mathrm{col}(u, F\omega)$.

We first check whether the robust output subspace property (Definition 4.4) holds by constructing the matrix $G(\delta)$ as outlined in Lemma 3.4. We construct a matrix $\mathcal{N}(\delta)$ satisfying $\mathbf{range}\,\mathcal{N}(\delta) = \mathbf{null}\begin{bmatrix} A(\delta) & B(\delta) \end{bmatrix}$. One may verify that choosing

$$\mathcal{N}(\delta) := \begin{bmatrix} \mathbb{1}_n & \mathbb{0} \\ \mathbb{0} & I_n \\ D(\delta)\mathbb{1}_n & \mathcal{A} \end{bmatrix} \tag{44}$$

yields the required property. Using (44) and (43), we calculate $G(\delta) = \begin{bmatrix} C & D \end{bmatrix}\mathcal{N}(\delta)$ to be

$$G(\delta) = \begin{bmatrix} D(\delta)\mathbb{1}_n & \mathcal{A} \\ \mathbb{1}_n & \mathbb{0} \end{bmatrix}.$$

The subspace $\mathbf{range}\,G(\delta)$ varies with δ, and therefore there cannot exist a fixed matrix G_0 such that $\mathbf{range}\,G(\delta) = \mathbf{range}\,G_0$ for all δ. However, it is still possible that the robust feasible subspace property holds. To check whether this is the case, we first construct a full-row-rank matrix $G_\perp(\delta) \in \mathbb{R}^{n\times 2n}$ satisfying $\mathbf{null}\,G_\perp(\delta) = \mathbf{range}\,G(\delta)$. We find that selecting

$$G_\perp(\delta) := \begin{bmatrix} \mathbb{1}_n\mathbb{1}_n^\mathsf{T} & -(\mathbb{1}_n^\mathsf{T}D(\delta)\mathbb{1}_n)I_n \end{bmatrix}$$

yields the required property. We identify the matrix H of the engineering equality constraints in (11) for the problem (42) as $H := \begin{bmatrix} \mathbb{0} & F \end{bmatrix}$. Following Definition 4.3, we now ask whether there exists a fixed matrix T_0 such that

$$\mathbf{range}\,T_0 = \mathbf{null}\begin{bmatrix} \mathbb{1}_n\mathbb{1}_n^\mathsf{T} & -(\mathbb{1}_n^\mathsf{T}D(\delta)\mathbb{1}_n)I_n \\ \mathbb{0} & F \end{bmatrix} \tag{45}$$

for all δ. The null space on the right-hand side of (45) is spanned by vectors of the form $\mathrm{col}(v, \mathbb{0})$ where $\mathbb{1}_n^\mathsf{T}v = 0$. Inspired by approaches in multi-agent control, we introduce a connected, weighted and directed communication graph $\mathcal{G}_{\mathrm{c}} = (\{1, \ldots, n\}, \mathcal{E}_{\mathrm{c}})$ between the buses, with associated *Laplacian matrix* $L_{\mathrm{c}} \in \mathbb{R}^{n\times n}$. We assume the directed graph $\mathcal{G}_{\mathrm{c}}$ contains a globally reachable node.[3] Under this assumption, (45) holds with $T_0 = \left[\begin{smallmatrix} L_{\mathrm{c}}^\mathsf{T} \\ \mathbb{0} \end{smallmatrix}\right]$. Therefore, the optimization problem satisfies the robust feasible subspace property.

The Laplacian matrix L_{c} has a left null space of dimension one spanned by some nonnegative vector $w \in \mathbb{R}^n$. Assuming that F is selected such that $w^\mathsf{T}F\mathbb{1}_n \neq \mathbb{0}$, the range condition of Proposition 4.11 is satisfied. Hence, with an appropriate choice of F, we may apply the reduced-error RFS-OM to obtain the optimality model

$$\epsilon = F\omega + L_{\mathrm{c}}\nabla J(u). \tag{46}$$

Therefore, one option for an OSS controller is

$$\dot{\eta} = F\omega + L_{\mathrm{c}}\nabla J(u) \tag{47a}$$

$$u = -K_1\eta_1 - K_2\eta_2 - K_3\omega, \tag{47b}$$

where K_1, K_2, and K_3 are gain matrices that should be selected for closed-loop stability/performance. If the objective function J is a positive definite quadratic, one can show that the augmented plant comprising (41) and (47a) is stabilizable and detectable using Proposition 4.16. With $F := I_n$, $K_1 = K_2 = \frac{1}{k}I_n$ for $k > 0$, and $K_3 = \mathbb{0}$, this design reduces to the *distributed-averaging proportional-integral* (DAPI) frequency control scheme; see [6], [46], [47].

