

# Voltage Collapse Stabilization in Star DC Networks

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**Abstract**—Voltage collapse is a type of blackout-inducing dynamic instability that occurs when the power demand exceeds the maximum power that can be transferred through the network. The traditional (preventive) approach to avoid voltage collapse is based on ensuring that the network never reaches its maximum capacity. However, such an approach leads to inefficiencies as it prevents operators to fully utilize the network resources and does not account for unprescribed events. To overcome this limitation, this paper seeks to initiate the study of *voltage collapse stabilization*.

More precisely, for a DC star network, we formulate the problem of voltage stability as a dynamic problem where each load seeks to achieve a constant power consumption by updating its conductance as the voltage changes. We show that such a system can be interpreted as a game, where each player (load) seeks to myopically maximize their utility using a gradient-based response.

Using this framework, we show that voltage collapse is the unique Nash Equilibrium of the induced game and is caused by the lack of cooperation between loads. Finally, we propose a *Voltage Collapse Stabilizer* (VCS) controller that uses (flexible) loads that are willing to cooperate and provides a fair allocation of the curtailed demand. Our solution stabilizes voltage collapse even in the presence of non-cooperative loads. Numerical simulations validate several features of our controllers.

## I. INTRODUCTION

Voltage collapse (VC) is a type of outage in power networks that arises when the aggregate power demand exceeds the capacity of the network to transfer the required power [1],[2],[3]. When such a point is achieved, (inflexible) constant power loads tend to rapidly reduce their effective impedance, bringing the voltage abruptly to zero. While this mechanism is intrinsically dynamic, associated with a saddle node bifurcation [4],[5], the inability to correct this behavior from the generation side has lead power engineers to take a rather static (preventive) approach to address it. That is, to ensure that the point of maximum network loading is never reached [6]. As a consequence, there has been a vast body of work trying to quantify voltage stability margins. This includes classical works, such as [7], [8],[9],[10] and more recently, [11], [12]. However, this approach leads to inefficiencies as it prevents operators to fully utilize the network resources and does not account for unprescribed events that can still produce a blackout.

This work seeks to initiate the study of *voltage collapse stabilization*. More specifically, we aim to investigate how to use (flexible) demand response to reduce consumption and match network capacity, when the total demand exceeds it. In this way we prevent inflexible demand from driving the system to voltage collapse. To the best of our knowledge, this work is the first effort on addressing the dynamic aspect of voltage collapse to design controllers aimed at preventing

it. Such a control scheme is required to overcome two main challenges. Firstly, it needs to stabilize an operating point that under inflexible load behavior is unstable.<sup>1</sup> Secondly, it needs to prevent collapse even in the presence of inflexible loads.

The work is motivated by the rapid development of power electronics and information technology [14] that, for the first time since the inception of the power system, has the potential to provide enough demand-side controllability that could allow us to envision the possibility of stabilizing voltage collapse. However, despite the additional flexibility that controllable demand provides, there are numerous questions that remain to be answered. Among them:

- Is voltage collapse stabilization possible?
- Can stabilization be achieved via decentralized actions?
- How should the necessary demand reduction be allocated among the flexible loads?

In this work, we provide an initial answer to some of these questions for a simple direct current DC network. In particular, we consider a star resistive network where each load seeks to consume a constant power by dynamically updating its conductance using a standard voltage droop. We show that the system can be interpreted as a game, where each player (load) seeks to (locally) maximize its utility by choosing a gradient-based response. Notably, voltage collapse can then be interpreted as the consequence of the selfish behavior of the players that drive the system towards the unique Nash equilibrium of the game. This observation hints at the need of introducing coordination to overcome voltage collapse, and motivates the proposed voltage collapse stabilizing control.

The rest of the paper is organized as follows. Section II introduces our DC network model of constant power loads as well as some required game theory terminology. Section III investigates the properties of (10) and frames our network model as a load satisfiability game where the unique Nash Equilibrium (NE) is the voltage collapse state. Section IV describes our voltage collapse stabilizer controller and studies its static and dynamic properties. We illustrate several features of our controllers using numerical simulations in Section V and conclude in Section VI.

## II. PRELIMINARIES

In this section we introduce the network model to be considered in this paper as well as the game theoretic framework to be used.

### A. DC Power Network Model

We consider the DC star network model described in Figure 1, where  $E$  denotes the source voltage,  $g_l$  the conductance of a transmission line that transfers power to  $n$  loads, and  $g_i$  denotes the  $i$ th load conductance,  $i \in N := \{1, \dots, n\}$ . We consider two types of loads, the *flexible*

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This work was supported by ARO through contract W911NF-17-1-0092, US DoE EERE award DE-EE0008006, and NSF through grants CNS 1544771, EPCN 1711188, AMPS 1736448, and CAREER 1752362.

<sup>1</sup>At a saddle node bifurcation a stable and an unstable equilibrium are merged, which leads to an unstable equilibrium [13].

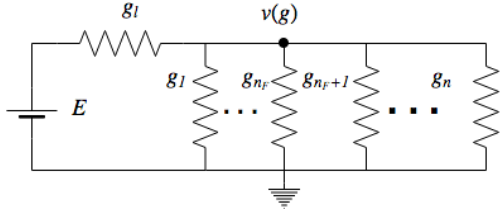


Fig. 1. Star DC Network with Several Dynamic loads

loads, belonging to the set  $F = \{1, \dots, n_F\}$  ( $n_F = |F|$ ), and the *inflexible* loads, belonging to  $I = \{n_F + 1, \dots, n\}$  ( $n_I = |I|$ ). Hence, the set of all loads is  $N = I \cup F = \{1, \dots, n\}$ . We further use  $g = (g_1, \dots, g_n) \in \mathbb{R}_{\geq 0}^n$  to denote the vector of conductances and  $g_{-i} \in \mathbb{R}^{n-1}$  the vector of all load conductances except  $g_i$ .

Using this notation, we can use Kirchoff's voltage and current laws (KVL and KCL) to compute the voltage applied to each load

$$v(g) = \frac{Eg_l}{\sum_{i \in N} g_i + g_l}. \quad (1)$$

Thus, the total power consumed by each load  $i \in N$  becomes

$$P_i(g) = v^2(g)g_i = \left( \frac{Eg_l}{g_{\text{eq}}(g) + g_l} \right)^2 g_i, \quad (2)$$

where  $g_{\text{eq}}(g) = \sum_{i \in N} g_i$  is the equivalent conductance. The difference between the power consumed by each load  $i \in N$  and its nominal demand  $P_{0,i} > 0$  is

$$\Delta P_i(g) = P_i(g) - P_{0,i}. \quad (3)$$

The total power consumed by all the loads in the system is

$$P_{\text{tot}}(g) = \sum_{i=1}^n P_i(g) = \frac{(Eg_l)^2}{(g_{\text{eq}}(g) + g_l)^2} g_{\text{eq}}(g). \quad (4)$$

For an arbitrary set  $S \subset N$  and its complement  $S^c = N \setminus S$ , we define  $g_S(g) = \sum_{i \in S} g_i$  and  $g_{S^c}(g) = \sum_{i \in S^c} g_i$ . The equivalent conductance can then be written as  $g_{\text{eq}}(g) = g_S(g) + g_{S^c}(g)$  and the aggregate power consumed by every load  $i \in S$  is defined as

$$P_S(g_S; g_{S^c}) = \sum_{i \in S} P_i(g) = \frac{(Eg_l)^2}{(g_S + g_{S^c} + g_l)^2} g_S. \quad (5)$$

Whenever  $S = N$  ( $S^c = \emptyset$ ) we drop the second argument in (5) and use  $P_N(g_N)$ .

**Network Capacity** ( $P_{S,\text{max}}$ ): Since voltage collapse is the result of the network reaching its maximum capacity [2], it is of interest to compute the maximum value that  $P_S(g)$  in (5) can achieve for fixed value of  $g_{S^c}$ .

