

# The Role of Convexity in Saddle-Point Dynamics: Lyapunov Function and Robustness

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**Abstract**—This paper studies the projected saddle-point dynamics associated to a convex–concave function, which we term saddle function. The dynamics consists of gradient descent of the saddle function in variables corresponding to convexity and (projected) gradient ascent in variables corresponding to concavity. We examine the role that the local and/or global nature of the convexity–concavity properties of the saddle function plays in guaranteeing convergence and robustness of the dynamics. Under the assumption that the saddle function is twice continuously differentiable, we provide a novel characterization of the omega-limit set of the trajectories of this dynamics in terms of the diagonal blocks of the Hessian. Using this characterization, we establish global asymptotic convergence of the dynamics under local strong convexity–concavity of the saddle function. When strong convexity–concavity holds globally, we establish three results. First, we identify a Lyapunov function (that decreases strictly along the trajectory) for the projected saddle-point dynamics when the saddle function corresponds to the Lagrangian of a general constrained convex optimization problem. Second, for the particular case when the saddle function is the Lagrangian of an equality-constrained optimization problem, we show input-to-state stability (ISS) of the saddle-point dynamics by providing an ISS Lyapunov function. Third, we use the latter result to design an opportunistic state-triggered implementation of the dynamics. Various examples illustrate our results.

**Index Terms**—Asymptotic stability, constrained convex optimization, input-to-state stability, primal-dual dynamics, saddle-point dynamics.

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## I. INTRODUCTION

SADDLE-POINT dynamics and its variations have been used extensively in the design and analysis of distributed feedback controllers and optimization algorithms in several domains, including power networks, network flow problems, and zero-sum games. The analysis of the global convergence of this class of dynamics typically relies on some global strong/strict convexity–concavity property of the saddle function defining the dynamics. The main aim of this paper is to refine this analysis by unveiling two ways in which convexity–concavity of the saddle function plays a role. First, we show that local strong convexity–concavity is enough to conclude global asymptotic convergence, thus generalizing previous results that rely on global strong/strict convexity–concavity instead. Second, we show that, if global strong convexity–concavity holds, then one can identify a novel Lyapunov function for the projected saddle-point dynamics for the case when the saddle function is the Lagrangian of a constrained optimization problem. This, in turn, implies a stronger form of convergence, that is, input-to-state stability (ISS) and has important implications in the practical implementation of the saddle-point dynamics.

### A. Literature Review

The analysis of the convergence properties of (projected) saddle-point dynamics to the set of saddle points goes back to [2] and [3], motivated by the study of nonlinear programming and optimization. These works employed direct methods, examining the approximate evolution of the distance of the trajectories to the saddle point and concluding attractivity by showing it to be decreasing. Subsequently, motivated by the extensive use of the saddle-point dynamics in congestion control problems, the literature on communication networks developed a Lyapunov-based and passivity-based asymptotic stability analysis, see, e.g., [4] and references therein. Motivated by network optimization, more recent works [5], [6] have employed indirect, LaSalle-type arguments to analyze asymptotic convergence. For this class of problems, the aggregate nature of the objective function and the local computability of the constraints make the saddle-point dynamics corresponding to the Lagrangian naturally distributed. Many other works exploit this dynamics to solve network optimization problems for various applications, e.g., distributed convex optimization [6], [7], distributed linear programming [8], bargaining problems [9], and power networks [10]–[14]. Another area of application is the game theory, where saddle-point dynamics is applied to find the Nash equilibria of two-person zero-sum games [15], [16]. In the context of distributed optimization, the recent work [17] employs a (strict) Lyapunov function approach to ensure asymptotic

convergence of saddle-point-like dynamics. The work [18] examines the asymptotic behavior of the saddle-point dynamics when the set of saddle points is not asymptotically stable, and instead, trajectories exhibit oscillatory behavior. Our previous work has established global asymptotic convergence of the saddle-point dynamics [19] and the projected saddle-point dynamics [20] under global strict convexity–concavity assumptions. The works mentioned previously require similar or stronger global assumptions on the convexity–concavity properties of the saddle function to ensure convergence. Our results here directly generalize the convergence properties reported previously. Specifically, we show that traditional assumptions on the problem setup can be relaxed if convergence of the dynamics is the desired property: global convergence of the projected saddle-point dynamics can be guaranteed under local strong convexity–concavity assumptions. Furthermore, if traditional assumptions do hold, then a stronger notion of convergence, that also implies robustness, is guaranteed: if strong convexity–concavity holds globally, the dynamics admits a Lyapunov function and in the absence of projection, the dynamics is ISS, admitting an ISS Lyapunov function.

### B. Statement of Contributions

Our starting point is the definition of the projected saddle-point dynamics for a differentiable convex–concave function, referred to as saddle function. The dynamics has three components: gradient descent, projected gradient ascent, and gradient ascent of the saddle function, where each gradient is with respect to a subset of the arguments of the function. This unified formulation encompasses all forms of the saddle-point dynamics mentioned in the literature review previously. Our contributions shed light on the effect that the convexity–concavity of the saddle function has on the convergence attributes of the projected saddle-point dynamics. Our first contribution is a novel characterization of the omega-limit set of the trajectories of the projected saddle-point dynamics in terms of the diagonal Hessian blocks of the saddle function. To this end, we use the distance to a saddle point as a LaSalle function, express the Lie derivative of this function in terms of the Hessian blocks, and show it is nonpositive using second-order properties of the saddle function. Building on this characterization, our second contribution establishes global asymptotic convergence of the projected saddle-point dynamics to a saddle point assuming only local strong convexity–concavity of the saddle function. Our third contribution identifies a novel Lyapunov function for the projected saddle-point dynamics for the case when strong convexity–concavity holds globally and the saddle function can be written as the Lagrangian of a constrained optimization problem. This discontinuous Lyapunov function can be interpreted as multiple continuously differentiable Lyapunov functions, one for each set in a particular partition of the domain determined by the projection operator of the dynamics. Interestingly, the identified Lyapunov function is the sum of two previously known and independently considered LaSalle functions. When the saddle function takes the form of the Lagrangian of an equality constrained optimization, then no projection is present. In such scenarios, if the saddle function satisfies global strong convexity–concavity, our fourth contribution establishes ISS of the dynamics with respect to the saddle point by providing an ISS Lyapunov function. Our last contribution uses this function to design an opportunistic state-triggered implementation

of the saddle-point dynamics. We show that the trajectories of this discrete-time system converge asymptotically to the saddle points and that executions are Zeno free, i.e., that the difference between any two consecutive triggering times is lower bounded by a common positive quantity. Examples illustrate our results.

## II. PRELIMINARIES

This section introduces our notation and preliminary notions on convex–concave functions, discontinuous dynamical systems, and ISS.

### A. Notation

Let  $\mathbb{R}$ ,  $\mathbb{R}_{\geq 0}$ , and  $\mathbb{N}$  denote the set of real, nonnegative real, and natural numbers, respectively. We let  $\|\cdot\|$  denote the 2-norm on  $\mathbb{R}^n$  and the respective induced norm on  $\mathbb{R}^{n \times m}$ . Given  $x, y \in \mathbb{R}^n$ ,  $x_i$  denotes the  $i$ th component of  $x$ , and  $x \leq y$  denotes  $x_i \leq y_i$ , for  $i \in \{1, \dots, n\}$ . For vectors  $u \in \mathbb{R}^n$  and  $w \in \mathbb{R}^m$ , the vector  $(u; w) \in \mathbb{R}^{n+m}$  denotes their concatenation. For  $a \in \mathbb{R}$  and  $b \in \mathbb{R}_{\geq 0}$ , we let

$$[a]_b^+ = \begin{cases} a, & \text{if } b > 0, \\ \max\{0, a\}, & \text{if } b = 0. \end{cases}$$

For vectors  $a \in \mathbb{R}^n$  and  $b \in \mathbb{R}_{\geq 0}^n$ ,  $[a]_b^+$  denotes the vector whose  $i$ th component is  $[a_i]_{b_i}^+$ , for  $i \in \{1, \dots, n\}$ . Given a set  $\mathcal{S} \subset \mathbb{R}^n$ , we denote by  $\text{cl}(\mathcal{S})$ ,  $\text{int}(\mathcal{S})$ , and  $|\mathcal{S}|$  its closure, interior, and cardinality, respectively. The distance of a point  $x \in \mathbb{R}^n$  to the set  $\mathcal{S} \subset \mathbb{R}^n$  in 2-norm is  $\|x\|_{\mathcal{S}} = \inf_{y \in \mathcal{S}} \|x - y\|$ . The projection of  $x$  onto a closed set  $\mathcal{S}$  is defined as the set  $\text{proj}_{\mathcal{S}}(x) = \{y \in \mathcal{S} \mid \|x - y\| = \|x\|_{\mathcal{S}}\}$ . When  $\mathcal{S}$  is also convex,  $\text{proj}_{\mathcal{S}}(x)$  is a singleton for any  $x \in \mathbb{R}^n$ . For a matrix  $A \in \mathbb{R}^{n \times n}$ , we use  $A \succeq 0$ ,  $A \succ 0$ ,  $A \preceq 0$ , and  $A \prec 0$  to denote that  $A$  is positive semidefinite, positive definite, negative semidefinite, and negative definite, respectively. For a symmetric matrix  $A \in \mathbb{R}^{n \times n}$ ,  $\lambda_{\min}(A)$  and  $\lambda_{\max}(A)$  denote the minimum and maximum eigenvalue of  $A$ . For a real-valued function  $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ ,  $(x, y) \mapsto F(x, y)$ , we denote by  $\nabla_x F$  and  $\nabla_y F$  the column vector of partial derivatives of  $F$  with respect to the first and second arguments, respectively. Higher order derivatives follow the convention  $\nabla_{xy} F = \frac{\partial^2 F}{\partial x \partial y}$ ,  $\nabla_{xx} F = \frac{\partial^2 F}{\partial x^2}$ , and so on. A function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is class  $\mathcal{K}$  if it is continuous, strictly increasing, and  $\alpha(0) = 0$ . The set of unbounded class  $\mathcal{K}$  functions are called  $\mathcal{K}_{\infty}$  functions. A function  $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is class  $\mathcal{KL}$  if for any  $t \in \mathbb{R}_{\geq 0}$ ,  $x \mapsto \beta(x, t)$  is class  $\mathcal{K}$  and for any  $x \in \mathbb{R}_{\geq 0}$ ,  $t \mapsto \beta(x, t)$  is continuous, decreasing with  $\beta(x, t) \rightarrow 0$  as  $t \rightarrow \infty$ .