We can obtain several other control schemes by instead applying the RFS-OM as our optimality model. Let $F := \mathsf{c}^\mathsf{T}$, where c is a vector of convex combination coefficients satisfying $\mathsf{c}_i \geq 0$ and $\sum_{i=1}^n \mathsf{c}_i = 1$. Define $\widetilde{L}_{\mathrm{c}} \in \mathbb{R}^{(n-1)\times n}$ as the matrix obtained by eliminating the first row from L_{c} and set

[3] See [45, Chapter 6] for details.

$T_0 := \begin{bmatrix} \widetilde{L}_{\mathrm{c}}^{\mathsf{T}} \\ \mathbb{0} \end{bmatrix}$. This choice of T_0 satisfies (45). The RFS-OM yields the optimality model

$$\epsilon = \begin{bmatrix} \mathsf{c}^{\mathsf{T}}\omega \\ \widetilde{L}_{\mathrm{c}} \nabla J(u) \end{bmatrix}. \tag{48}$$

By integrating both components of ϵ and choosing to use pure integral feedback for the stabilizer, one obtains the controller

$$\dot{\eta}_1 = \mathsf{c}^{\mathsf{T}}\omega \tag{49a}$$

$$\dot{\eta}_2 = \widetilde{L}_{\mathrm{c}} \nabla J(u) \tag{49b}$$

$$u = -K_1\eta_1 - K_2\eta_2 - K_3\omega. \tag{49c}$$

The interpretation of this (novel) controller is that one agent collects frequency measurements and implements the integral control (49a), while the other agents average their marginal costs via (49b). If the objective function J is a positive definite quadratic, one can show that the augmented plant comprising (41), (49a), and (49b) is stabilizable and detectable using Proposition 4.14.

We can also recover the gather-and-broadcast scheme of [8] from the optimality model (48) as follows. Assume that each J_i is strictly convex, and retain the integral controller (49a). Next, using the fact that $\mathbf{null}\, \widetilde{L}_{\mathrm{c}} = \mathrm{span}(\mathbb{1}_n)$, select the input u to zero the second component of ϵ:

$$\begin{aligned} \widetilde{L}_{\mathrm{c}} \nabla J(u) = \mathbb{0} \quad &\iff \quad \exists\, \alpha \in \mathbb{R} \text{ s.t. } \nabla J(u) = \alpha \mathbb{1}_n \\ &\iff \quad \exists\, \alpha \in \mathbb{R} \text{ s.t. } u = (\nabla J)^{-1}(\alpha \mathbb{1}_n). \end{aligned}$$

Selecting $\alpha = \eta$ leads to the hierarchical gather-and-broadcast controller

$$\dot{\eta} = \sum\nolimits_{i=1}^{n} \mathsf{c}_i \omega_i, \quad u_i(t) = (\nabla J_i)^{-1}(\eta(t)). \tag{50}$$

In summary, several recent frequency control schemes, and the novel scheme (49), can be recovered as special cases of our general control framework. The full potential of our methodology for the design of improved power system control will be an area for future study.

VI. Conclusions

We have studied in detail the linear-convex OSS control problem, wherein we design a controller to guide an LTI system to the solution of an optimization problem despite unknown, constant exogenous disturbances and parametric uncertainty in the plant model. We introduced the idea of an optimality model, the existence of which allows us to reduce the OSS control problem to a stabilization problem, and discussed methodologies for the design of optimality models and stabilizers. We then demonstrated the increased generality of the OSS control framework by comparing it to existing frameworks in the literature.

Future work will present the analogous discrete-time and sampled-data OSS control problems, along with a more detailed study of applications in power system control. A large number of open problems and directions, including but not limited to: OSS control for nonlinear systems subject to time-varying disturbances, flexibility of the framework for distributed/decentralized control, formulations and solutions of hierarchical, competitive, and approximate OSS control problems, and the application of the OSS control framework to the design of new optimization algorithms.

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APPENDIX

Proof of Theorem 4.2: By assumption, the closed-loop system (15) and (16) is well-posed and possesses a globally asymptotically stable equilibrium point for each (w, δ); hence, the first two requirements of the OSS control problem are satisfied. It remains to show that $\lim_{t\to\infty} y(t) = y^\star(w, \delta)$ for each (w, δ) and every initial condition.