A straightforward calculation shows that for all  $i \in S$

$$\frac{\partial}{\partial g_i} P_i(g) = \frac{(Eg_l)^2}{(g_{\text{eq}}(g) + g_l)^3} (g_l + g_{\text{eq}}(g) - 2g_i), \quad (6)$$

and similarly we get

$$\frac{\partial}{\partial g_S} P_S(g_S; g_{S^c}) = \frac{(Eg_l)^2}{(g_S + g_{S^c} + g_l)^3} (g_l + g_{S^c} - g_S). \quad (7)$$

From (7), it is easy to see that  $P_S(g_S; g_{S^c})$  is an increasing function of  $g_S$  whenever  $g_S < g_{S,\text{max}}(g_{S^c}) := g_l + g_{S^c}$ , and

decreasing when  $g_S > g_{S,\text{max}}(g_{S^c})$ . Therefore, the maximum power that can be supplied to the loads in  $S$  is given by

$$P_{S,\text{max}} = P_S(g_{S,\text{max}}; g_{S^c}) = \frac{E^2 g_l}{4} \frac{g_l}{g_l + g_{S^c}} \quad (8)$$

and is achieved when  $g_S = g_{S,\text{max}}(g_{S^c})$ .

In the special case where  $S = N$ , (8) becomes:

$$P_{\text{max}} = P_N(g_{N,\text{max}}) = \frac{E^2 g_l}{4} \quad (9)$$

where  $g_{N,\text{max}} := g_l$ .

**Dynamic Load Model:** We assume that each load  $i \in N$  has a constant power demand  $P_{0,i}$ . For an *inflexible* load  $i \in I$ , this demand  $P_{0,i}$  must always be satisfied. This is achieved by dynamically changing the conductance  $g_i$  in order to change the power consumption  $P_i(g)$ . Following [2], we use the following dynamic model

$$\dot{g}_i = -(v^2(g)g_i - P_{0,i}) = -\Delta P_i(g), \quad i \in I. \quad (10)$$

Notice that  $\mathbb{R}_{\geq 0}^n$  is invariant, since whenever  $g_i = 0$  then (10) implies that  $\dot{g}_i > 0$ .

**Definition 1 (Voltage Collapse):** The system (10) undergoes voltage collapse whenever  $v(g(t)) \rightarrow 0$  as  $t \rightarrow +\infty$ .

For the case of *flexible loads*, we assume that although they aim to satisfy their own constant power demand  $P_{0,i}$ , at the same time they are willing to consume less than  $P_{0,i}$  whenever  $P_{0,\text{tot}} := \sum_{i \in N} P_{0,i} > P_{\text{max}}$ . Thus, our goal is to design a control law

$$\dot{g}_i = u_i, \quad i \in F, \quad (11)$$

where the input  $u_i$  is such that in equilibrium  $\Delta P_i(g) = 0$  whenever  $P_{0,\text{tot}} < P_{\text{max}}$ .

**Power Flow Solutions:** Given an equilibrium  $g^*$  of (10)-(11), there exists a unique voltage  $v(g^*)$  and power consumption  $P(g^*) = (P_i(g^*), i \in N)$ . The pair  $(v, P)$  is referred as *power flow solution*. Thus, given the one-to-one relationship between  $g$  and the pair  $(v, P)$ , we refer to  $g^*$  as a power flow solution.

## B. Game Theory

We now present the game theoretical preliminaries that will allow us to better grasp the level of coordination required to prevent voltage collapse.

**Definition 2 (Normal Form Game [15]):** A Normal Form Game is given by the triple  $\langle N, S, \mathbf{u} \rangle$  where:

- 1)  $N = \{1, \dots, n\}$ , is the set of players.
- 2)  $S := S_1 \times \dots \times S_n$ , with  $S_i$  being the strategy set of player  $i \in N$ , is the set of strategies.
- 3)  $\mathbf{u} = \{u_i, i \in N\}$ , where  $u_i : S := S_1 \times \dots \times S_n \rightarrow \mathbb{R}$ ,  $\forall i \in N$ , is the set of payoff functions.

Given a game  $\langle N, S, \mathbf{u} \rangle$ , we seek to understand the set of strategies  $s = (s_1, \dots, s_n) \in S$  for which every player has no incentive to move. Moreover, since in our context it is in general difficult to understand the best response of each player, we focus on locally optimal strategies.

**Definition 3 (Nash Equilibrium [15]):** A strategy  $s^* = (s_1^*, \dots, s_n^*) \in S$  is a (strict) Nash Equilibrium (NE) if and only if for each  $i \in N$

$$u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*), \quad \forall s_i \in S_i. \quad (12)$$

**Definition 4 (Local Nash Equilibrium [16]):** A strategy  $s^* = (s_1^*, \dots, s_n^*) \in S$  is a (strict) Local Nash Equilibrium

(LNE) if and only if for each  $i \in N$  there exists an open set  $W_i \subset S_i$  such that:

$$\mathbf{u}_i(s_i^*, s_{-i}^*) > \mathbf{u}_i(s_i, s_{-i}^*), \quad \forall s_i \in W_i \setminus \{s_i^*\}. \quad (13)$$

Whenever the payoff functions  $\mathbf{u}_i$  are sufficiently smooth, it is possible to verify (13) using first and second order derivatives.

**Lemma 1 (Criterion for LNE [16]):** Given a game  $\langle N, \{S_i, i \in N\}, \{\mathbf{u}_i, i \in N\} \rangle$  with doubly continuously differentiable payoff functions, a strategy  $s^* = (s_1^*, \dots, s_n^*) \in S$  is a strict LNE whenever

$$\frac{\partial}{\partial s_i} \mathbf{u}_i(s^*) = 0 \text{ and } \frac{\partial^2}{\partial s_i^2} \mathbf{u}_i(s^*) < 0, \quad \forall i \in N. \quad (14)$$

### III. SYSTEM ANALYSIS WITH INFLEXIBLE LOADS

In this section we characterize the region of stable equilibria of (10) when  $P_{0,\text{tot}} < P_{\max}$  and prove that the system undergoes voltage collapse when  $P_{0,\text{tot}} > P_{\max}$ . We then build a game theoretical framework that provides a deeper insight on the voltage collapse phenomenon and further suggests the necessity of coordination among resources in order to prevent voltage collapse without incurring unnecessary inefficiencies. Throughout this section we assume  $I = N$  in (10).

#### A. Stability Analysis and Voltage Collapse

We start by first characterizing the region of stable equilibria of (10). For this reason, we consider the set

$$M := \{g \in \mathbb{R}^n : \sum_{i \in N} g_i < g_l\}. \quad (15)$$

**Lemma 2 (Characterization of Stable Region):** A hyperbolic equilibrium<sup>2</sup> point  $g^*$  of (10) is stable if and only if  $g^* \in M$ .

*Proof:* Let  $g^*$  be an equilibrium of (10), i.e.,  $\Delta P_i(g_i^*) = 0$  for all  $i \in N$ . The Jacobian of the system is given by

$$J(g^*) = \frac{2v^2(g^*)}{g_{\text{eq}}(g^*) + g_l} g^* \mathbb{1}_n^T - v^2(g^*) \mathbb{I}_n \quad (16)$$

where  $\mathbb{I}_n \in \mathbb{R}^{n \times n}$  is the identity matrix and  $\mathbb{1}_n^T \in \mathbb{R}^{1 \times n}$  is a vector of all ones.