### B. Saddle Points and Convex–Concave Functions

Here, we review notions of convexity, concavity, and saddle points from [21]. A function  $f : \mathcal{X} \rightarrow \mathbb{R}$  is *convex* if

$$f(\lambda x + (1 - \lambda)x') \leq \lambda f(x) + (1 - \lambda)f(x')$$

for all  $x, x' \in \mathcal{X}$  (where  $\mathcal{X}$  is a convex domain) and all  $\lambda \in [0, 1]$ . A convex differentiable  $f$  satisfies the following *first-order convexity condition*:

$$f(x') \geq f(x) + (x' - x)^{\top} \nabla f(x)$$

for all  $x, x' \in \mathcal{X}$ . A twice differentiable function  $f$  is *locally strongly convex* at  $x \in \mathcal{X}$  if  $f$  is convex and  $\nabla^2 f(x) \succeq mI$  for

some  $m > 0$  (note that this is equivalent to having  $\nabla^2 f \succ 0$  in a neighborhood of  $x$ ). Moreover, a twice differentiable  $f$  is *strongly convex* if  $\nabla^2 f(x) \succeq mI$  for all  $x \in \mathcal{X}$  for some  $m > 0$ . A function  $f : \mathcal{X} \rightarrow \mathbb{R}$  is *concave*, *locally strongly concave*, or *strongly concave* if  $-f$  is convex, locally strongly convex, or strongly convex, respectively. A function  $F : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$  is *convex–concave* (on  $\mathcal{X} \times \mathcal{Y}$ ) if, given any point  $(\tilde{x}, \tilde{y}) \in \mathcal{X} \times \mathcal{Y}$ ,  $x \mapsto F(x, \tilde{y})$  is convex and  $y \mapsto F(\tilde{x}, y)$  is concave. When the space  $\mathcal{X} \times \mathcal{Y}$  is clear from the context, we refer to this property as  $F$  being convex–concave in  $(x, y)$ . A point  $(x_*, y_*) \in \mathcal{X} \times \mathcal{Y}$  is a *saddle point* of  $F$  on the set  $\mathcal{X} \times \mathcal{Y}$  if  $F(x_*, y) \leq F(x_*, y_*) \leq F(x, y_*)$  for all  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$ . The set of saddle points of a convex–concave function  $F$  is convex. The function  $F$  is *locally strongly convex–concave* at a saddle point  $(x, y)$  if it is convex–concave and either  $\nabla_{xx} F(x, y) \succeq mI$  or  $\nabla_{yy} F(x, y) \preceq -mI$  for some  $m > 0$ . Finally,  $F$  is *globally strongly convex–concave* if it is convex–concave and either  $x \mapsto F(x, y)$  is strongly convex for all  $y \in \mathcal{Y}$  or  $y \mapsto F(x, y)$  is strongly concave for all  $x \in \mathcal{X}$ .

### C. Discontinuous Dynamical Systems

Here, we present notions of discontinuous dynamical systems [22], [23]. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be Lebesgue measurable and locally bounded. Consider the differential equation

$$\dot{x} = f(x). \quad (1)$$

A map  $\gamma : [0, T) \rightarrow \mathbb{R}^n$  is a (*Caratheodory*) *solution* of (1) on the interval  $[0, T)$  if it is absolutely continuous on  $[0, T)$  and satisfies  $\dot{\gamma}(t) = f(\gamma(t))$  almost everywhere in  $[0, T)$ . We use the terms solution and trajectory interchangeably. A set  $\mathcal{S} \subset \mathbb{R}^n$  is *invariant* under (1) if every solution starting in  $\mathcal{S}$  remains in  $\mathcal{S}$ . For a solution  $\gamma$  of (1) defined on the time interval  $[0, \infty)$ , the *omega-limit* set  $\Omega(\gamma)$  is defined by

$$\Omega(\gamma) = \{y \in \mathbb{R}^n \mid \exists \{t_k\}_{k=1}^{\infty} \subset [0, \infty) \text{ with } \lim_{k \rightarrow \infty} t_k = \infty \\ \text{and } \lim_{k \rightarrow \infty} \gamma(t_k) = y\}.$$

If the solution  $\gamma$  is bounded, then  $\Omega(\gamma) \neq \emptyset$  by the Bolzano–Weierstrass theorem [24, p. 33]. Given a continuously differentiable function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$ , the *Lie derivative of  $V$  along (1)* at  $x \in \mathbb{R}^n$  is  $\mathcal{L}_f V(x) = \nabla V(x)^\top f(x)$ . The next result is a simplified version of [22, Proposition 3].

*Proposition 2.1 (Invariance principle for discontinuous Caratheodory systems):* Let  $\mathcal{S} \in \mathbb{R}^n$  be compact and invariant. Assume that, for each point  $x_0 \in \mathcal{S}$ , there exists a unique solution of (1) starting at  $x_0$  and that its omega-limit set is invariant too. Let  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuously differentiable map such that  $\mathcal{L}_f V(x) \leq 0$  for all  $x \in \mathcal{S}$ . Then, any solution of (1) starting at  $\mathcal{S}$  converges to the largest invariant set in  $\text{cl}(\{x \in \mathcal{S} \mid \mathcal{L}_f V(x) = 0\})$ .

### D. Input-to-State Stability (ISS)

Here, we review the notion of ISS following [25]. Consider a system

$$\dot{x} = f(x, u) \quad (2)$$

where  $x \in \mathbb{R}^n$  is the state,  $u : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$  is the input that is measurable and locally essentially bounded, and  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is locally Lipschitz. Assume that starting from any

point in  $\mathbb{R}^n$ , the trajectory of (2) is defined on  $\mathbb{R}_{\geq 0}$  for any given control. Let  $\text{Eq}(f) \subset \mathbb{R}^n$  be the set of equilibrium points of the unforced system. Then, the system (2) is *input-to-state stable* with respect to  $\text{Eq}(f)$  if there exists  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}$  such that each trajectory  $t \mapsto x(t)$  of (2) satisfies

$$\|x(t)\|_{\text{Eq}(f)} \leq \beta(\|x(0)\|_{\text{Eq}(f)}, t) + \gamma(\|u\|_{\infty})$$

for all  $t \geq 0$ , where  $\|u\|_{\infty} = \text{ess sup}_{t \geq 0} \|u(t)\|$  is the essential supremum (see [24, p. 185] for the definition) of  $u$ . This notion captures the graceful degradation of the asymptotic convergence properties of the unforced system as the size of the disturbance input grows. One convenient way of showing ISS is by finding an ISS-Lyapunov function. An *ISS-Lyapunov function* with respect to the set  $\text{Eq}(f)$  for system (2) is a differentiable function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  such that

1) there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$  such that for all  $x \in \mathbb{R}^n$ ,

$$\alpha_1(\|x\|_{\text{Eq}(f)}) \leq V(x) \leq \alpha_2(\|x\|_{\text{Eq}(f)}) \quad (3)$$

2) there exists a continuous, positive definite function  $\alpha_3 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  and  $\gamma \in \mathcal{K}_{\infty}$  such that

$$\nabla V(x)^\top f(x, v) \leq -\alpha_3(\|x\|_{\text{Eq}(f)}) \quad (4)$$

for all  $x \in \mathbb{R}^n$ ,  $v \in \mathbb{R}^m$  for which  $\|x\|_{\text{Eq}(f)} \geq \gamma(\|v\|)$ .

*Proposition 2.2. (ISS-Lyapunov function implies ISS):* If (2) admits an ISS-Lyapunov function, then it is ISS.