Because the closed-loop system possesses a globally asymptotically stable equilibrium point for each (w, δ), there exists a unique solution $(\bar{x}, \bar{\xi}, \bar{\eta}, \bar{x}_{\mathrm{s}})$ to the steady-state equations

$$\begin{aligned} \mathbb{0} &= A(\delta)\bar{x} + B(\delta)\bar{u} + B_w(\delta)w \qquad & \mathbb{0} &= \varphi(\bar{\xi}, \bar{y}_{\mathrm{m}}) \\ \bar{y}_{\mathrm{m}} &= h_{\mathrm{m}}(\bar{x}, \bar{u}, w, \delta) & \mathbb{0} &= h_\epsilon(\bar{\xi}, \bar{y}_{\mathrm{m}}) \end{aligned}$$

$$\begin{aligned} \mathbb{0} &= f_{\mathrm{s}}(\bar{x}_{\mathrm{s}}, \bar{\eta}, \bar{\xi}, \bar{y}_{\mathrm{m}}, \mathbb{0}) \\ \bar{u} &= h_{\mathrm{s}}(\bar{x}_{\mathrm{s}}, \bar{\eta}, \bar{\xi}, \bar{y}_{\mathrm{m}}, \mathbb{0}) \end{aligned}$$

for each (w, δ). Since (φ, h_ϵ) is an optimality model, the pair $(\bar{x}, \bar{u})$ satisfies $y^\star(w, \delta) = C(\delta)\bar{x} + D(\delta)\bar{u} + Q(\delta)w$. Because this equilibrium point attracts all trajectories of the closed-loop system and $y(t)$ is continuous since the system is well-posed, it must be the case that $\lim_{t\to\infty} y(t) = y^\star(w, \delta)$ for every (w, δ) and every initial condition. Therefore, the controller (15b), (15c), (16a), (16b) solves the OSS control problem. □

Proof of Proposition 4.11: For each (w, δ), consider the solutions $(\bar{x}, \bar{\nu}, \bar{u})$ to

$$\mathbb{0} = A(\delta)\bar{x} + B(\delta)\bar{u} + B_w(\delta)w \tag{51a}$$

$$\bar{y} = C(\delta)\bar{x} + D(\delta)\bar{u} + Q(\delta)w \tag{51b}$$

$$\mathbb{0} = \varphi_\nu(\bar{\nu}, \mathsf{f}(y; w)) \tag{51c}$$

$$\mathbb{0} = H\bar{y} - Lw + T_0^{\mathsf{T}}\left(\nabla f_0(\bar{y}; w) + \sum_{i=1}^{n_{\mathrm{ic}}} \bar{\nu}_i \nabla f_i(\bar{y}; w)\right). \tag{51d}$$

The equations (51) correspond to the equations (14) in the definition of an optimality model. By assumption, the feasible region of the optimization problem (11) is non-empty: hence, there exists a $\mathsf{y}(w, \delta)$ such that

$$G_\perp(\delta)\mathsf{y}(w, \delta) = b(w, \delta), \quad H\mathsf{y}(w, \delta) = Lw.$$

Equations (51a) and (51b) imply that $G_\perp(\delta)\bar{y} = b(w, \delta)$. Hence, there exists a v such that $\bar{y} = \mathsf{y}(w, \delta) + G(\delta)v$. Equation (51d) and the fact that $H\mathsf{y}(w, \delta) = Lw$ then imply

$$\mathbb{0} = HG(\delta)v + T_0^{\mathsf{T}}\left(\nabla f_0(\bar{y}; w) + \sum_{i=1}^{n_{\mathrm{ic}}} \bar{\nu}_i \nabla f_i(\bar{y}; w)\right). \tag{52}$$

Since

$$\mathbf{range}\, HG(\delta) \cap \mathbf{range}\, T_0^{\mathsf{T}} = \{\mathbb{0}\} \tag{53}$$

for all $\delta \in \boldsymbol{\delta}$, (52) and (53) imply

$$\begin{aligned} \mathbb{0} &= HG(\delta)v \\ \mathbb{0} &= T_0^{\mathsf{T}}\left(\nabla f_0(\bar{y}; w) + \textstyle\sum_{i=1}^{n_{\mathrm{ic}}} \bar{\nu}_i \nabla f_i(\bar{y}; w)\right) \end{aligned}$$

for every (w, δ). Since $H\bar{y} - Lw = HG(\delta)v$,

$$\begin{aligned} \mathbb{0} &= H\bar{y} - Lw \\ \mathbb{0} &= T_0^{\mathsf{T}}\left(\nabla f_0(\bar{y}; w) + \textstyle\sum_{i=1}^{n_{\mathrm{ic}}} \bar{\nu}_i \nabla f_i(\bar{y}; w)\right) \end{aligned}$$

for every (w,δ). The remainder of the proof proceeds like the proof of Proposition 4.9. □

Proof of Proposition 4.14: We evaluate stabilizability and detectability of the augmented system via Theorem 2.2. We will make use of the following lemma.