Let  $K_1(g^*) = \frac{2v^2(g^*)}{g_{\text{eq}}(g^*) + g_l} g^* \mathbb{1}_n^T$ . Since  $K_1(g^*)$  is a rank 1 matrix, it has  $n-1$  eigenvalues  $\lambda_i(K_1) = 0$ ,  $i \in \{1, \dots, n-1\}$ , and one non-zero eigenvalue

$$\lambda_n(K_1) = \frac{2v^2(g^*)}{g_{\text{eq}}(g^*) + g_l} \mathbb{1}_n^T g^* = \frac{2v^2(g^*)}{g_{\text{eq}}(g^*) + g_l} g_{\text{eq}}(g^*).$$

The second term in (16) is an identity matrix and for that the eigenvalues of  $J(g^*)$  are shifted from the eigenvalues of  $K_1(g^*)$  by  $-v^2(g^*)$ , i.e.,  $\lambda_i(J) = \lambda_i(K_1) - v^2(g^*)$ , which gives

$$\lambda_i(J) = \begin{cases} -v^2(g^*), & i \in \{1, \dots, n-1\}; \\ \frac{2v^2(g^*)}{g_{\text{eq}}(g^*) + g_l} g_{\text{eq}}(g^*) - v^2(g^*), & i = n. \end{cases}$$

We can now prove the statement of the lemma.

( $\Rightarrow$ ) If  $g^*$  is an asymptotically stable hyperbolic equilibrium, then  $J(g^*)$  is Hurwitz and thus:  $\lambda_n(J) < 0 \Rightarrow v^2(g^*) \left( \frac{2g_{\text{eq}}(g^*)}{g_{\text{eq}}(g^*) + g_l} - 1 \right) < 0 \Rightarrow g_{\text{eq}}(g^*) < g_l$ .

<sup>2</sup>An equilibrium is hyperbolic if its Jacobian is nonsingular.

( $\Leftarrow$ ) If  $g^* \in M$ , then:  $\lambda_n(J) < 0$ . Since all eigenvalues of  $J(g^*)$  are negative, by Lyapunov's Indirect Method [13, Theorem 3.5]  $g^*$  is asymptotically stable. ■

We now show how in the overload regime ( $P_{0,\text{tot}} > P_{\max}$ ), the system is led to voltage collapse.

**Theorem 1 (Voltage Collapse with Inflexible Loads):**

The dynamic load model (10) with  $I = N$  undergoes a voltage collapse whenever  $\varepsilon := P_{0,\text{tot}} - P_{\max} > 0$ .

*Proof:* We have already pointed out that  $\mathbb{R}_{>0}^n$  is invariant. Also, it is easy to check that (10) is globally Lipschitz on  $\mathbb{R}_{>0}^n$  since  $g_l > 0$ . Thus, by [13, Theorem 3.2], there is a unique solution to (10),  $g(t)$ , that is defined  $\forall t \geq 0$ . Now, consider the function  $V(g) = \sum_{i \in N} g_i$ , and let  $S_V^+(a) = \{g \in \mathbb{R}_{>0}^n : V(g) \leq a\}$ . By taking the time derivative of  $V$  we get

$$\dot{V}(g) = \sum_{i=1}^n \dot{g}_i = - \sum_{i=1}^n P_i(g) - P_{0,i} \geq P_{0,\text{tot}} - P_{\max} = \varepsilon > 0.$$

Therefore,  $\forall a \geq 0$  if  $g(0) \in S_V^+(a)$ , then  $g(t)$  escapes  $S_V^+(a)$  in finite time and therefore  $\|g(t)\| \rightarrow \infty$  as  $t \rightarrow \infty$ . It follows then that  $g_{\text{eq}}(t)$  grows unboundedly and by (1)  $v(g(t)) \rightarrow 0$ , i.e., the system's voltage collapses. ■

**Remark 1:** Theorem 1 illustrates that our model for inflexible loads successfully captures the fundamental property that excess on power demand beyond the network implies voltage collapse and thus imitates the system's dynamics.

#### B. A Game for Inflexible Constant Power Loads

We now provide a game theoretical interpretation to voltage collapse. In our formulation, the set of players is the set of loads, both conveniently denoted by  $N$ . We consider here the case of inflexible loads, that is,  $I = N$  and  $F = \emptyset$ . For each player  $i \in N$  the strategy  $s_i$  is given by its conductance  $g_i \geq 0$ . Therefore, the strategy set  $S := \mathbb{R}_{>0}^n$ .

Following Definition 2, it remains to define the utility function of each agent  $i \in N$ . The following proposition motivates a particular choice of payoff function.

**Proposition 1:** Consider the game  $\langle N, S, \mathbf{u} \rangle$ , where  $S = \mathbb{R}_{>0}^n$  and for each load  $i \in N$  the utility function is given by

$$\mathbf{u}_i(g_i; g_{-i}) = P_{0,i} g_i + (E g_l)^2 \ln \left( \frac{g_{-i} + g_l}{g_i + g_{-i} + g_l} \right) - (E g_l)^2 \left( \frac{g_{-i} + g_l}{g_i + g_{-i} + g_l} - 1 \right), \quad (17)$$

where  $g_{-i}$  denotes  $g_{N \setminus \{i\}} = \sum_{j \neq i} g_j$ . Then, the inflexible load dynamics (10) amount to the myopic gradient dynamics

$$\dot{g}_i = \frac{\partial}{\partial s_i} \mathbf{u}_i(s), \quad i \in I. \quad (18)$$

As a consequence, if  $g^* \in \mathbb{R}_{>0}^n$  is a LNE of  $\langle N, S, \mathbf{u} \rangle$ , then it is an equilibrium of (10).

*Proof:*

From equation (10), it follows that if  $\mathbf{u}$  is the set of payoff functions of the game  $\langle N, S, \mathbf{u} \rangle$ , then

$$\frac{\partial \mathbf{u}_i(g_i; g_{-i})}{\partial g_i} = - \left( \left( \frac{E g_l}{g_{\text{eq}}(g) + g_l} \right)^2 g_i - P_{0,i} \right)$$

Integrating the above expression with respect to  $g_i$  gives

$$\mathbf{u}_i = \int_0^{g_i} \left( P_{0,i} - \frac{(E g_l)^2}{(s + g_{-i} + g_l)^2} s \right) ds$$

$$\begin{aligned}
&= \int_0^{g_i} \left( P_{0,i} - (Eg_l)^2 \frac{s + g_{-i} + g_l}{(s + g_{-i} + g_l)^2} \right) ds \\
&\quad + \int_0^{g_i} (Eg_l)^2 \frac{g_{-i} + g_l}{(s + g_{-i} + g_l)^2} ds \\
&= \left[ P_{0,i}s - (Eg_l)^2 \left( \ln(s + g_{-i} + g_l) + \frac{g_{-i} + g_l}{s + g_{-i} + g_l} \right) \right]_0^{g_i}
\end{aligned}$$

We retrieve (17) by substituting for the limit values. ■

Proposition 1 reverse engineers a utility function for each load such that any equilibrium of (10) is a stationary point of the game. While this clearly hints that *some* of the equilibria of (10) may constitute a LNE the following theorem unveils a more surprising fact.

**Theorem 2 (Voltage Collapse is the Unique NE):** Given the induced game  $\langle N, \mathbb{R}_{>0}^n, \mathbf{u} \rangle$  with utility given by (17), the strategy  $g_i \rightarrow +\infty \forall i \in N$  is the unique Nash Equilibrium.