## III. PROBLEM STATEMENT

In this section, we provide a formal statement of the problem of interest. Consider a twice continuously differentiable function  $F : \mathbb{R}^n \times \mathbb{R}_{\geq 0}^p \times \mathbb{R}^m \rightarrow \mathbb{R}$ ,  $(x, y, z) \mapsto F(x, y, z)$ , which we refer to as *saddle function*. With the notation of Section II-B, we set  $\mathcal{X} = \mathbb{R}^n$  and  $\mathcal{Y} = \mathbb{R}_{\geq 0}^p \times \mathbb{R}^m$ , and assume that  $F$  is convex–concave on  $(\mathbb{R}^n) \times (\mathbb{R}_{\geq 0}^p \times \mathbb{R}^m)$ . Let  $\text{Saddle}(F)$  denote its (nonempty) set of saddle points. We define the *projected saddle-point dynamics* for  $F$  as

$$\dot{x} = -\nabla_x F(x, y, z), \quad (5a)$$

$$\dot{y} = [\nabla_y F(x, y, z)]_y^+, \quad (5b)$$

$$\dot{z} = \nabla_z F(x, y, z). \quad (5c)$$

When convenient, we use the map  $X_{\text{p-sp}} : \mathbb{R}^n \times \mathbb{R}_{\geq 0}^p \times \mathbb{R}^m \rightarrow \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m$  to refer to the dynamics (5). Note that the domain  $\mathbb{R}^n \times \mathbb{R}_{\geq 0}^p \times \mathbb{R}^m$  is invariant under  $X_{\text{p-sp}}$  (this follows from the definition of the projection operator) and its set of equilibrium points precisely corresponds to  $\text{Saddle}(F)$  (this follows from the defining property of saddle points and the first-order condition for convexity–concavity of  $F$ ). Thus, a saddle point  $(x_*, y_*, z_*)$  satisfies

$$\nabla_x F(x_*, y_*, z_*) = 0, \quad \nabla_z F(x_*, y_*, z_*) = 0, \quad (6a)$$

$$\nabla_y F(x_*, y_*, z_*) \leq 0, \quad y_*^\top \nabla_y F(x_*, y_*, z_*) = 0. \quad (6b)$$

Our interest in the dynamics (5) is motivated by two bodies of work in the literature: one that analyzes primal-dual dynamics, corresponding to (5a) together with (5b), for solving inequality constrained network optimization problems, see, e.g., [3], [5], [11], and [14]; and the other one analyzing saddle-point dynamics, corresponding to (5a) together with (5c), for solving equality constrained problems and finding Nash equilibrium of

zero-sum games, see, e.g., [19] and references therein. By considering (5a)–(5c) together, we aim to unify these lines of work. Later, we explain further the significance of the dynamics in solving specific network optimization problems.

*Remark 3.1 (Motivating examples):* Consider the following constrained convex optimization problem:

$$\min\{f(x) \mid g(x) \leq 0, Ax = b\}$$

where  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  and  $g: \mathbb{R}^n \rightarrow \mathbb{R}^p$  are convex continuously differentiable functions,  $A \in \mathbb{R}^{m \times n}$ , and  $b \in \mathbb{R}^m$ . Under zero duality gap, saddle points of the associated Lagrangian  $L(x, y, z) = f(x) + y^\top g(x) + z^\top (Ax - b)$  correspond to the primal-dual optimizers of the problem. This observation motivates the search for the saddle points of the Lagrangian, which can be done via the projected saddle-point dynamics (5). In many network optimization problems,  $f$  is the summation of individual costs of agents and the constraints, defined by  $g$  and  $A$ , are such that each of its components is computable by one agent interacting with its neighbors. This structure renders the projected saddle-point dynamics of the Lagrangian implementable in a distributed manner. Motivated by this, the dynamics is widespread in network optimization scenarios. For example, in optimal dispatch of power generators [11]–[14], the objective function is the sum of the individual cost function of each generator, the inequalities consist of generator capacity constraints and line limits, and the equality encodes the power balance at each bus. In congestion control of communication networks [4], [5], [26], the cost function is the summation of the negative of the utility of the communicated data, the inequalities define constraints on channel capacities, and equalities encode the data balance at each node. •

Our main objectives are to identify conditions that guarantee that the set of saddle points is globally asymptotically stable under the dynamics (5) and formally characterize the robustness properties using the concept of ISS. The rest of this paper is structured as follows. Section IV investigates novel conditions that guarantee global asymptotic convergence relying on LaSalle-type arguments. Section V instead identifies a strict Lyapunov function for constrained convex optimization problems. This finding allows us in Section VI to go beyond convergence guarantees and explore the robustness properties of the saddle-point dynamics.

#### IV. LOCAL PROPERTIES OF THE SADDLE FUNCTION IMPLY GLOBAL CONVERGENCE

Our first result of this section provides a novel characterization of the omega-limit set of the trajectories of the projected saddle-point dynamics (5).

*Proposition 4.1 (Characterization of the omega-limit set of solutions of  $X_{p\text{-sp}}$ ):* Given a twice continuously differentiable, convex–concave function  $F$ , each point in the set  $\text{Saddle}(F)$  is stable under the projected saddle-point dynamics  $X_{p\text{-sp}}$  and the omega-limit set of every solution is contained in the largest invariant set  $\mathcal{M}$  in  $\mathcal{E}(F)$ , where

$$\begin{aligned} \mathcal{E}(F) = \{ & (x, y, z) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}^p \times \mathbb{R}^m \mid \\ & (x - x_*, y - y_*, z - z_*) \in \ker(\overline{H}(x, y, z, x_*, y_*, z_*)) \\ & \text{for all } (x_*, y_*, z_*) \in \text{Saddle}(F) \} \end{aligned} \quad (7)$$

and

$$\begin{aligned} \overline{H}(x, y, z, x_*, y_*, z_*) &= \int_0^1 H(x(s), y(s), z(s)) ds, \\ (x(s), y(s), z(s)) &= (x_*, y_*, z_*) + s(x - x_*, y - y_*, z - z_*), \\ H(x, y, z) &= \begin{bmatrix} -\nabla_{xx} F & 0 & 0 \\ 0 & \nabla_{yy} F & \nabla_{yz} F \\ 0 & \nabla_{zy} F & \nabla_{zz} F \end{bmatrix}_{(x, y, z)}. \end{aligned} \quad (8)$$

*Proof:* The proof follows from the application of the LaSalle invariance principle for discontinuous Caratheodory systems (cf., Proposition 2.1). Let  $(x_*, y_*, z_*) \in \text{Saddle}(F)$  and  $V_1: \mathbb{R}^n \times \mathbb{R}_{\geq 0}^p \times \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}$  be defined as

$$V_1(x, y, z) = \frac{1}{2} (\|x - x_*\|^2 + \|y - y_*\|^2 + \|z - z_*\|^2). \quad (9)$$

The Lie derivative of  $V_1$  along (5) is

$$\begin{aligned} \mathcal{L}_{X_{p\text{-sp}}} V_1(x, y, z) &= -(x - x_*)^\top \nabla_x F(x, y, z) + (y - y_*)^\top [\nabla_y F(x, y, z)]_y^+ \\ &\quad + (z - z_*)^\top \nabla_z F(x, y, z) \\ &= -(x - x_*)^\top \nabla_x F(x, y, z) + (y - y_*)^\top \nabla_y F(x, y, z) \\ &\quad + (z - z_*)^\top \nabla_z F(x, y, z) \\ &\quad + (y - y_*)^\top ([\nabla_y F(x, y, z)]_y^+ - \nabla_y F(x, y, z)) \\ &\leq -(x - x_*)^\top \nabla_x F(x, y, z) + (y - y_*)^\top \nabla_y F(x, y, z) \\ &\quad + (z - z_*)^\top \nabla_z F(x, y, z), \end{aligned} \quad (10)$$

where the last inequality follows from the fact that  $T_i = (y - y_*)_i ([\nabla_y F(x, y, z)]_y^+ - \nabla_y F(x, y, z))_i \leq 0$ , for each  $i \in \{1, \dots, p\}$ . Indeed if  $y_i > 0$ , then  $T_i = 0$ , and if  $y_i = 0$ , then  $(y - y_*)_i \leq 0$  and  $([\nabla_y F(x, y, z)]_y^+ - \nabla_y F(x, y, z))_i \geq 0$ , which implies that  $T_i \leq 0$ . Next, denoting  $\lambda = (y; z)$  and  $\lambda_* = (y_*; z_*)$ , we simplify the above inequality as

$$\begin{aligned} \mathcal{L}_{X_{p\text{-sp}}} V_1(x, y, z) &\leq -(x - x_*)^\top \nabla_x F(x, \lambda) + (\lambda - \lambda_*)^\top \nabla_\lambda F(x, \lambda) \\ &\stackrel{(a)}{=} -(x - x_*)^\top \int_0^1 \left( \nabla_{xx} F(x(s), \lambda(s)) (x - x_*) \right. \\ &\quad \left. + \nabla_{\lambda x} F(x(s), \lambda(s)) (\lambda - \lambda_*) \right) ds \\ &\quad + (\lambda - \lambda_*)^\top \int_0^1 \left( \nabla_{x\lambda} F(x(s), \lambda(s)) (x - x_*) \right. \\ &\quad \left. + \nabla_{\lambda\lambda} F(x(s), \lambda(s)) (\lambda - \lambda_*) \right) ds \\ &\stackrel{(b)}{=} [x - x_*; \lambda - \lambda_*]^\top \overline{H}(x, \lambda, x_*, \lambda_*) \stackrel{(c)}{\leq} 0, \end{aligned}$$