*Lemma A.1 (***Unique Solution***):* Suppose the optimization problem

$$\begin{aligned} \underset{y\in\mathbb{R}^p}{\text{minimize}} \quad & \tfrac{1}{2}y^\mathsf{T}My - y^\mathsf{T}Nw \\ \text{subject to} \quad & G_\perp(\delta)y = b(w,\delta) \\ & Hy = Lw \end{aligned} \tag{54}$$

with $M \succeq \mathbb{0}$ is feasible and satisfies the robust feasible subspace property. Let $T_0 \in \mathbb{R}^{p\times\bullet}$ be any matrix satisfying $\mathbf{range}\, T_0 = \mathbf{null}\left[\begin{smallmatrix} G_\perp(\delta) \\ H\end{smallmatrix}\right]$. Then (54) has a unique optimizer if and only if $v^\mathsf{T}Mv > 0$ for all non-zero $v \in \mathbf{range}\, T_0$.

Proof: Fix a member $\tilde{y}(w,\delta)$ of the feasible set of (54). Since $\mathbf{range}\, T_0 = \mathbf{null}\left[\begin{smallmatrix} G_\perp(\delta) \\ H\end{smallmatrix}\right]$, we can rewrite the optimization problem (54) as

$$\begin{aligned} \underset{y\in\mathbb{R}^p,\, v\in\mathbf{range}\, T_0}{\text{minimize}} \quad & \tfrac{1}{2}y^\mathsf{T}My - y^\mathsf{T}Nw \\ \text{subject to} \quad & y = \tilde{y}(w,\delta) + v\,, \end{aligned} \tag{55}$$

where $v \in \mathbf{range}\, T_0$ is a new decision variable which is in one-to-one correspondence with y. Eliminating y we obtain the equivalent problem

$$\begin{aligned} \underset{v\in\mathbf{range}\, T_0}{\text{minimize}} \quad & \tfrac{1}{2}v^\mathsf{T}Mv + v^\mathsf{T}(M\tilde{y}(w,\delta) - Nw) \\ & + \tilde{y}(w,\delta)^\mathsf{T}\left(M\tilde{y}(w,\delta) - Nw\right). \end{aligned} \tag{56}$$

The unconstrained QP (56) has a unique optimizer $v^\star$ if and only if $M \succ \mathbb{0}$ on $\mathbf{range}\, T_0$ and the result follows. □

We move on to the main proof. Condition (i) of Proposition 4.14 is exactly condition (i) of Theorem 2.2. We show the remaining conditions of Proposition 4.14 are equivalent to condition (ii) of Theorem 2.2. Fix a $\delta \in \boldsymbol{\delta}$ and define the matrices $\mathcal{N}(\delta)$, $G(\delta)$, and $G_\perp(\delta)$ as done in Section III. The augmented plant using the RFS-OM is

$$\begin{aligned} \dot{x} &= A(\delta)x + B(\delta)u + B_w(\delta)w \\ \dot{\eta} &= \begin{bmatrix} HC(\delta) \\ T_0^\mathsf{T}MC(\delta)\end{bmatrix}x + \begin{bmatrix} HD(\delta) \\ T_0^\mathsf{T}MD(\delta)\end{bmatrix}u - \begin{bmatrix} L \\ T_0^\mathsf{T}N\end{bmatrix}w. \end{aligned}$$

Therefore, we examine whether the matrix

$$\mathcal{R}_{\mathrm{RFS}} := \left[\begin{array}{c|c} I_n & \mathbb{0} \\ \hline \mathbb{0} & \left[\begin{smallmatrix} H \\ T_0^\mathsf{T}M\end{smallmatrix}\right]\end{array}\right]\begin{bmatrix} A(\delta) & B(\delta) \\ C(\delta) & D(\delta)\end{bmatrix} \tag{57}$$

is full row rank. Let $\mathrm{col}(\alpha,\beta,\gamma) \in \mathbf{null}\,\mathcal{R}_{\mathrm{RFS}}^\mathsf{T}$, so that

$$\begin{bmatrix} \alpha \\ H^\mathsf{T}\beta + MT_0\gamma\end{bmatrix}^\mathsf{T}\begin{bmatrix} A(\delta) & B(\delta) \\ C(\delta) & D(\delta)\end{bmatrix} = \mathbb{0}. \tag{58}$$