*Proof:* We first show that each player maximizes their utility by setting  $g_i \rightarrow +\infty$ . Using (17),

$$\begin{aligned}
\lim_{g_i \rightarrow +\infty} \mathbf{u}_i(g_i; g_{-i}) &= \lim_{g_i \rightarrow +\infty} (Eg_l)^2 \ln \left( \frac{g_{-i} + g_l}{g_i + g_{-i} + g_l} \right) \\
&\quad - \lim_{g_i \rightarrow +\infty} (Eg_l)^2 \left( \frac{g_{-i} + g_l}{g_i + g_{-i} + g_l} - 1 \right) + \lim_{g_i \rightarrow +\infty} P_{0,i}g_i \\
&= \lim_{g_i \rightarrow +\infty} P_{0,i}g_i - (Eg_l)^2 \left( \ln \left( \frac{g_{-i} + g_l}{g_i + g_{-i} + g_l} \right) - 1 \right) = +\infty.
\end{aligned}$$

The previous derivation assumes that all other agents decide finite conductances. In the case where any other agent  $j \neq i$  is also choosing  $g_j \rightarrow \infty$ , then a similar computation using (17) gives

$$\lim_{g_j \rightarrow \infty} \mathbf{u}_i(g_i; g_{-i}) = P_{0,i}g_i + (Eg_l)^2,$$

which implies that

$$\lim_{g_j \rightarrow \infty} \lim_{g_i \rightarrow \infty} \mathbf{u}_i(g_i; g_{-i}) = \lim_{g_i \rightarrow \infty} P_{0,i}g_i + (Eg_l)^2 = +\infty$$

Therefore, choosing  $g_i \rightarrow +\infty$  is a strictly dominant strategy for agent  $i$ , i.e. it is the best possible strategy regardless of the strategy chosen by all other agents. Since  $i$  was chosen arbitrarily, results follows. ■

Theorem 2 unveils an unusual phenomenon. A game representation of (10) for which power flow solutions can provide some notion of (local) optimality (LNE) naturally leads to voltage collapse as a dominant strategy. This suggests that it is the selfish myopic behavior of each player (load) –that seeks to maximize their own payoff– that produces voltage collapse. This behavior is reminiscent of the *tragedy of the commons* [17], and further suggests that certain level of coordination may be required in order to prevent voltage collapse. This is the basis of the solution proposed next.

#### IV. VOLTAGE COLLAPSE STABILIZER CONTROL

We now focus our attention to the task of preventing voltage collapse. Thus, we assume that there exists a subset of the loads  $F \subseteq N$ ,  $F \neq \emptyset$ , that are receptive to curtailment. However, from an efficiency perspective, such curtailment should only occur whenever the total demand exceeds the network capacity ( $P_{0,\text{tot}} > P_{\text{max}}$ ). Moreover, if curtailment does occur, it should be fairly allocated among the flexible loads. In our setting this translates into designing a controller that allows for proportional sharing of load shedding among flexible loads. These design objectives are summarized in the following problem formulation.

**Problem 1 (Voltage Collapse Stabilization):** Design a control signal  $u_i$ ,  $i \in F$ , such that:

- **Load Satisfaction:** Whenever  $P_{0,\text{tot}} < P_{\text{max}}$  the equilibrium  $g^* \in M \cap \{g : \Delta P_i(g) = 0, i \in N\}$

is the unique asymptotically stable equilibrium within  $M$ .

- **Efficient Allocation:** Whenever  $P_{0,\text{tot}} > P_{\text{max}}$ , there exists a unique equilibrium within  $\text{cl}(M)$  given by  $g^* \in E_F \cap \text{cl}(M)$  that is asymptotically stable and leads to a *fair* curtailment, i.e., it is the optimal solution to

$$\begin{aligned}
&\text{minimize}_{\Delta P_i, i \in F} \sum_{i \in F} \frac{\theta_i}{2} (\Delta P_i)^2 \\
&\text{subject to} \quad \sum_{i \in F} \Delta P_i = P_{\text{max}} - P_{0,\text{tot}}.
\end{aligned} \tag{19}$$

**Remark 2:** Since by Lemma 2 the set  $M$  in (15) characterizes the region where the system typically operates in the absence of flexible loads, the goal of Problem 1 is to maintain this behavior and ensure that if the operating point gets to the boundary  $M$ , then it is stable and efficient.

We will call a control law  $u$  that solves Problem 1 a *Voltage Collapse Stabilizing (VCS) control*. The rest of this section is devoted to showing that the following is a VCS control

$$\dot{g} = -A(g)(P(g) - P_0) \tag{20}$$

where  $A(g) = \text{diag}\{\alpha_i(g), i \in N\}$ ,

$$\alpha_i(g) = \begin{cases} \frac{\kappa(\bar{g}_i - g_i)}{1 + \kappa(\bar{g}_i - g_i)}, & \forall i \in F; \\ 1, & \forall i \in I; \end{cases} \tag{21}$$

and

$$\bar{g}_i = \frac{P_{0,i}}{(E/2)^2} + \frac{P_{\text{max}} - P_{0,\text{tot}}}{\gamma_i(E/2)^2}, \tag{22}$$

with  $\gamma_i = \theta_i \sum_{j \in F} \frac{1}{\theta_j}$  and  $\kappa$  is a positive parameter:  $0 < \kappa < \infty$ .

**Remark 3:** The term  $\alpha_i(g)$  aims to introduce a new equilibrium point  $g^*$  when  $P_{0,\text{tot}} > P_{\text{max}}$  such that, whenever  $g^*$  satisfies  $\alpha_i(g^*) = 0 \forall i \in F$ , then  $(\Delta P_i(g^*), i \in F)$  is a solution to (19). However, as we show in the next section, this can tentatively introduce new equilibria.

#### A. Characterization of Equilibria

We now proceed to characterize the set of equilibria of (20). Given a set of loads  $G \subseteq N$ , consider

$$E_G := \{g : \alpha_i(g) = 0, i \in G, \Delta P_i(g) = 0, i \in G^c\} \tag{23}$$

It is easy to see that the set  $\cup_{G \subseteq F} E_G$  compactly encapsulates every equilibrium of (20). The following lemma will allow us to further characterize each set (23).

**Lemma 3 (Intermediate Value Theorem [18]):** Let  $f \in C[a, b]$ . Then for any  $\psi \in (f(a), f(b))$  there exists  $\xi \in [a, b]$  such that  $f(\xi) = \psi$ .

**Lemma 4 (Characterization of  $E_G$ ):** Given any set  $G \subseteq F$ , the set  $E_G$  comprises of two finite equilibria  $g_1^*, g_2^*$  such that

$$g_{1,i}^* = g_{2,i}^* = \bar{g}_i \quad \forall i \in G,$$

and

$$\sum_{i \in G^c} g_{1,i}^* < g_l + \sum_{i \in G} \bar{g}_i < \sum_{i \in G^c} g_{2,i}^* \tag{24}$$

if and only if  $0 < P_{0,G^c} < \frac{E^2 g_l}{4} \frac{g_l}{g_G + g_l}$ .

Moreover, whenever  $G = F$ , then  $g_1^* \in \text{cl}(M)$  is such that

$$g_{\text{eq}}(g_1^*) = g_l, g_{1,i}^* = \frac{P_{0,i}}{\left(\frac{E}{2}\right)^2}, \forall i \in I, \text{ and } v(g_1^*) = \frac{E}{2}. \quad (25)$$

*Proof:* We assume that (20) has 2 equilibria  $g_1^*, g_2^* \in E_G$  with the properties as described in the statement of the Lemma. Since for  $i \in G^c$ ,  $\Delta P_i(g_k^*) = 0$ ,  $k \in \{1, 2\}$ , then substituting  $S = G^c$  into (5) and evaluating the expression at  $g_{k,G^c}$ ,  $k \in \{1, 2\}$ , yields

$$P_{G^c}(g_{k,G^c}^*; g_{k,G}^*) - P_{0,G^c} = \frac{(Eg_l)^2 g_{k,G^c}^*}{(g_{k,G^c}^* + \bar{g}_G + g_l)^2} - P_{0,G^c} = 0 \quad (26)$$

where we used the fact  $g_{k,G}^* = \sum_{i \in G} \bar{g}_i = \bar{g}_G$  (by assumption). Since  $P_{0,i} > 0$  by definition, we can divide by  $P_{0,G^c}$  in (26) to get

$$-(g_{k,G^c}^*)^2 + \left(\frac{(Eg_l)^2}{P_{0,G^c}} - 2(\bar{g}_G + g_l)\right) g_{k,G^c}^* - (\bar{g}_G + g_l)^2 = 0 \quad (27)$$