where (a) follows from the fundamental theorem of calculus using the notation  $x(s) = x_* + s(x - x_*)$  and  $\lambda(s) = \lambda_* + s(\lambda - \lambda_*)$  and recalling from (6) that  $\nabla_x F(x_*, \lambda_*) = 0$  and  $(\lambda - \lambda_*)^\top \nabla_\lambda F(x_*, \lambda_*) \leq 0$ ; (b) follows from the definition of  $\overline{H}$  using  $(\nabla_{\lambda x} F(x, \lambda))^\top = \nabla_{x\lambda} F(x, \lambda)$ ; and (c) follows from the fact

that  $\bar{H}$  is negative semidefinite. Now using this fact that  $\mathcal{L}_{X_{p\text{-sp}}} V_1$  is nonpositive at any point, one can deduce, see, e.g., [20, Lemmas 4.2–4.4], that starting from any point  $(x(0), y(0), z(0))$  a unique trajectory of  $X_{p\text{-sp}}$  exists, is contained in the compact set  $V_1^{-1}(V_1(x(0), y(0), z(0))) \cap (\mathbb{R}^n \times \mathbb{R}_{\geq 0}^p \times \mathbb{R}^m)$  at all times, and its omega-limit set is invariant. These facts imply that the hypotheses of Proposition 2.1 hold and so, we deduce that the solutions of the dynamics  $X_{p\text{-sp}}$  converge to the largest invariant set where the Lie derivative is zero, that is, the set

$$\mathcal{E}(F, x_*, y_*, z_*) = \{(x, y, z) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}^p \times \mathbb{R}^m \mid (x; y; z) - (x_*; y_*; z_*) \in \ker(\bar{H}(x, y, z, x_*, y_*, z_*))\}. \quad (11)$$

Finally, since  $(x_*, y_*, z_*)$  was chosen arbitrary, we get that the solutions converge to the largest invariant set  $\mathcal{M}$  contained in  $\mathcal{E}(F) = \bigcap_{(x_*, y_*, z_*) \in \text{Saddle}(F)} \mathcal{E}(F, x_*, y_*, z_*)$ , concluding the proof.  $\blacksquare$

Note that the proof of Proposition 4.1 shows that the Lie derivative of the function  $V_1$  is negative, but not strictly negative, outside the set  $\text{Saddle}(F)$ . From Proposition 4.1 and the definition (7), we deduce that if a point  $(x, y, z)$  belongs to the omega-limit set (and is not a saddle point), then the line integral of the Hessian block matrix (8) from any saddle point to  $(x, y, z)$  cannot be full rank. Elaborating further,

- 1) if  $\nabla_{xx} F$  is full rank at a saddle point  $(x_*, y_*, z_*)$  and if the point  $(x, y, z) \notin \text{Saddle}(F)$  belongs to the omega-limit set, then  $x = x_*$ ;
- 2) if  $\begin{bmatrix} \nabla_{yy} F & \nabla_{yz} F \\ \nabla_{zy} F & \nabla_{zz} F \end{bmatrix}$  is full rank at a saddle point  $(x_*, y_*, z_*)$ , then  $(y, z) = (y_*, z_*)$ .

These properties are used in the next result, which shows that local strong convexity–concavity at a saddle point together with global convexity–concavity of the saddle function are enough to guarantee global convergence proving Theorem 4.2.

**Theorem 4.2** (*Global asymptotic stability of the set of saddle points under  $X_{p\text{-sp}}$* ): Given a twice continuously differentiable, convex–concave function  $F$ , which is locally strongly convex–concave at a saddle point, the set  $\text{Saddle}(F)$  is globally asymptotically stable under the projected saddle-point dynamics  $X_{p\text{-sp}}$  and the convergence of trajectories is to a point.

*Proof:* Our proof proceeds by characterizing the set  $\mathcal{E}(F)$  defined in (7). Let  $(x_*, y_*, z_*)$  be a saddle point at which  $F$  is locally strongly convex–concave. Without loss of generality, assume that  $\nabla_{xx} F(x_*, y_*, z_*) \succ 0$  (the case of negative definiteness of the other Hessian block can be reasoned analogously). Let  $(x, y, z) \in \mathcal{E}(F, x_*, y_*, z_*)$  [recall the definition of this set in (11)]. Since  $\nabla_{xx} F(x_*, y_*, z_*) \succ 0$  and  $F$  is twice continuously differentiable, we have that  $\nabla_{xx} F$  is positive definite in a neighborhood of  $(x_*, y_*, z_*)$  and so

$$\int_0^1 \nabla_{xx} F(x(s), y(s), z(s)) ds \succ 0,$$

where  $x(s) = x_* + s(x - x_*)$ ,  $y(s) = y_* + s(y - y_*)$ , and  $z(s) = z_* + s(z - z_*)$ . Therefore, by definition of  $\mathcal{E}(F, x_*, y_*, z_*)$ , it follows that  $x = x_*$  and so,  $\mathcal{E}(F, x_*, y_*, z_*) \subseteq \{x_*\} \times (\mathbb{R}_{\geq 0}^p \times \mathbb{R}^m)$ . From Proposition 4.1, the trajectories of  $X_{p\text{-sp}}$  converge to the largest invariant set  $\mathcal{M}$  contained in  $\mathcal{E}(F, x_*, y_*, z_*)$ . To characterize this set, let  $(x_*, y, z) \in \mathcal{M}$  and  $t \mapsto (x_*, y(t), z(t))$  be a trajectory of  $X_{p\text{-sp}}$  that is contained in

$\mathcal{M}$ , and hence, in  $\mathcal{E}(F, x_*, y_*, z_*)$ . From (10), we get

$$\begin{aligned} & \mathcal{L}_{X_{p\text{-sp}}} V_1(x, y, z) \\ & \leq -(x - x_*)^\top \nabla_x F(x, y, z) + (y - y_*)^\top \nabla_y F(x, y, z) \\ & \quad + (z - z_*)^\top \nabla_z F(x, y, z) \\ & \leq F(x, y, z) - F(x, y_*, z_*) + F(x_*, y, z) - F(x, y, z) \\ & \leq F(x_*, y_*, z_*) - F(x, y_*, z_*) + F(x_*, y, z) \\ & \quad - F(x_*, y_*, z_*) \leq 0, \end{aligned} \quad (12)$$

where in the second inequality, we have used the first-order convexity and concavity property of the maps  $x \mapsto F(x, y, z)$  and  $(y, z) \mapsto F(x, y, z)$ . Now since  $\mathcal{E}(F, x_*, y_*, z_*) = \{(x_*, y, z) \mid \mathcal{L}_{X_{p\text{-sp}}} V_1(x_*, y, z) = 0\}$ , using the above inequality, we get  $F(x_*, y(t), z(t)) = F(x_*, y_*, z_*)$  for all  $t \geq 0$ . Thus, for all  $t \geq 0$ ,  $\mathcal{L}_{X_{p\text{-sp}}} F(x_*, y(t), z(t)) = 0$ , which yields

$$\begin{aligned} & \nabla_y F(x_*, y(t), z(t))^\top [\nabla_y F(x_*, y(t), z(t))]_{y(t)}^+ \\ & \quad + \|\nabla_z F(x_*, y(t), z(t))\|^2 = 0. \end{aligned}$$

Note that both terms in the above expression are non-negative, and so, we get  $[\nabla_y F(x_*, y(t), z(t))]_{y(t)}^+ = 0$  and  $\nabla_z F(x_*, y(t), z(t)) = 0$  for all  $t \geq 0$ . In particular, this holds at  $t = 0$  and so,  $(x, y, z) \in \text{Saddle}(F)$ , and we conclude  $\mathcal{M} \subset \text{Saddle}(F)$ . Hence,  $\text{Saddle}(F)$  is globally asymptotically stable. Combining this with the fact that individual saddle points are stable, one deduces the pointwise convergence of trajectories along the same lines as in [27, Corollary 5.2].  $\blacksquare$

A closer look at the proof of the above result reveals that the same conclusion also holds under milder conditions on the saddle function. In particular,  $F$  need only be twice continuously differentiable in a neighborhood of the saddle point and the local strong convexity–concavity can be relaxed to a condition on the line integral of Hessian blocks of  $F$ . We state next this stronger result.

**Theorem 4.3** (*Global asymptotic stability of the set of saddle points under  $X_{p\text{-sp}}$* ): Let  $F$  be convex–concave and continuously differentiable with locally Lipschitz gradient. Suppose there is a saddle point  $(x_*, y_*, z_*)$  and a neighborhood of this point  $\mathcal{U}_* \subset \mathbb{R}^n \times \mathbb{R}_{\geq 0}^p \times \mathbb{R}^m$  such that  $F$  is twice continuously differentiable on  $\mathcal{U}_*$  and either of the following holds:

- 1) for all  $(x, y, z) \in \mathcal{U}_*$ ,

$$\int_0^1 \nabla_{xx} F(x(s), y(s), z(s)) ds \succ 0,$$

- 2) for all  $(x, y, z) \in \mathcal{U}_*$ ,

$$\int_0^1 \begin{bmatrix} \nabla_{yy} F & \nabla_{yz} F \\ \nabla_{zy} F & \nabla_{zz} F \end{bmatrix}_{(x(s), y(s), z(s))} ds \prec 0,$$

where  $(x(s), y(s), z(s))$  are given in (8). Then,  $\text{Saddle}(F)$  is globally asymptotically stable under the projected saddle-point dynamics  $X_{p\text{-sp}}$  and the convergence of trajectories is to a point.