Multiplying on the right by $\mathcal{N}(\delta)$ and recalling that $\mathbf{range}\,\mathcal{N}(\delta) = \mathbf{null}\begin{bmatrix} A(\delta) & B(\delta)\end{bmatrix}$ and also that $G(\delta) = \begin{bmatrix} C(\delta) & D(\delta)\end{bmatrix}\mathcal{N}(\delta)$, we find

$$(H^\mathsf{T}\beta + MT_0\gamma)^\mathsf{T}G(\delta) = \mathbb{0}. \tag{59}$$

Hence, $H^\mathsf{T}\beta + MT_0\gamma \in (\mathbf{range}\, G(\delta))^\perp$. Because $(\mathbf{range}\, G(\delta))^\perp = \mathbf{range}\, G_\perp(\delta)^\mathsf{T}$ by the definition of $G_\perp(\delta)$, the above is equivalent to the existence of a vector v such that

$$H^\mathsf{T}\beta + MT_0\gamma = G_\perp(\delta)^\mathsf{T}v. \tag{60}$$

Recall that $\mathbf{range}\, T_0 = (\mathbf{null}\, G_\perp(\delta)) \cap (\mathbf{null}\, H)$, so $G_\perp(\delta)T_0 = \mathbb{0}$ and $HT_0 = \mathbb{0}$. Multiplying (60) on the left by $\gamma^\mathsf{T}T_0^\mathsf{T}$ we find

$$\gamma^\mathsf{T}T_0^\mathsf{T}MT_0\gamma = 0. \tag{61}$$

For the sufficient direction, we show that if conditions (ii),(iii),(iv) hold, then $\mathrm{col}(\alpha,\beta,\gamma) = \mathbb{0}$, i.e., the left null space of the matrix (57) is empty, and therefore the matrix is full row rank. From conditions (iii) and (iv), it follows by Lemma A.1 that the matrix $T_0^\mathsf{T}MT_0$ is positive definite and hence $\gamma = \mathbb{0}$. Equation (60) then implies that

$$\begin{bmatrix} v & -\beta\end{bmatrix}^\mathsf{T}\begin{bmatrix} G_\perp(\delta) \\ H\end{bmatrix} = \mathbb{0}. \tag{62}$$

By condition (ii) the constraints of the problem (29) are nonredundant, and hence (62) implies that $v = \mathbb{0}$ and $\beta = \mathbb{0}$. Equation (58) then implies that

$$\alpha^\mathsf{T}\begin{bmatrix} A(\delta) & B(\delta)\end{bmatrix} = \mathbb{0}.$$

Since $(A(\delta), B(\delta))$ is stabilizable, the left null space of $\begin{bmatrix} A(\delta) & B(\delta)\end{bmatrix}$ is empty. Therefore $\alpha = \mathbb{0}$ and we conclude that $\mathcal{R}_{\mathrm{RFS}}$ has full row rank.

For the necessary direction, we show that if any one of the conditions (ii),(iii),(iv) fail, then we can construct $\mathrm{col}(\alpha,\beta,\gamma) \neq \mathbb{0}$ satisfying (58). Suppose (ii) fails, so there exists a nonzero solution to (62). It cannot be the case that $\beta = \mathbb{0}$, for then v would be zero since $G_\perp(\delta)$ is full row rank by construction. As a result, if we set $\gamma := \mathbb{0}$, (59) implies that there exists a $\bar{\beta} \neq \mathbb{0}$ such that $\bar{\beta}^\mathsf{T}HG(\delta) = \mathbb{0}$. We observe

$$\bar{\beta}^\mathsf{T}HG(\delta) = \begin{bmatrix} \bar{\beta}^\mathsf{T}HC(\delta) & \bar{\beta}^\mathsf{T}HD(\delta)\end{bmatrix}\mathcal{N}(\delta) = \mathbb{0}. \tag{63}$$

Since $\mathbf{range}\,\mathcal{N}(\delta) = \mathbf{null}\begin{bmatrix} A(\delta) & B(\delta)\end{bmatrix}$, the preceding implies that

$$\begin{bmatrix} C(\delta)^\mathsf{T}H^\mathsf{T}\bar{\beta} \\ D(\delta)^\mathsf{T}H^\mathsf{T}\bar{\beta}\end{bmatrix} \in \left(\mathbf{null}\begin{bmatrix} A(\delta) & B(\delta)\end{bmatrix}\right)^\perp = \mathbf{range}\begin{bmatrix} A(\delta)^\mathsf{T} \\ B(\delta)^\mathsf{T}\end{bmatrix}.$$

As a result, a solution $\bar{\alpha}$ exists to

$$\begin{bmatrix} C(\delta)^\mathsf{T}H^\mathsf{T}\bar{\beta} \\ D(\delta)^\mathsf{T}H^\mathsf{T}\bar{\beta}\end{bmatrix} = \begin{bmatrix} A(\delta)^\mathsf{T} \\ B(\delta)^\mathsf{T}\end{bmatrix}\bar{\alpha}.$$