This is a second order polynomial and has 2 real roots if and only if

$$\begin{aligned} \Delta &= \left(\frac{(Eg_l)^2}{P_{0,G^c}} - 2(\bar{g}_G + g_l)\right)^2 - 4(\bar{g}_G + g_l)^2 > 0 \\ \iff P_{0,G^c} &< \frac{(Eg_l)^2}{4(\bar{g}_G + g_l)} = \frac{E^2 g_l}{4} \frac{g_l}{\bar{g}_G + g_l} \end{aligned} \quad (28)$$

( $\Leftarrow$ ) Given  $G$ , let  $\hat{g} \in \mathbb{R}^n$  such that  $\hat{g}_i = \bar{g}_i \forall i \in G$ . For  $\hat{g}$  to be an equilibrium, (27) must hold. We know by assumption that  $0 < P_{0,G^c} < \frac{E^2 g_l}{4} \frac{g_l}{\bar{g}_G + g_l}$  which implies that (27) has two real roots  $\hat{g}_{k,G^c} = g_{k,G^c}^*$ ,  $k = \{1, 2\}$ . Each root  $g_{k,G^c}^*$ ,  $k \in \{1, 2\}$ , defines an equivalent conductance  $g_{k,\text{eq}} = g_{k,G^c}^* + \bar{g}_G$  and a voltage level  $v_k = \frac{Eg_l}{g_{k,\text{eq}} + g_l}$ . Therefore, there will exist two equilibria  $\hat{g}_k \in E_G$ , with

$$\begin{aligned} \hat{g}_{1,i} &= \hat{g}_{2,i} = \bar{g}_i & \forall i \in G, \\ \hat{g}_{1,i} &= \frac{P_{0,i}}{v_1^2}, \quad \hat{g}_{2,i} = \frac{P_{0,i}}{v_2^2} & \forall i \in G^c. \end{aligned}$$

For the next property, we define the function

$$f(g_{G^c}) = P_{G^c}(g_{G^c}; \bar{g}_G) - P_{0,G^c}$$

and observe that  $f$  is equivalent to  $P_{G^c}(g_{G^c}; \bar{g}_G)$  shifted by a constant. Therefore,  $g_{G^c,\text{max}}$  is also a maximizer of  $f$  and:

$$\frac{\partial f(g_{G^c}; \bar{g}_G)}{\partial g_{G^c}} = \frac{\partial P_{G^c}(g_{G^c}; \bar{g}_G)}{\partial g_{G^c}} \stackrel{(7)}{=} \begin{cases} > 0 & g_{G^c} < g_{G^c,\text{max}} \\ < 0 & g_{G^c} > g_{G^c,\text{max}} \end{cases}$$

We can prove by contradiction that the two roots of  $f$  satisfy the condition  $g_{1,G^c}^* < g_{G^c,\text{max}} < g_{2,G^c}^*$ . If this condition is not met, then both equilibria  $g_{k,G^c}^*$ ,  $k \in \{1, 2\}$ , are either in  $(0, g_{G^c,\text{max}})$  or in  $(g_{G^c,\text{max}}, +\infty)$ . In the first interval  $f$  is strictly increasing, whereas in the second  $f$  is strictly decreasing. In this case,

$$0 = f(g_{1,G^c}^*) \neq f(g_{2,G^c}^*) = 0,$$

contradiction. Therefore, the condition holds.

Finally, when  $G = F$  (hence  $G^c = I$ ), let  $\tilde{g} \in \mathbb{R}^n$  such that:  $\tilde{g}_i = \bar{g}_i, \forall i \in F$  and  $\tilde{g}_i = \frac{P_{0,i}}{\left(\frac{E}{2}\right)^2}, \forall i \in I$ .

We substitute  $\tilde{g}$  into  $g_{\text{eq}}(g)$

$$\begin{aligned} g_{\text{eq}}(\tilde{g}) &\stackrel{(22)}{=} \sum_{i \in F} \frac{P_{0,i}}{\left(\frac{E}{2}\right)^2} + \sum_{i \in F} \frac{1}{\gamma_i} \frac{P_{\text{max}} - P_{0,\text{tot}}}{\left(\frac{E}{2}\right)^2} + \sum_{i \in I} \frac{P_{0,i}}{\left(\frac{E}{2}\right)^2} \\ &= \frac{P_{\text{max}}}{\left(\frac{E}{2}\right)^2} \stackrel{(9)}{=} g_l. \end{aligned}$$

Moreover, substituting  $g_{\text{eq}}(\tilde{g})$  into (1) yields  $v(\tilde{g}) = \frac{E}{2}$ . In this case, we can verify that  $\alpha_i(\tilde{g}) = 0$  for all  $i \in F$  and

$$P_i(\tilde{g}) = v^2(\tilde{g})\tilde{g}_i = \left(\frac{E}{2}\right)^2 \tilde{g}_i = P_{0,i} \quad \forall i \in I$$

Therefore,  $\tilde{g} \in E_G$ . The point  $\tilde{g}$  can be either of  $g_1^*, g_2^*$ . However,  $g_{\text{eq}}(\tilde{g}) = g_l < g_l + \sum_{i \in G} \bar{g}_i$ , so  $\tilde{g} = g_1^*$ . ■

We now show that our controller (20) does in fact guarantee the existence of an equilibrium that solves (19).

**Theorem 3 (Efficient Allocation under Extreme Loading):** Consider the system (20) with equilibria characterized by the set  $E_F$  as shown in (23). Then, the conductance  $g^* \in E_F \cap \text{cl}(M) = \{g^*\}$  leads to a curtailment  $\{\Delta P_i(g^*), i \in F\}$  that is optimal w.r.t. (19).

*Proof:* We have shown in Lemma 4 that there exists  $g_1^* \in E_F$  such that  $g_{\text{eq}}(g_1^*) = g_l$ , i.e.,  $g_1^* \in E_F \cap \text{cl}(M)$ . For this equilibrium, the total power is

$$P_{\text{tot}}(g_1^*) = v^2(g_1^*)g_{\text{eq}}(g_1^*) \stackrel{(25)}{=} \left(\frac{E}{2}\right)^2 g_l \stackrel{(9)}{=} P_{\text{max}}$$

We can then compute the allocation of the curtailment among loads  $i \in F$  for  $g = g_1^*$ :

$$\begin{aligned} \Delta P_i(g_1^*) &= v^2(g_1^*)\bar{g}_i - P_{0,i} \\ &= \left(\frac{E}{2}\right)^2 \left(\frac{P_{0,i}}{\left(\frac{E}{2}\right)^2} + \frac{P_{\text{max}} - P_{0,\text{tot}}}{\gamma_i \left(\frac{E}{2}\right)^2}\right) - P_{0,i} \\ &= \frac{P_{\text{max}} - P_{0,\text{tot}}}{\gamma_i}, \quad \forall i \in F. \end{aligned}$$

We can easily check that the allocation of the curtailment  $\Delta P_i(g_1^*)$  is proportional to  $\theta_i$

$$\frac{\Delta P_i(g_1^*)}{\Delta P_j(g_1^*)} = \frac{\gamma_j}{\gamma_i} = \frac{\theta_j}{\theta_i}, \quad i, j \in F, \quad (29)$$

and thus is an Efficient Allocation. ■

**Remark 4:** Theorem 3 only guarantees that one of the equilibria of  $E_F$  solves (19). However, it does not provide any information regarding all the possible additional equilibria  $E_G$ . We will show that the remaining equilibria either do not exist, are unstable, or do not belong to the region  $M$ .

We conclude this subsection with showing a case where  $E_G \cap \text{cl}(M) = \emptyset$ .