We omit the proof of this result for space reasons: The argument is analogous to the proof of Theorem 4.2, where one replaces the integral of Hessian blocks by the integral of generalized Hessian blocks (see [28, ch. 2] for the definition of the

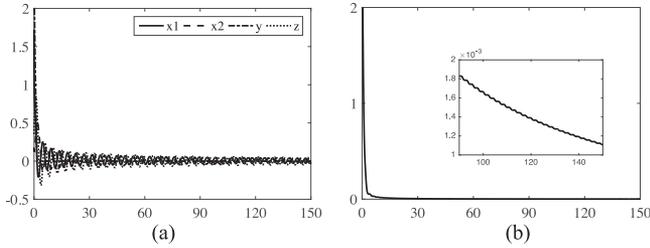


Fig. 1. Execution of the projected saddle-point dynamics (5) starting from  $(1.7256, 0.1793, 2.4696, 0.3532)$  for Example 4.4. Panel (a) and (b) show the evolution of  $(x, y, z)$  and  $V_1$ , respectively. As guaranteed by Theorem 4.3, the trajectory converges to the unique saddle point 0 and the function  $V_1$  defined in (9) decreases monotonically.

latter), as the function is not twice continuously differentiable everywhere.

*Example 4.4 (Illustration of global asymptotic convergence):* Consider  $F : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \times \mathbb{R} \rightarrow \mathbb{R}$  given as

$$F(x, y, z) = f(x) + y(-x_1 - 1) + z(x_1 - x_2) \quad (13)$$

where

$$f(x) = \begin{cases} \|x\|^4, & \text{if } \|x\| \leq \frac{1}{2}, \\ \frac{1}{16} + \frac{1}{2} \left( \|x\| - \frac{1}{2} \right), & \text{if } \|x\| \geq \frac{1}{2}. \end{cases}$$

Note that  $F$  is convex-concave on  $(\mathbb{R}^2) \times (\mathbb{R}_{\geq 0} \times \mathbb{R})$  and  $\text{Saddle}(F) = \{0\}$ . Also,  $F$  is continuously differentiable on the entire domain and its gradient is locally Lipschitz. Finally,  $F$  is twice continuously differentiable on the neighborhood  $\mathcal{U}_* = B_{1/2}(0) \cap (\mathbb{R}^2 \times \mathbb{R}_{\geq 0} \times \mathbb{R})$  of the saddle point 0 and hypothesis (1) of Theorem 4.3 holds on  $\mathcal{U}_*$ . Therefore, we conclude from Theorem 4.3 that the trajectories of the projected saddle-point dynamics of  $F$  converge globally asymptotically to the saddle point 0. Fig. 1 shows an execution. •

*Remark 4.5 (Comparison with the literature):* Theorems 4.2 and 4.3 complement the available results in the literature concerning the asymptotic convergence properties of saddle-point [3], [17], [19] and primal-dual dynamics [5], [20]. The former dynamics corresponds to (5) when the variable  $y$  is absent and the later to (5) when the variable  $z$  is absent. For both saddle-point and primal-dual dynamics, existing global asymptotic stability results require assumptions on the global properties of  $F$ , in addition to the global convexity-concavity of  $F$ , such as global strong convexity-concavity [3], global strict convexity-concavity, and its generalizations [19]. In contrast, the novelty of our results lies in establishing that certain local properties of the saddle function are enough to guarantee global asymptotic convergence. •

## V. LYAPUNOV FUNCTION FOR CONSTRAINED CONVEX OPTIMIZATION PROBLEMS

Our above discussion has established the global asymptotic stability of the set of saddle points resorting to LaSalle-type arguments (because the function  $V_1$  defined in (9) is not a strict Lyapunov function). In this section, we identify instead a strict Lyapunov function for the projected saddle-point dynamics when the saddle function  $F$  corresponds to the Lagrangian

of a constrained optimization problem, cf., Remark 3.1. The relevance of this result stems from two facts. On the one hand, the projected saddle-point dynamics has been employed profusely to solve network optimization problems. On the other hand, although the conclusions on the asymptotic convergence of this dynamics that can be obtained with the identified Lyapunov function are the same as in the previous section, having a Lyapunov function available is advantageous for a number of reasons, including the study of robustness against disturbances, the characterization of the algorithm convergence rate, or as a design tool for developing opportunistic state-triggered implementations. We come back to this point in Section VI.

*Theorem 5.1 (Lyapunov function for  $X_{p\text{-sp}}$ ):* Let  $F : \mathbb{R}^n \times \mathbb{R}_{\geq 0}^p \times \mathbb{R}^m \rightarrow \mathbb{R}$  be defined as

$$F(x, y, z) = f(x) + y^\top g(x) + z^\top (Ax - b), \quad (14)$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is strongly convex, twice continuously differentiable,  $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$  is convex, twice continuously differentiable,  $A \in \mathbb{R}^{m \times n}$ , and  $b \in \mathbb{R}^m$ . For each  $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}^p \times \mathbb{R}^m$ , define the index set of active constraints

$$\mathcal{J}(x, y, z) = \{j \in \{1, \dots, p\} \mid y_j = 0 \text{ and } (\nabla_y F(x, y, z))_j < 0\}.$$

Then, the function  $V_2 : \mathbb{R}^n \times \mathbb{R}_{\geq 0}^p \times \mathbb{R}^m \rightarrow \mathbb{R}$ ,

$$\begin{aligned} V_2(x, y, z) = & \frac{1}{2} \left( \|\nabla_x F(x, y, z)\|^2 + \|\nabla_z F(x, y, z)\|^2 \right. \\ & + \sum_{j \in \{1, \dots, p\} \setminus \mathcal{J}(x, y, z)} ((\nabla_y F(x, y, z))_j)^2 \Big) \\ & + \frac{1}{2} \|(x, y, z)\|_{\text{Saddle}(F)}^2 \end{aligned}$$

is nonnegative everywhere in its domain and  $V_2(x, y, z) = 0$  if and only if  $(x, y, z) \in \text{Saddle}(F)$ . Moreover, for any trajectory  $t \mapsto (x(t), y(t), z(t))$  of  $X_{p\text{-sp}}$ , the map  $t \mapsto V_2(x(t), y(t), z(t))$

- 1) is differentiable almost everywhere and if  $(x(t), y(t), z(t)) \notin \text{Saddle}(F)$  for some  $t \geq 0$ , then  $\frac{d}{dt} V_2(x(t), y(t), z(t)) < 0$  provided the derivative exists. Furthermore, for any sequence of times  $\{t_k\}_{k=1}^\infty$  such that  $t_k \rightarrow t$  and  $\frac{d}{dt} V_2(x(t_k), y(t_k), z(t_k))$  exists for every  $t_k$ , we have  $\limsup_{k \rightarrow \infty} \frac{d}{dt} V_2(x(t_k), y(t_k), z(t_k)) < 0$ ;
- 2) is right continuous and at any point of discontinuity  $t' \geq 0$ , we have  $V_2(x(t'), y(t'), z(t')) \leq \lim_{t \uparrow t'} V_2(x(t), y(t), z(t))$ .

As a consequence,  $\text{Saddle}(F)$  is globally asymptotically stable under  $X_{p\text{-sp}}$  and convergence of trajectories is to a point.

*Proof:* We start by partitioning the domain based on the active constraints. Let  $\mathcal{I} \subset \{1, \dots, p\}$  and

$$\mathcal{D}(\mathcal{I}) = \{(x, y, z) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}^p \times \mathbb{R}^m \mid \mathcal{J}(x, y, z) = \mathcal{I}\}.$$

Note that for  $\mathcal{I}_1, \mathcal{I}_2 \subset \{1, \dots, p\}$ ,  $\mathcal{I}_1 \neq \mathcal{I}_2$ , we have  $\mathcal{D}(\mathcal{I}_1) \cap \mathcal{D}(\mathcal{I}_2) = \emptyset$ . Moreover

$$\mathbb{R}^n \times \mathbb{R}_{\geq 0}^p \times \mathbb{R}^m = \bigcup_{\mathcal{I} \subset \{1, \dots, p\}} \mathcal{D}(\mathcal{I}).$$

For each  $\mathcal{I} \subset \{1, \dots, p\}$ , define the function

$$\begin{aligned} V_2^{\mathcal{I}}(x, y, z) &= \frac{1}{2} \left( \|\nabla_x F(x, y, z)\|^2 + \|\nabla_z F(x, y, z)\|^2 \right. \\ &\quad \left. + \sum_{j \in \{1, \dots, p\} \setminus \mathcal{I}} ((\nabla_y F(x, y, z))_j)^2 \right) \\ &\quad + \frac{1}{2} \|(x, y, z)\|_{\text{Saddle}(F)}^2. \end{aligned} \quad (15)$$

These functions will be used later for analyzing the evolution of  $V_2$ . Consider a trajectory  $t \mapsto (x(t), y(t), z(t))$  of  $X_{\text{p-sp}}$  starting at some point  $(x(0), y(0), z(0)) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}^p \times \mathbb{R}^m$ . Our proof strategy consists of proving assertions (1) and (2) for two scenarios, depending on whether or not there exists  $\delta > 0$  such that the difference between two consecutive time instants when the trajectory switches from one partition set to another is lower bounded by  $\delta$ .