Let $\bar{\alpha}$ satisfy the above. Then $\mathrm{col}(\alpha,\beta,\gamma) := \mathrm{col}(-\bar{\alpha},\bar{\beta},\mathbb{0})$ satisfies (58). Next, if (iii) fails, then there exists a $\bar{\gamma} \neq \mathbb{0}$ such that $\bar{\gamma}^\mathsf{T}T_0^\mathsf{T}MT_0\bar{\gamma} = 0$ by Lemma A.1. Moreover, this $\bar{\gamma}$ satisfies $MT_0\bar{\gamma} = \mathbb{0}$. To see this, note that since M is positive semidefinite, M has a positive semidefinite square root P satisfying $M = P^\mathsf{T}P$ [48]. Hence

$$\bar{\gamma}^\mathsf{T}T_0^\mathsf{T}MT_0\bar{\gamma} = \bar{\gamma}^\mathsf{T}T_0^\mathsf{T}P^\mathsf{T}PT_0\bar{\gamma} = \|PT_0\bar{\gamma}\|^2\,,$$

from which we can infer that $MT_0\bar{\gamma} = P^\mathsf{T}(PT_0\bar{\gamma}) = \mathbb{0}$. It follows that the vector $\mathrm{col}(\alpha,\beta,\gamma) := \mathrm{col}(\mathbb{0},\mathbb{0},\bar{\gamma})$ satisfies (58). Finally, if (iv) fails, then there exists a $\bar{\gamma} \neq \mathbb{0}$ such that $T_0\bar{\gamma} = \mathbb{0}$. It follows that the vector $\mathrm{col}(\alpha,\beta,\gamma) := \mathrm{col}(\mathbb{0},\mathbb{0},\bar{\gamma})$ satisfies (58). □

Proof of Proposition 4.15: We evaluate stabilizability and detectability of the augmented system via Theorem 2.2. Condition (i) of Proposition 4.15 is exactly condition (i) of Theorem 2.2. We show the remaining conditions of Proposition 4.15 are equivalent to condition (ii) of Theorem 2.2.

Fix a $\delta \in \boldsymbol{\delta}$. Define the matrices $\mathcal{N}(\delta)$, $G(\delta)$, and $G_\perp(\delta)$ as done in Section III. Let T_0 be a full column rank matrix satisfying (20).

The augmented plant using the ROS-OM is

$$\begin{aligned}\dot{x} &= A(\delta)x + \mathbb{0}\mu + B(\delta)u + B_w(\delta)w \\ \dot{\mu} &= HC(\delta)x + \mathbb{0}\mu + HD(\delta)u - Lw \\ \dot{\eta} &= G_0^\mathsf{T} MC(\delta)x + G_0^\mathsf{T} H^\mathsf{T}\mu + G_0^\mathsf{T} MD(\delta)u - G_0^\mathsf{T} Nw.\end{aligned}$$

Therefore, we examine whether the matrix

$$\begin{bmatrix} A(\delta) & \mathbb{0} & B(\delta) \\ HC(\delta) & \mathbb{0} & HD(\delta) \\ G_0^\mathsf{T} MC(\delta) & G_0^\mathsf{T} H^\mathsf{T} & G_0^\mathsf{T} MD(\delta) \end{bmatrix} \tag{64}$$

is full row rank. Let $\mathrm{col}(\alpha, \beta, \gamma)$ be a member of the left null space of (64), so that

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}^\mathsf{T} \begin{bmatrix} A(\delta) & \mathbb{0} & B(\delta) \\ HC(\delta) & \mathbb{0} & HD(\delta) \\ G_0^\mathsf{T} MC(\delta) & G_0^\mathsf{T} H^\mathsf{T} & G_0^\mathsf{T} MD(\delta) \end{bmatrix} = \mathbb{0}. \tag{65}$$

One may rewrite the above equivalently as

$$\begin{bmatrix} \alpha \\ H^\mathsf{T}\beta + MG_0\gamma \end{bmatrix}^\mathsf{T} \begin{bmatrix} A(\delta) & B(\delta) \\ C(\delta) & D(\delta) \end{bmatrix} = \mathbb{0} \tag{66a}$$

$$HG_0\gamma = \mathbb{0}. \tag{66b}$$

Multiplying (66a) on the right by $\mathcal{N}(\delta)$ and recalling that $\mathbf{range}\,\mathcal{N}(\delta) = \mathbf{null}\begin{bmatrix} A(\delta) & B(\delta)\end{bmatrix}$ and also that $G(\delta) = \begin{bmatrix} C(\delta) & D(\delta)\end{bmatrix}\mathcal{N}(\delta)$, we find

$$(H^\mathsf{T}\beta + MG_0\gamma)^\mathsf{T} G(\delta) = \mathbb{0}.$$

Hence, $H^\mathsf{T}\beta + MT_0\gamma \in (\mathbf{range}\, G(\delta))^\perp$. Recalling that $\mathbf{range}\, G_0 = \mathbf{range}\, G(\delta)$, the vector pair (β, γ) satisfies the above if and only if (β, γ) satisfies