**Theorem 4 (Empty  $E_G$  under overloading conditions):** When  $G \subsetneq F \subseteq N$  and  $P_{0,\text{tot}} > P_{\text{max}}$ , then  $E_G \cap \text{cl}(M) = \emptyset$ .

*Proof:* Let  $g^* \in E_G \cap \text{cl}(M)$  and  $G \subset F$ . Then, there will exist a non empty set  $I_F \subset F$  such that  $I \cup I_F = G^c$  (and  $G = F \setminus I_F$ ). From Lemma 4,  $g_G^* = \sum_{i \in G} \bar{g}_i = \bar{g}_G$ . Let  $\bar{g}_{G^c} = g_l - \bar{g}_G$ . Since  $g^* \in \text{cl}(M)$ , it holds that:

$$g_{G^c}^* + \bar{g}_G < g_l = \bar{g}_G + \bar{g}_{G^c} \Rightarrow g_{G^c}^* < \bar{g}_{G^c}.$$

The first inequality is strict, otherwise we would conclude that:  $g_{I_G} = \frac{P_{0,I_G}}{\left(\frac{E}{2}\right)^2} = \bar{g}_{I_G} \Rightarrow G = F$ , contradiction. We will now look at how  $P_{G^c}(g_{G^c}^*; \bar{g}_G)$  behaves with respect to  $g_{G^c}^*$ . Since  $g^* \in E_G \cap \text{cl}(M)$ , then  $\Delta P_{G^c}(g_{G^c}^*; \bar{g}_G) = 0$ . From

(7), with  $S = G^c$ , we obtain that  $P_{G^c}(g_{G^c}^*; \bar{g}_G)$  is a strictly increasing function for  $g_{G^c}^* < g_l + \bar{g}_G$ . Therefore:

$$\begin{aligned} 0 &= P_{G^c}(g_{G^c}^*; \bar{g}_G) - P_{0,G^c} \stackrel{g_{G^c}^* < \bar{g}_G}{<} P_{G^c}(\bar{g}_G; \bar{g}_G) - P_{0,G^c} \\ &= \left(\frac{E}{2}\right)^2 \bar{g}_G - P_{0,G^c} \stackrel{(9)}{=} P_{\max} - \left(\frac{E}{2}\right)^2 \bar{g}_G - P_{0,G^c} \\ &\stackrel{(22)}{=} (P_{\max} - P_{0,\text{tot}}) \left(1 - \sum_{i \in G} \frac{1}{\gamma_i}\right) < 0, \end{aligned}$$

where in the second equality we have substituted for  $\bar{g}_G = g_l - \bar{g}_G$  and in the last step we have used (22) and the fact that  $\gamma_i > 0$  and  $\sum_{i \in F} \gamma_i^{-1} = 1$ . The above contradiction implies that  $\nexists g^* \in E_G \cap \text{cl}(M)$ , i.e.,  $E_G \cap \text{cl}(M) = \emptyset$  for  $G \subseteq F$ . ■

So far we have shown that whenever  $P_{0,\text{tot}} > P_{\max}$ , the only feasible set  $E_G$  in  $\text{cl}(M)$  is  $E_F$  (Theorem 4), and  $E_F$  contains the equilibrium that solves the efficient curtailment problem (19) (Lemma 4). However, there can also exist more equilibria in  $E_G \cap \text{cl}(M)^c$ . The next section will show that  $g_1^* \in \text{cl}(M)$  is in fact a *stable* equilibrium under extreme loading conditions.

Finally, since this equilibrium can change for different choice of  $\theta_i$ , we could span different equilibria for a certain level of demand, simply by varying the relative values of  $\theta_i$  and  $\theta_j$ .

### B. Stability Analysis

In this Section we will study the stability of the different equilibria with the objective of showing that the only equilibrium chosen by the controller solves Problem 1 and thus qualifies as a VCS Controller.

The following lemma will be of use in the eigenvalue computation.

**Lemma 5 (Matrix Determinant Lemma [19]):** If  $D$  is an invertible  $n \times n$  matrix and  $x, y \in \mathbb{R}^n$ , then:

$$\det(D + xy^T) = (1 + y^T D^{-1}x) \det(D) \quad (30)$$

We can now compute the eigenvalues of the Jacobian of (20).

**Lemma 6 (Computation of Eigenvalues of (20)):**

Consider the system (20) and a point  $g \in E_G$ ,  $G \subseteq F$ . Then, the eigenvalues of its Jacobian  $J_C(g)$  at each point  $g$  satisfy:

$$\begin{cases} \lambda_i = \Delta P_i(g) \kappa, & i \in G, \\ \lambda_i : c(g, \lambda_i) = 0, & \text{o.w.} \end{cases} \quad (31)$$

where

$$c(g, \lambda) := \left(1 + \frac{2v^2(g)}{g_{\text{eq}}(g) + g_l} \sum_{i \in N} \frac{\alpha_i(g) g_i}{d_i(g) - \lambda}\right) \quad (32)$$

and  $d_i(g) = -\alpha_i(g)v^2(g) + \Delta P_i(g) \frac{\kappa}{(1 + \kappa(\bar{g}_i - g_i))^2}$ .

*Proof:* The Jacobian of this system is

$$J_C(g) = A(g) \left( \frac{2v^2(g)}{g_{\text{eq}}(g) + g_l} g \mathbf{1}_n^T - v^2(g) \mathbb{I}_n \right) - D_{\Delta P}(g) D_\kappa(g) \quad (33)$$

where

$$D_{\Delta P}(g) = \text{diag}\{\Delta P(g)\}, \quad D_\kappa(g) = \text{diag}\left\{\frac{\partial}{\partial g_i} \alpha_i(g)\right\},$$

$$\text{with } \frac{\partial}{\partial g_i} \alpha_i(g) = \begin{cases} \frac{-\kappa}{(1 + \kappa(\bar{g}_i - g_i))^2} & \forall i \in F \\ 0 & \forall i \in I \end{cases}$$

The eigenvalues of  $J_C(g)$  are given as the solution of  $\det(J_C(g) - \lambda \mathbb{I}_n) = 0$ . Notice that  $J_C(g) - \lambda \mathbb{I}_n$  is composed by a diagonal matrix

$$D(g, \lambda) := -D_{\Delta P}(g) D_\kappa(g) - v^2(g) A(g) - \lambda \mathbb{I}_n$$

plus a rank 1 matrix  $-\frac{2v^2(g)}{g_{\text{eq}}(g) + g_l} (A(g)g) \mathbf{1}_n^T = xy^T$ , with

$$x = -\frac{2v^2(g)}{g_{\text{eq}}(g) + g_l} (A(g)g), \quad y = \mathbf{1}_n.$$

Moreover, the entries of  $D(g, \lambda)$  can be written as  $d_i(g) - \lambda$ . Therefore, using [19], we can compute

$$\det(J_C(g) - \lambda \mathbb{I}_n) = c(g, \lambda) \det(D(g, \lambda)),$$

which implies that  $\lambda_i$  is either equal to  $\Delta P_i(g) \kappa$  or is a solution to  $c(g, \lambda) = 0$ . Result follows. ■

Having characterized the eigenvalues of  $J_C(g)$ , we now analyze the stability of the equilibria of (23).

**Theorem 5 (Stability of VCS Controller):** Consider the system (20). Then, for  $0 < \kappa < \infty$  the following holds:

- (1) When  $\sum_{i \in N} P_{0,i} > P_{\max}$ , then the only equilibrium within  $\text{cl}(M)$ , given by  $g^* \in E_F \cap \text{cl}(M) = \{g^*\}$ , is asymptotically stable and guarantees fair curtailment.
- (2) When  $\sum_{i \in N} P_{0,i} < P_{\max}$ , then the only asymptotically stable equilibrium in  $\text{cl}(M)$  is given by  $g^* \in E_\emptyset \cap \text{cl}(M) = \{g^*\}$ .