*Scenario 1: Time elapsed between consecutive switches is lower bounded:* Let  $(a, b) \subset \mathbb{R}_{\geq 0}$ ,  $b - a \geq \delta$ , be a time interval for which the trajectory belongs to a partition  $\mathcal{D}(\mathcal{I}')$ ,  $\mathcal{I}' \subset \{1, \dots, p\}$  for all  $t \in (a, b)$ . In the following, we show that  $\frac{d}{dt} V_2(x(t), y(t), z(t))$  exists for almost all  $t \in (a, b)$  and its value is negative whenever  $(x(t), y(t), z(t)) \notin \text{Saddle}(F)$ . Consider the function  $V_2^{\mathcal{I}'}$  defined in (15) and note that  $t \mapsto V_2^{\mathcal{I}'}(x(t), y(t), z(t))$  is absolutely continuous as  $V_2^{\mathcal{I}'}$  is continuously differentiable on  $\mathbb{R}^n \times \mathbb{R}_{\geq 0}^p \times \mathbb{R}^m$  and the trajectory is absolutely continuous. Employing Rademacher's Theorem [28], we deduce that the map  $t \mapsto V_2^{\mathcal{I}'}(x(t), y(t), z(t))$  is differentiable almost everywhere. By definition,  $V_2(x(t), y(t), z(t)) = V_2^{\mathcal{I}'}(x(t), y(t), z(t))$  for all  $t \in (a, b)$ . Therefore,

$$\frac{d}{dt} V_2(x(t), y(t), z(t)) = \frac{d}{dt} V_2^{\mathcal{I}'}(x(t), y(t), z(t)) \quad (16)$$

for almost all  $t \in (a, b)$ . Further, since  $V_2^{\mathcal{I}'}$  is continuously differentiable, we have

$$\frac{d}{dt} V_2^{\mathcal{I}'}(x(t), y(t), z(t)) = \mathcal{L}_{X_{\text{p-sp}}} V_2^{\mathcal{I}'}(x(t), y(t), z(t)). \quad (17)$$

Now consider any  $(x, y, z) \in \mathcal{D}(\mathcal{I}') \setminus \text{Saddle}(F)$ . Our next computation shows that  $\mathcal{L}_{X_{\text{p-sp}}} V_2^{\mathcal{I}'}(x, y, z) < 0$ . We have

$$\begin{aligned} \mathcal{L}_{X_{\text{p-sp}}} V_2^{\mathcal{I}'}(x, y, z) &= -\nabla_x F(x, y, z)^\top \nabla_{xx} F(x, y, z) \nabla_x F(x, y, z) \\ &\quad + \begin{bmatrix} [\nabla_y F(x, y, z)]_y^+ \\ \nabla_z F(x, y, z) \end{bmatrix}^\top \begin{bmatrix} \nabla_{yy} F & \nabla_{yz} F \\ \nabla_{zy} F & \nabla_{zz} F \end{bmatrix}_{(x, y, z)} \\ &\quad \begin{bmatrix} [\nabla_y F(x, y, z)]_y^+ \\ \nabla_z F(x, y, z) \end{bmatrix} \\ &\quad + \mathcal{L}_{X_{\text{p-sp}}} \left( \frac{1}{2} \|(x, y, z)\|_{\text{Saddle}(F)}^2 \right). \end{aligned} \quad (18)$$

The first two terms in the above expression are the Lie derivative of  $(x, y, z) \mapsto V_2^{\mathcal{I}'}(x, y, z) - \frac{1}{2} \|(x, y, z)\|_{\text{Saddle}(F)}^2$ . This computation can be shown using the properties of the operator  $[\cdot]_y^+$ . Now let  $(x_*, y_*, z_*) = \text{proj}_{\text{Saddle}(F)}(x, y, z)$ . Then, by Danskin's Theorem [29, p. 99], we have

$$\nabla \|(x, y, z)\|_{\text{Saddle}(F)}^2 = 2(x - x_*; y - y_*; z - z_*). \quad (19)$$

Using this expression, we get

$$\begin{aligned} \mathcal{L}_{X_{\text{p-sp}}} \left( \frac{1}{2} \|(x, y, z)\|_{\text{Saddle}(F)}^2 \right) &= -(x - x_*)^\top \nabla_x F(x, y, z) + (y - y_*)^\top [\nabla_y F(x, y, z)]_y^+ \\ &\quad + (z - z_*)^\top \nabla_z F(x, y, z) \\ &\leq F(x_*, y, z) - F(x_*, y_*, z_*) + F(x_*, y_*, z_*) \\ &\quad - F(x, y_*, z_*), \end{aligned}$$

where the last inequality follows from (12). Now using the above expression in (18), we get

$$\begin{aligned} \mathcal{L}_{X_{\text{p-sp}}} V_2^{\mathcal{I}'}(x, y, z) &\leq -\nabla_x F(x, y, z) \nabla_{xx} F(x, y, z) \nabla_x F(x, y, z) \\ &\quad + \begin{bmatrix} [\nabla_y F(x, y, z)]_y^+ \\ \nabla_z F(x, y, z) \end{bmatrix}^\top \begin{bmatrix} \nabla_{yy} F & \nabla_{yz} F \\ \nabla_{zy} F & \nabla_{zz} F \end{bmatrix}_{(x, y, z)} \\ &\quad \begin{bmatrix} [\nabla_y F(x, y, z)]_y^+ \\ \nabla_z F(x, y, z) \end{bmatrix} \\ &\quad + F(x_*, y, z) - F(x_*, y_*, z_*) + F(x_*, y_*, z_*) \\ &\quad - F(x, y_*, z_*) \leq 0. \end{aligned}$$

If  $\mathcal{L}_{X_{\text{p-sp}}} V_2^{\mathcal{I}'}(x, y, z) = 0$ , then (a)  $\nabla_x F(x, y, z) = 0$ ; (b)  $x = x_*$ ; and (c)  $F(x_*, y, z) = F(x_*, y_*, z_*)$ . From (b) and (c), we conclude that  $\nabla_z F(x, y, z) = 0$ . From (c) and (14), we deduce that  $(y - y_*)^\top g(x_*) = 0$ . Note that for each  $i \in \{1, \dots, p\}$ , we have  $(y_i - (y_*)_i)(g(x_*)_i) \leq 0$ . This is because either  $(g(x_*)_i) = 0$  in which case it is trivial or  $(g(x_*)_i) < 0$  in which case  $(y_*)_i = 0$  (as  $y_*$  maximizes the map  $y \mapsto y^\top g(x_*)$ ) thereby making  $y_i - (y_*)_i \geq 0$ . Since,  $(y_i - (y_*)_i)(g(x_*)_i) \leq 0$  for each  $i$  and  $(y - y_*)^\top g(x_*) = 0$ , we get that for each  $i \in \{1, \dots, p\}$ , either  $(g(x_*)_i) = 0$  or  $y_i = (y_*)_i$ . Thus,  $[\nabla_y F(x, y, z)]_y^+ = 0$ . These facts imply that  $(x, y, z) \in \text{Saddle}(F)$ . Therefore, if  $(x, y, z) \in \mathcal{D}(\mathcal{I}') \setminus \text{Saddle}(F)$ , then  $\mathcal{L}_{X_{\text{p-sp}}} V_2^{\mathcal{I}'}(x, y, z) < 0$ . Combining this with (16) and (17), we deduce

$$\frac{d}{dt} V_2(x(t), y(t), z(t)) < 0$$

for almost all  $t \in (a, b)$ . Therefore, between any two switches in the partition, the evolution of  $V_2$  is differentiable and the value of the derivative is negative. Since the number of time instances when a switch occurs is countable, the first part of assertion 1) holds. To show the limit condition, consider  $t \geq 0$  such that  $(x(t), y(t), z(t)) \notin \text{Saddle}(F)$ . Let  $\{t_k\}_{k=1}^\infty$  be such that  $t_k \rightarrow t$  and  $\frac{d}{dt} V_2(x(t_k), y(t_k), z(t_k))$  exists for every  $t_k$ . By continuity,  $\lim_{k \rightarrow \infty} (x(t_k), y(t_k), z(t_k)) = (x(t), y(t), z(t))$ . Let  $\mathcal{B} \subset \mathbb{R}^n \times \mathbb{R}_{\geq 0}^p \times \mathbb{R}^m$  be a compact neighborhood of  $(x(t), y(t), z(t))$  such that  $\mathcal{B} \cap \text{Saddle}(F) = \emptyset$ . Without loss of generality, assume that  $\{(x(t_k), y(t_k), z(t_k))\}_{k=1}^\infty \subset \mathcal{B}$ . Define

$$S = \max \{ \mathcal{L}_{X_{\text{p-sp}}} V_2^{\mathcal{J}(x, y, z)}(x, y, z) \mid (x, y, z) \in \mathcal{B} \}.$$

The Lie derivatives in the above expression are well-defined and continuous as each  $V_2^{\mathcal{J}(x, y, z)}$  is continuously differentiable. Note that  $S < 0$  as  $\mathcal{B} \cap \text{Saddle}(F) = \emptyset$ . Moreover, as established previously, for each  $k$ ,  $\frac{d}{dt} V_2(x(t_k), y(t_k), z(t_k)) =$

$\mathcal{L}_{X_{p\text{-sp}}} V_2^{\mathcal{J}(x(t_k), y(t_k), z(t_k))}(x(t_k), y(t_k), z(t_k)) \leq S$ . Thus, we get  $\limsup_{k \rightarrow \infty} \frac{d}{dt} V_2(x(t_k), y(t_k), z(t_k)) \leq S' < 0$ , establishing 1) for Scenario 1.