$$(H^\mathsf{T}\beta + MG_0\gamma)^\mathsf{T} G_0 = \mathbb{0}.$$

We multiply on the left and right by γ and make use of (66b) to find

$$\gamma^\mathsf{T} G_0^\mathsf{T} MG_0\gamma = 0. \tag{67}$$

By (66b) we have that $G_0\gamma \in \mathbf{null}\, H$. By definition, $G_0\gamma \in \mathbf{null}\, G_\perp(\delta)$ also. Since $\mathbf{range}\, T_0 = (\mathbf{null}\, G_\perp(\delta)) \cap (\mathbf{null}\, H)$, there exists a vector v such that $G_0\gamma = T_0 v$. Using (67), this v satisfies

$$v^\mathsf{T} T_0^\mathsf{T} MT_0 v = \mathbb{0}.$$

The remainder of the proof proceeds like the proof of Proposition 4.14 following equation (61). ☐

Proof of Proposition 4.16: To begin we proceed as in the proof of Proposition 4.14. The augmented plant using the reduced-error RFS-OM is

$$\begin{aligned}\dot{x} &= A(\delta)x + B(\delta)u + B_w(\delta)w \\ \dot{\eta} &= (HC(\delta) + T_0^\mathsf{T} MC(\delta))x + (HD(\delta) + T_0^\mathsf{T} MD(\delta))u \\ &\quad - (Lw + T_0^\mathsf{T} N)w.\end{aligned}$$

Therefore, we examine whether the matrix

$$\mathcal{R}_{\mathrm{re}} := \begin{bmatrix} I & \mathbb{0} \\ \mathbb{0} & H + T_0^\mathsf{T} M \end{bmatrix} \begin{bmatrix} A(\delta) & B(\delta) \\ C(\delta) & D(\delta) \end{bmatrix} \tag{68}$$

is full row rank. Let $\mathrm{col}(\alpha, \beta) \in \mathbf{null}\, \mathcal{R}_{\mathrm{re}}^\mathsf{T}$, which is equivalent to the equations

$$\begin{bmatrix} \alpha^\mathsf{T} \\ \beta^\mathsf{T}(H + T_0^\mathsf{T} M) \end{bmatrix} \begin{bmatrix} A(\delta) & B(\delta) \\ C(\delta) & D(\delta) \end{bmatrix} = \mathbb{0} \tag{69}$$

Multiplying on the right by $\mathcal{N}(\delta)$ and recalling that $\mathbf{range}\,\mathcal{N}(\delta) = \mathbf{null}\begin{bmatrix} A(\delta) & B(\delta)\end{bmatrix}$ and also that $G(\delta) = \begin{bmatrix} C(\delta) & D(\delta)\end{bmatrix}\mathcal{N}(\delta)$, we find

$$\beta^\mathsf{T}(H + T_0^\mathsf{T} M)G(\delta) = \mathbb{0}. \tag{70}$$

Hence, $H^\mathsf{T}\beta + MT_0\beta \in (\mathbf{range}\, G(\delta))^\perp$. Because $(\mathbf{range}\, G(\delta))^\perp = \mathbf{range}\, G_\perp(\delta)^\mathsf{T}$ by the definition of $G_\perp(\delta)$, (70) is equivalent to the existence of a vector v such that

$$H^\mathsf{T}\beta + MT_0\beta = G_\perp(\delta)^\mathsf{T} v. \tag{71}$$

Recall that $\mathbf{range}\, T_0 = (\mathbf{null}\, G_\perp(\delta)) \cap (\mathbf{null}\, H)$, so $G_\perp(\delta)T_0 = \mathbb{0}$ and $HT_0 = \mathbb{0}$. Multiplying (71) on the left by $\beta^\mathsf{T} T_0^\mathsf{T}$, we find that $\beta^\mathsf{T} T_0^\mathsf{T} MT_0\beta = \mathbb{0}$.

For the sufficient direction, we show that if conditions (ii),(iii) hold then $\mathrm{col}(\alpha, \beta) = \mathbb{0}$, i.e., the left null space of the matrix (68) is empty, and therefore the matrix is full row rank. By Lemma A.1, condition (ii) implies M is positive definite on $\mathbf{range}\, T_0$, so it follows from the above that $T_0\beta = \mathbb{0}$, or equivalently that $\beta \in (\mathbf{range}\, T_0^\mathsf{T})^\perp$. It follows then from (70) that $\beta^\mathsf{T} HG(\delta) = \mathbb{0}$, implying that $\beta \in (\mathbf{range}\, HG(\delta))^\perp$ also. From condition (iii) we have $(\mathbf{range}\, HG(\delta))^\perp \cap (\mathbf{range}\, T_0^\mathsf{T})^\perp = \{\mathbb{0}\}$, so we conclude that $\beta = \mathbb{0}$. Equation (69) then reads $\alpha^\mathsf{T}\begin{bmatrix} A(\delta) & B(\delta)\end{bmatrix} = \mathbb{0}$, from which we conclude $\alpha = \mathbb{0}$ since $(A(\delta), B(\delta))$ is stabilizable.