*Proof:* (1) Let  $\sum_{i \in N} P_{0,i} > P_{\max}$ . Then, by Theorem 1, there does not exist  $g^*$  such that  $\Delta P_i(g^*) = 0$ ,  $\forall i \in N$ . That is, if  $G = \emptyset$  then  $E_\emptyset = \emptyset$ .

Let  $g^* \in E_G \cap \text{cl}(M)$ . By Theorem 4, if  $\emptyset \neq G \subseteq F$ , then again  $E_G \cap \text{cl}(M) = \emptyset$ . Therefore, when  $P_{0,\text{tot}} > P_{\max}$ ,  $E_G \cap \text{cl}(M)$  is nonempty only for  $G = F$ .

For  $g^* \in E_F \cap \text{cl}(M)$ , Lemma 4 implies

$$g_{\text{eq}}(g^*) = g_l, \quad g_i^* = \bar{g}_i \quad i \in F, \quad g_i^* = \frac{P_{0,i}}{\left(\frac{E}{2}\right)^2} \quad i \in I.$$

We will compute the eigenvalues of  $J_c(g^*)$  using Lemma 6. For  $i \in F$ , the eigenvalues are  $\lambda_i = \Delta P_i(g^*) \kappa$ . Since

$$\begin{aligned} P_i(g^*) &= v^2(g^*) g_i^* = \left(\frac{E}{2}\right)^2 \left(\frac{P_{0,i}}{\left(\frac{E}{2}\right)^2} + \frac{1}{\gamma_i} \frac{P_{\max} - P_{0,\text{tot}}}{\left(\frac{E}{2}\right)^2}\right) \\ &= P_{0,i} + \frac{1}{\gamma_i} (P_{\max} - P_{0,\text{tot}}), \end{aligned}$$

it follows that  $\lambda_i = \Delta P_i(g^*) \kappa = \frac{1}{\gamma_i} (P_{\max} - P_{0,\text{tot}}) \kappa < 0$ , for all  $i \in F$ .

The rest of the eigenvalues are computed from (32) by substituting  $\alpha_i(g^*) = 0$  for  $i \in F$  and  $\alpha_i(g^*) = 1$  for  $i \in I$ :

$$c(g^*, \lambda) = 1 + \frac{2v^2(g^*)}{g_{\text{eq}}(g^*) + g_l} \sum_{i \in I} \frac{g_i^*}{-v^2(g^*) - \lambda} = 0 \quad (34)$$

It is easy to show, following the analysis of [20], that  $c(g^*, \lambda)$  always has real roots. We will examine the sign of the roots of  $c(g^*, \lambda)$  by first looking at its derivative:

$$\frac{\partial}{\partial \lambda} c(g^*, \lambda) = \frac{2v^2(g^*)}{g_{\text{eq}}(g^*) + g_l} \sum_{i \in I} \frac{g_i^*}{(-v^2(g^*) - \lambda)^2} > 0$$

Hence, since the denominator is non-singular for  $\lambda \geq 0$ ,  $c(g^*, \lambda)$  is continuous and strictly increasing for  $\lambda \in [0, +\infty)$ . Moreover,  $c(g^*, 0) = 1 - \frac{2 \sum_{i \in I} g_i}{g_{\text{eq}}(g^*) + g_l} > 0$ . Therefore,

there does not exist  $\lambda \geq 0$  such that  $c(g^*, \lambda) = 0$ . Consequently,  $\lambda_i < 0$  for all  $i \in I$  and we have shown above that  $\lambda_i < 0$  also for all  $i \in F$ . Therefore, by Lyapunov's Indirect Method [13, Theorem 3.2]  $g^* \in E_F \cap \text{cl}(M)$  is an asymptotically stable equilibrium of (20).

(2) Let  $\sum_{i \in N} P_{0,i} < P_{\max}$  and  $g^* \in E_\emptyset \cap \text{cl}(M)$ . In this case  $g_i^* \neq \bar{g}_i \forall i \in N$  (otherwise  $g^* \notin E_\emptyset$ ). Therefore  $\alpha_i(g^*) \neq 0$ . From (31) the eigenvalues of the system satisfy:

$$c(g^*, \lambda) = 1 + \frac{2v^2(g^*)}{g_{\text{eq}}(g^*) + g_l} \sum_{i \in N} \frac{\alpha_i(g^*)g_i^*}{d_i(g^*) - \lambda} = 0 \quad (35)$$

Since  $g^* \in E_\emptyset \cap \text{cl}(M)$ , then from (1)  $v(g^*) \geq \frac{E}{2}$ . Therefore:

$$\begin{aligned} g_i^* &= \frac{P_{0,i}}{v^2(g^*)} \leq \frac{P_{0,i}}{(\frac{E}{2})^2} < \frac{P_{0,i}}{(\frac{E}{2})^2} + \frac{P_{\max} - P_{0,\text{tot}}}{\gamma_i(\frac{E}{2})^2} \\ &= \bar{g}_i \Rightarrow \alpha_i(g^*) > 0 \end{aligned}$$

where in the second inequality we have used that  $P_{\max} - P_{0,\text{tot}} > 0$ .

From [20], when  $\alpha_i(g^*) > 0$ , equation (35) has  $n-1$  roots that satisfy  $\lambda_i < \max_i \{-\alpha_i(g^*)v^2(g^*)\} = d_M < 0$ . For the  $n$ th eigenvalue, we observe that  $c(g^*, \lambda) \in C^\infty(d_M, 0]$  and:

$$\begin{aligned} c(g^*, d_M^-) &= \lim_{\lambda \rightarrow d_M^-} c(g^*, \lambda) \rightarrow -\infty < 0 \\ c(g^*, 0) &= 1 - \sum_{i \in N} \frac{2g_i^*}{g_{\text{eq}}(g^*) + g_l} \underset{> 0}{g \in \text{cl}(M)} > 0 \end{aligned}$$

From Lemma 3, there exists  $\lambda_n \in (d_M, 0)$  s.t.  $c(g^*, \lambda_n) = 0$ . Therefore, the  $n$ th eigenvalue is also negative and from [13, Theorem 3.2],  $g^* \in E_\emptyset \cap \text{cl}(M)$  is asymptotically stable.

If  $g^* \in E_G \cap \text{cl}(M)$  for an arbitrary  $G \neq \emptyset$ , then the only equilibrium that could satisfy this condition (when it exists) is  $g_1^* \in E_G$ . From (1),  $v(g^*) \geq \frac{E}{2}$ . Substituting into (31) for  $i \in G$ :

$$\begin{aligned} \lambda_i &= \Delta P_i(g^*)\kappa = \kappa(v^2(g^*)\bar{g}_i - P_{0,i}) \\ &= \kappa P_{0,i} \left( \left( \frac{v(g^*)}{(\frac{E}{2})} \right)^2 - 1 \right) + \frac{\kappa}{\gamma_i} \frac{P_{\max} - P_{0,\text{tot}}}{(\frac{E}{2})^2} > 0 \end{aligned}$$

Since there exists at least one positive eigenvalue, [13, Theorem 3.2] implies that the equilibrium is unstable. Therefore, the only stable eigenvalue is  $g_1^* \in E_\emptyset \cap \text{cl}(M)$ . ■

Theorem 5 shows that (20) is indeed a Voltage Collapse Stabilizing Control. That is, within the region  $\text{cl}(M)$ , the only stable equilibrium either satisfies  $\Delta P_i(g) = 0 \forall i \in N$  when  $P_{\text{tot}} < P_{\max}$  or is given by  $g^* \in E_F \cap \text{cl}(M)$  and is efficient. However, there may be some other equilibria that are unstable within  $\text{cl}(M)$  (Figure 2) or within  $\text{cl}(M)^c$  (Figure 3).