To prove assertion 2), note that discontinuity in  $V_2$  can only happen when the trajectory switches the partition. In order to analyze this, consider any time instant  $t' \geq 0$  and let  $(x(t'), y(t'), z(t')) \in \mathcal{D}(\mathcal{I}')$  for some  $\mathcal{I}' \subset \{1, \dots, p\}$ . Looking at times  $t \geq t'$ , following two cases arise.

- There exists  $\tilde{\delta} > 0$  such that  $(x(t), y(t), z(t)) \in \mathcal{D}(\mathcal{I}')$  for all  $t \in [t', t' + \tilde{\delta})$ .
- There exists  $\tilde{\delta} > 0$  and  $\mathcal{I} \neq \mathcal{I}'$  such that  $(x(t), y(t), z(t)) \in \mathcal{D}(\mathcal{I})$  for all  $t \in (t', t' + \tilde{\delta})$ .

One can show that for Scenario 1, the trajectory cannot show any behavior other than the above two cases. We proceed to show that in both the previously outlined cases,  $t \mapsto V_2(x(t), y(t), z(t))$  is right continuous at  $t'$ . Case a) is straightforward as  $V_2$  is continuous in the domain  $\mathcal{D}(\mathcal{I}')$  and the trajectory is absolutely continuous. In case b),  $\mathcal{I} \neq \mathcal{I}'$  implies that there exists  $j \in \{1, \dots, p\}$  such that either  $j \in \mathcal{I} \setminus \mathcal{I}'$  or  $j \in \mathcal{I}' \setminus \mathcal{I}$ . Note that the later scenario, i.e.,  $j \in \mathcal{I}'$  and  $j \notin \mathcal{I}$  cannot happen. Indeed by definition  $(y(t'))_j = 0$  and  $(\nabla_y F(x(t'), y(t'), z(t')))_j < 0$ , and by continuity of the trajectory and the map  $\nabla_y F$ , these conditions also hold for some finite time interval starting at  $t'$ . Therefore, we focus on the case that  $j \in \mathcal{I} \setminus \mathcal{I}'$ . Then, either  $(y(t'))_j > 0$  or  $(\nabla_y F(x(t'), y(t'), z(t')))_j \geq 0$ . The former implies, due to continuity of trajectories, that it is not possible to have  $j \in \mathcal{I}$ . Similarly, by continuity if  $(\nabla_y F(x(t'), y(t'), z(t')))_j > 0$ , then one cannot have  $j \in \mathcal{I}$ . Therefore, the only possibility is  $(y(t'))_j = 0$  and  $(\nabla_y F(x(t'), y(t'), z(t')))_j = 0$ . This implies that the term  $t \mapsto (\nabla_y F(x(t), y(t), z(t)))_j^2$  is right continuous at  $t'$ . Since this holds for any  $j \in \mathcal{I} \setminus \mathcal{I}'$ , we conclude right continuity of  $V_2$  at  $t'$ . Therefore, for both cases a) and b), we conclude right continuity of  $V_2$ .

Next we show the limit condition of assertion 2). Let  $t' \geq 0$  be a point of discontinuity. Then, from the preceding discussion, there must exist  $\mathcal{I}, \mathcal{I}' \subset \{1, \dots, p\}$ ,  $\mathcal{I} \neq \mathcal{I}'$ , such that  $(x(t'), y(t'), z(t')) \in \mathcal{D}(\mathcal{I}')$  and  $(x(t), y(t), z(t)) \in \mathcal{D}(\mathcal{I})$  for all  $t \in (t' - \delta, t')$ . By continuity,  $\lim_{t \uparrow t'} V_2(x(t), y(t), z(t))$  exists. Note that if  $j \in \mathcal{I}$  and  $j \notin \mathcal{I}'$ , then the term getting added to  $V_2$  at time  $t'$ , which was absent at times  $t \in (t' - \delta, t')$ , i.e.,  $(\nabla_y F(x(t), y(t), z(t)))_j^2$ , is zero at  $t'$ . Therefore, the discontinuity at  $t'$  can only happen due to the existence of  $j \in \mathcal{I}' \setminus \mathcal{I}$ . That is, a constraint becomes active at time  $t'$ , which was inactive in the time interval  $(t' - \delta, t')$ . Thus, the function  $V_2$  loses a nonnegative term at time  $t'$ . This can only mean at  $t'$ , the value of  $V_2$  decreases. Hence, the limit condition of assertion 2) holds.

*Scenario 2: Time elapsed between consecutive switches is not lower bounded:* Observe that three cases arise. First is when there are only a finite number of switches in partition in any compact time interval. In this case, the analysis of Scenario 1 applies to every compact time interval and so assertions 1) and 2) hold. The second case is when there exist time instants  $t' > 0$  where there is the absence of “finite dwell time,” that is, there exist index sets  $\mathcal{I}_1 \neq \mathcal{I}_2$  and  $\mathcal{I}_2 \neq \mathcal{I}_3$  such that  $(x(t), y(t), z(t)) \in \mathcal{D}(\mathcal{I}_1)$  for all  $t \in (t' - \epsilon_1, t')$  and some  $\epsilon_1 > 0$ ;  $(x(t'), y(t'), z(t')) \in \mathcal{D}(\mathcal{I}_2)$ ; and  $(x(t), y(t), z(t)) \in \mathcal{D}(\mathcal{I}_3)$  for all  $t \in (t', t' + \epsilon_2)$  and some  $\epsilon_2 > 0$ . Again using the arguments of Scenario 1, one can show that both assertions 1) and

2) hold for this case if there is no accumulation point of such time instants  $t'$ .

The third case instead is when there are infinite switches in a finite-time interval. We analyze this case in parts. Assume that there exists a sequence of times  $\{t_k\}_{k=1}^\infty$ ,  $t_k \uparrow t'$ , such that trajectory switches partition at each  $t_k$ . The aim is to show left continuity of  $t \mapsto V_2(x(t), y(t), z(t))$  at  $t'$ . Let  $\mathcal{I}^s \subset \{1, \dots, p\}$  be the set of indices that switch between being active and inactive an infinite number of times along the sequence  $\{t_k\}$  (note that the set is nonempty as there are an infinite number of switches and a finite number of indices). To analyze the left continuity at  $t'$ , we only need to study the possible occurrence of discontinuity due to terms in  $V_2$  corresponding to the indices in  $\mathcal{I}^s$ , since all other terms do not affect the continuity. Pick any  $j \in \mathcal{I}^s$ . Then, the term in  $V_2$  corresponding to the index  $j$  satisfies

$$\lim_{k \rightarrow \infty} (\nabla_y F(x(t_k), y(t_k), z(t_k)))_j^2 = 0. \quad (20)$$

In order to show this, assume the contrary. This implies the existence of  $\epsilon > 0$  such that

$$\limsup_{k \rightarrow \infty} (\nabla_y F(x(t_k), y(t_k), z(t_k)))_j^2 \geq \epsilon.$$

As a consequence, the set of  $k$  for which  $(\nabla_y F(x(t_k), y(t_k), z(t_k)))_j^2 \geq \epsilon/2$  is infinite. Recall that if the constraint  $j$  becomes active at  $t_k$ , then  $V_2$  loses the term  $(\nabla_y F(x(t_k), y(t_k), z(t_k)))_j^2$  at  $t_k$ . Further, at  $t_k$ , if some other constraint  $j'$  becomes inactive while being active at times just before  $t_k$ , then it follows by the definition of active constraint that  $(\nabla_y F(x(t_k), y(t_k), z(t_k)))_{j'}^2 = 0$ . Finally, if some other constraint becomes active at  $t_k$  apart from  $j$ , then this only decreases the value of  $V_2$  at  $t_k$ . Collecting all this reasoning, we deduce that  $V_2$  decreases by at least  $\epsilon/2$  at each  $t_k$ . From what we showed before,  $V_2$  decreases monotonically between any consecutive  $t_k$ 's. These facts lead to the conclusion that  $V_2$  tends to  $-\infty$  as  $t_k \rightarrow t'$ . However,  $V_2$  takes nonnegative values, yielding a contradiction. Hence, (20) is true for all  $j \in \mathcal{I}^s$  and so

$$\lim_{k \rightarrow \infty} V_2(x(t_k), y(t_k), z(t_k)) = V_2(x(t'), y(t'), z(t'))$$

proving left continuity of  $V_2$  at  $t'$ . Using this reasoning, one can also conclude that if the infinite number of switches happen on a sequence  $\{t_k\}_{k=1}^\infty$  with  $t_k \downarrow t'$ , then one has right continuity at  $t'$ . Therefore, at each time instant when a switch happens, we have right continuity of  $t \mapsto V_2(x(t), y(t), z(t))$  and at points where there is accumulation of switches, we have continuity (depending on which side of the time instance the accumulation takes place). This proves assertion 2). Note that in this case too, we have a countable number of time instants where the partition set switches and so the map  $t \mapsto V_2(x(t), y(t), z(t))$  is differentiable almost everywhere. Moreover, one can also analyze, as done in Scenario 1, that the limit condition of assertion 1) holds in this case. These facts together establish the condition of assertion 2). Thus, we have shown that trajectories converge to a saddle point and since  $\text{Saddle}(F)$  is stable under  $X_{p\text{-sp}}$  (cf., Proposition 4.1), we conclude the global asymptotic stability of  $\text{Saddle}(F)$ . ■