For the necessary direction, we show that if any one of the conditions (ii),(iii) fails, then we can construct a vector $\mathrm{col}(\alpha, \beta) \neq \mathbb{0}$ satisfying (69). Suppose (ii) fails, so that by Lemma A.1, there exists a $\bar{\beta} \neq \mathbb{0}$ such that $\bar{\beta}^\mathsf{T} T_0^\mathsf{T} MT_0\bar{\beta} = 0$ but $T_0\bar{\beta} \neq \mathbb{0}$. Equation (70) implies that a solution $\bar{\alpha}$ exists to

$$\begin{bmatrix} C(\delta)^\mathsf{T}(H^\mathsf{T}\bar{\beta} + MT_0\bar{\beta}) \\ D(\delta)^\mathsf{T}(H^\mathsf{T}\bar{\beta} + MT_0\bar{\beta}) \end{bmatrix} = \begin{bmatrix} A(\delta)^\mathsf{T} \\ B(\delta)^\mathsf{T} \end{bmatrix}\bar{\alpha}$$

using the same reasoning as in the proof of Proposition 4.14 following (63). With such an $\bar{\alpha}$, $\mathrm{col}(\alpha, \beta) := \mathrm{col}(-\bar{\alpha}, \bar{\beta})$ satisfies (69).

Now suppose (iii) fails. Then there exists a $\bar{\beta} \neq \mathbb{0}$ such that $T_0\bar{\beta} = \mathbb{0}$ and $\bar{\beta}^\mathsf{T} HG(\delta) = \mathbb{0}$. The same reasoning as the proof of Proposition 4.14 following (63) shows that a solution $\bar{\alpha}$ exists to

$$\begin{bmatrix} C(\delta)^\mathsf{T} H^\mathsf{T}\bar{\beta} \\ D(\delta)^\mathsf{T} H^\mathsf{T}\bar{\beta} \end{bmatrix} = \begin{bmatrix} A(\delta)^\mathsf{T} \\ B(\delta)^\mathsf{T} \end{bmatrix}\bar{\alpha}.$$

With such an $\bar{\alpha}$, $\mathrm{col}(\alpha, \beta) := \mathrm{col}(-\bar{\alpha}, \bar{\beta})$ satisfies (69). ☐

Liam Lawrence (S'17) received the B. Eng. degree in engineering physics from McMaster University, Hamilton, ON, Canada in 2017 and the M.A.Sc. degree in electrical engineering from the University of Waterloo, Waterloo, ON, Canada in 2019. He is currently a doctoral student in the Medical Biophysics department at the University of Toronto, Toronto, ON, Canada. He is a recipient of the NSERC Canada Graduate Scholarship at both the master's and doctoral level.

John W. Simpson-Porco (S'11–M'18) received the B.Sc. degree in engineering physics from Queen's University, Kingston, ON, Canada in 2010, and the Ph.D. degree in mechanical engineering from the University of California at Santa Barbara, Santa Barbara, CA, USA in 2015.

He is currently an Assistant Professor of Electrical and Computer Engineering at the University of Waterloo, Waterloo, ON, Canada. He was previously a visiting scientist with the Automatic Control Laboratory at ETH Zürich, Zürich, Switzerland. His research focuses on feedback control theory and applications of control in modernized power grids.

Prof. Simpson-Porco is a recipient of the 2012–2014 IFAC Automatica Prize and the Center for Control, Dynamical Systems and Computation Best Thesis Award and Outstanding Scholar Fellowship.

Enrique Mallada (S'09-M'18) is an Assistant Professor of Electrical and Computer Engineering at Johns Hopkins University. Prior to joining Hopkins in 2016, he was a Post-Doctoral Fellow in the Center for the Mathematics of Information at Caltech from 2014 to 2016. He received his Ingeniero en Telecomunicaciones degree from Universidad ORT, Uruguay, in 2005 and his Ph.D. degree in Electrical and Computer Engineering with a minor in Applied Mathematics from Cornell University in 2014. Dr. Mallada was awarded the NSF CAREER award in 2018, the ECE Director's PhD Thesis Research Award for his dissertation in 2014, the Center for the Mathematics of Information (CMI) Fellowship from Caltech in 2014, and the Cornell University Jacobs Fellowship in 2011. His research interests lie in the areas of control, dynamical systems and optimization, with applications to engineering networks such as power systems and the Internet.