## V. NUMERICAL ILLUSTRATIONS

In this section, we validate our theoretical results using numerical illustrations. We consider a DC grid as in Figure 1 with three loads. In all the experiments we start the simulations with initial set-points such that  $P_{0,\text{tot}} < P_{\max}$  and with conductances close to the equilibrium  $g^*$  where all demands are met, i.e.,  $g^* \in E_\emptyset \cap M$ . We explore the parameter space by *slowly* varying the demand ( $P_0$ ) with time and observing the changes in the equilibria. We use  $\kappa = 10$ .

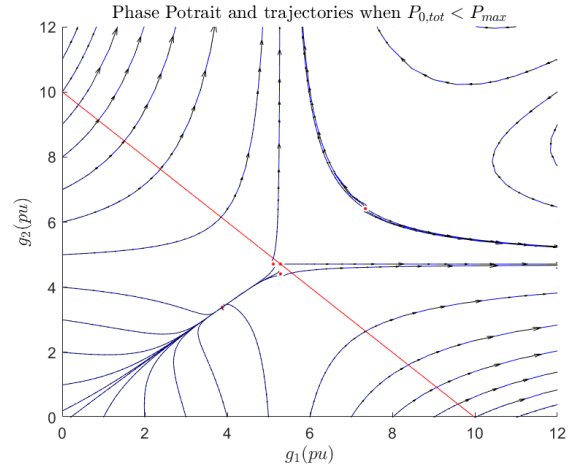


Fig. 2. Phase Portrait of 2 flexible loads, when  $P_{0,1} = 1.57$ p.u. and  $P_{0,2} = 1.37$ p.u. (with  $P_{0,\text{tot}} < P_{\max} = 3.025$ p.u. and  $\kappa = 0.05$ ). The system converges to the stable equilibrium (3.88, 3.38) for different initial conditions. Notice that the trajectories starting close to all other equilibria diverge.

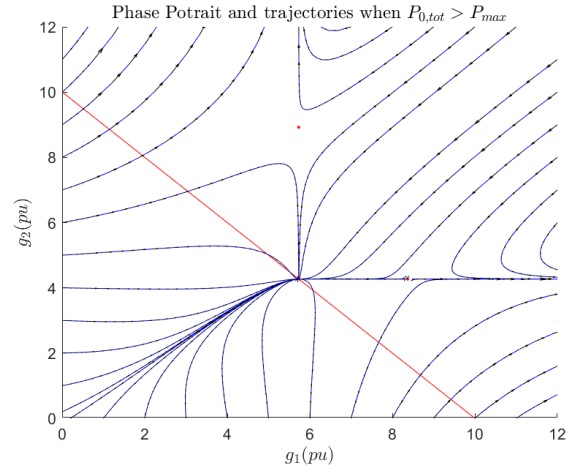


Fig. 3. Phase Portrait of 2 flexible loads, when  $P_{0,1} = 1.98$ p.u. and  $P_{0,2} = 1.78$ p.u. (with  $P_{0,\text{tot}} > P_{\max} = 3.025$ p.u. and  $\kappa = 0.05$ ). The system converges to the stable equilibrium (5.73, 4.27) for different initial conditions 'close enough' to the equilibrium. The trajectories with initial conditions too far from the equilibrium diverge.

**Case 1 ( $\mathbf{I}=\mathbf{N}=\{1,2,3\}$ ):** Figure 4 illustrates the behavior of the system (10)-(11) consisting of only inflexible loads. We can see that as soon as the aggregate demand reaches  $P_{\max}$ , the system undergoes a voltage collapse.

**Case 2 ( $\mathbf{F}=\mathbf{N}=\{1,2,3\}$ ):** The case where all loads are flexible is illustrated in Figure 5. In comparison with Case 1, here our VCS controller forces the consumption of *all* loads to adjust proportionally to their assigned weight  $\theta_i$ , preventing in this way voltage collapse.

**Case 3 ( $\mathbf{F}=\{1,2\}, \mathbf{I}=\{3\}$ ):** Finally we illustrate a case with mixed load types where load 3 is inflexible, and our VCS controller is executed in loads 1 and 2. We observe in Figure 6 that the flexible loads (1, 2) adjust their demand proportionally to their assigned weights in order to accommodate the increasing demand of the inflexible load, again, preventing voltage collapse. However, when the system runs out of flexible demand then it will eventually undergo a voltage



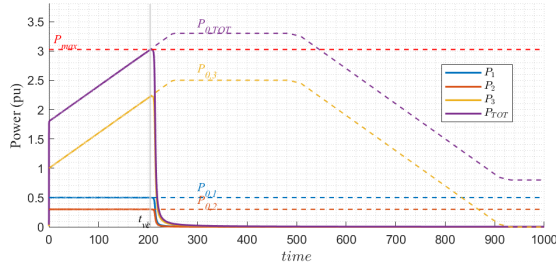


Fig. 4. Voltage collapse illustration with inflexible loads.

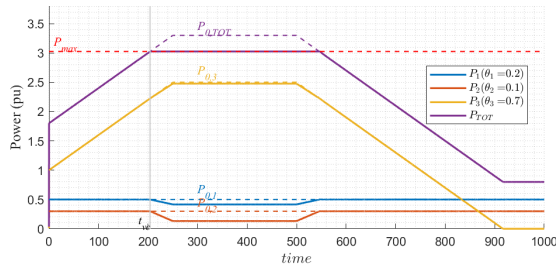


Fig. 5. VCS Controller acting on all loads. Supply tracks demand under varying loading.

collapse, as predicted in Lemma 4. We observe exactly this behavior in Figure 7.

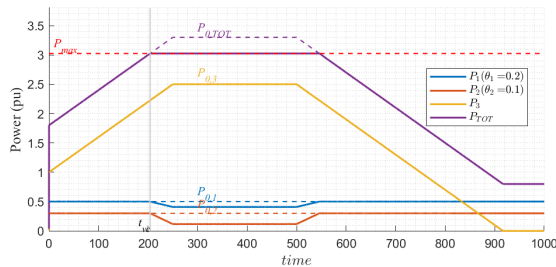


Fig. 6. Loads 1,2 are flexible and Load 3 is inflexible. Supply adjusts according to the assigned weight  $\theta_i$

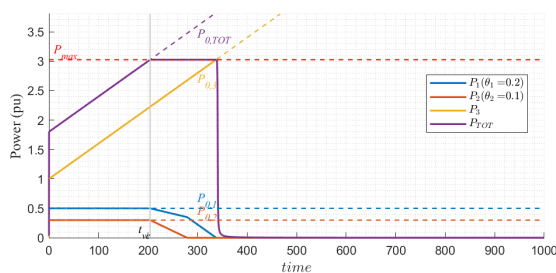


Fig. 7. Loads 1,2 are flexible and Load 3 is inflexible. VC happens when  $\sum_{i \in I} P_{0,i} > P_{\max}$

In all cases we verify that the proposed controller tracks the desired equilibria, whenever it exists.

## VI. CONCLUSIONS

This work seeks to initiate the study of voltage collapse stabilization as a mechanism to provide a more efficient

and reliable operation of electric power grids. We develop a game theoretical framework that sheds light on the behavioral mechanism that leads to voltage collapse and suggests the need of cooperation as a means to prevent it. Based on this insight, we propose a Voltage Collapse Stabilizer controller that is able to not only prevent voltage collapse, but also fairly distribute the curtailment among the flexible loads. Further research needs to be conducted to fully characterize the behavior of our solution. In particular, the point where  $P_{0,\text{tot}} = P_{\max}$  is a non-trivial point, where the Jacobian of the system is identically zero and thus requires the treatment of higher order dynamics. We identify two desired extensions of this work that are subject of current research: (a) extending the analysis to a general DC network and (b) extending the analysis to a general AC network.

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