*Remark 5.2 (Multiple Lyapunov functions):* The Lyapunov function  $V_2$  is discontinuous on the domain  $\mathbb{R}^n \times \mathbb{R}_{\geq 0}^p \times \mathbb{R}^m$ . However, it can be seen as multiple (continuously differentiable) Lyapunov functions [30], each valid on a domain, patched

together in an appropriate way such that along the trajectories of  $X_{p\text{-sp}}$ , the evolution of  $V_2$  is continuously differentiable with negative derivative at intervals where it is continuous, and at times of discontinuity, the value of  $V_2$  only decreases. Note that in the absence of the projection in  $X_{p\text{-sp}}$  (that is, no  $y$ -component of the dynamics), the function  $V_2$  takes a much simpler form with no discontinuities and is continuously differentiable on the entire domain. •

*Remark 5.3 (Connection with the literature: II):* The two functions whose sum defines  $V_2$  are, individually by themselves, sufficient to establish asymptotic convergence of  $X_{p\text{-sp}}$  using LaSalle Invariance arguments, see, e.g., [5] and [20]. However, the fact that their combination results in a strict Lyapunov function for the projected saddle-point dynamics is a novelty of our analysis here. In [17], a different Lyapunov function is proposed and an exponential rate of convergence is established for a saddle-point-like dynamics, which is similar to  $X_{p\text{-sp}}$  but without projection components. •

## VI. ISS AND SELF-TRIGGERED IMPLEMENTATION OF THE SADDLE-POINT DYNAMICS

Here, we build on the novel Lyapunov function identified in Section V to explore other properties of the projected saddle-point dynamics beyond global asymptotic convergence. Throughout this section, we consider saddle functions  $F$  that corresponds to the Lagrangian of an equality-constrained optimization problem, i.e.,

$$F(x, z) = f(x) + z^\top (Ax - b) \quad (21)$$

where  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$ , and  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ . The reason behind this focus is that, in this case, the dynamics (5) is smooth and the Lyapunov function identified in Theorem 5.1 is continuously differentiable. These simplifications allow us to analyze ISS of the dynamics using the theory of ISS-Lyapunov functions (cf., Section II-D). On the other hand, we do not know of such a theory for projected systems, which precludes us from carrying out ISS analysis for dynamics (5) for a general saddle function. The projected saddle-point dynamics (5) for the class of saddle functions given in (21) takes the form

$$\dot{x} = -\nabla_x F(x, z) = -\nabla f(x) - A^\top z, \quad (22a)$$

$$\dot{z} = \nabla_z F(x, z) = Ax - b, \quad (22b)$$

corresponding to (5a) and (5c). We term these dynamics simply *saddle-point dynamics* and denote it as  $X_{sp}: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ .

### A. Input-to-State Stability (ISS)

Here, we establish that the saddle-point dynamics (22) is ISS with respect to the set  $\text{Saddle}(F)$  when disturbance inputs affect it additively. Disturbance inputs can arise when implementing the saddle-point dynamics as a controller of a physical system because of a variety of malfunctions, including errors in the gradient computation, noise in state measurements, and errors in the controller implementation. In such scenarios, the following result shows that the dynamics (22) exhibits a graceful degradation of its convergence properties, one that scales with the size of the disturbance.

*Theorem 6.1 (ISS of saddle-point dynamics):* Let the saddle function  $F$  be of the form (21), with  $f$  strongly convex, twice continuously differentiable, and satisfying  $mI \preceq \nabla^2 f(x) \preceq MI$  for all  $x \in \mathbb{R}^n$  and some constants  $0 < m \leq M < \infty$ . Then, the dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\nabla_x F(x, z) \\ \nabla_z F(x, z) \end{bmatrix} + \begin{bmatrix} u_x \\ u_z \end{bmatrix} \quad (23)$$

where  $(u_x, u_z): \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n \times \mathbb{R}^m$  is a measurable and locally essentially bounded map, is ISS with respect to  $\text{Saddle}(F)$ .

*Proof:* For notational convenience, we refer to (23) by  $X_{sp}^p: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ . Our proof consists of establishing that the function  $V_3: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}$ ,

$$V_3(x, z) = \frac{\beta_1}{2} \|X_{sp}^p(x, z)\|^2 + \frac{\beta_2}{2} \|(x, z)\|_{\text{Saddle}(F)}^2 \quad (24)$$

with  $\beta_1 > 0$ ,  $\beta_2 = \frac{4\beta_1 M^4}{m^2}$ , is an ISS-Lyapunov function with respect to  $\text{Saddle}(F)$  for  $X_{sp}^p$ . The statement then directly follows from Proposition 2.2.

We first show (3) for  $V_3$ , that is, there exist  $\alpha_1, \alpha_2 > 0$  such that  $\alpha_1 \|(x, z)\|_{\text{Saddle}(F)}^2 \leq V_3(x, z) \leq \alpha_2 \|(x, z)\|_{\text{Saddle}(F)}^2$  for all  $(x, z) \in \mathbb{R}^n \times \mathbb{R}^m$ . The lower bound follows by choosing  $\alpha_1 = \beta_2/2$ . For the upper bound, define the function  $U: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  by

$$U(x_1, x_2) = \int_0^1 \nabla^2 f(x_1 + s(x_2 - x_1)) ds. \quad (25)$$

By assumption, it holds that  $mI \preceq U(x_1, x_2) \preceq MI$  for all  $x_1, x_2 \in \mathbb{R}^n$ . Also, from the fundamental theorem of calculus, we have  $\nabla f(x_2) - \nabla f(x_1) = U(x_1, x_2)(x_2 - x_1)$  for all  $x_1, x_2 \in \mathbb{R}^n$ . Now pick any  $(x, z) \in \mathbb{R}^n \times \mathbb{R}^m$ . Let  $(x_*, z_*) = \text{proj}_{\text{Saddle}(F)}(x, z)$ , that is, the projection of  $(x, z)$  on the set  $\text{Saddle}(F)$ . This projection is unique as  $\text{Saddle}(F)$  is convex. Then, one can write

$$\begin{aligned} \nabla_x F(x, z) &= \nabla_x F(x_*, z_*) + \int_0^1 \nabla_{xx} F(x(s), z(s))(x - x_*) ds \\ &\quad + \int_0^1 \nabla_{xz} F(x(s), z(s))(z - z_*) ds \\ &= U(x_*, x)(x - x_*) + A^\top (z - z_*), \end{aligned} \quad (26)$$

where  $x(s) = x_* + s(x - x_*)$  and  $z(s) = z_* + s(z - z_*)$ . Also, note that

$$\begin{aligned} \nabla_z F(x, z) &= \nabla_z F(x_*, z_*) + \int_0^1 \nabla_{xz} F(x(s), z(s))(x - x_*) ds \\ &= A(x - x_*). \end{aligned} \quad (27)$$

The expressions (26) and (27) use  $\nabla_x F(x_*, z_*) = 0$ ,  $\nabla_z F(x_*, z_*) = 0$ , and  $\nabla_{zx} F(x, z) = \nabla_{xz} F(x, z)^\top = A^\top$  for all  $(x, z)$ . From (26) and (27), we get

$$\begin{aligned} \|X_{sp}(x, z)\|^2 &\leq \tilde{\alpha}_2 (\|x - x_*\|^2 + \|z - z_*\|^2) \\ &= \tilde{\alpha}_2 \|(x, z)\|_{\text{Saddle}(F)}^2, \end{aligned}$$

where  $\tilde{\alpha}_2 = \frac{3}{2}(M^2 + \|A\|^2)$ . In the above computation, we have used the inequality  $(a + b)^2 \leq 3(a^2 + b^2)$  for any  $a, b \in$

$\mathbb{R}$ . The above inequality gives the upper bound  $V_3(x, z) \leq \alpha_2 \|(x, z)\|_{\text{Saddle}(F)}^2$ , where  $\alpha_2 = \frac{3\beta_1}{2}(M^2 + \|A\|^2) + \frac{\beta_2}{2}$ .

The next step is to show that the Lie derivative of  $V_3$  along the dynamics  $X_{\text{sp}}^p$  satisfies the ISS property (4). Again, pick any  $(x, z) \in \mathbb{R}^n \times \mathbb{R}^m$  and let  $(x_*, z_*) = \text{proj}_{\text{Saddle}(F)}(x, z)$ . Then, by Danskin's Theorem [29, p. 99], we get

$$\nabla \|(x, z)\|_{\text{Saddle}(F)}^2 = 2(x - x_*; z - z_*).$$

Using the above expression, one can compute the Lie derivative of  $V_3$  along the dynamics  $X_{\text{sp}}^p$  as

$$\begin{aligned} \mathcal{L}_{X_{\text{sp}}^p} V_3(x, z) &= -\beta_1 \nabla_x F(x, z) \nabla_{xx} F(x, z) \nabla_x F(x, z) \\ &\quad - \beta_2 (x - x_*)^\top \nabla_x F(x, z) + \beta_2 (z - z_*)^\top \nabla_z F(x, z) \\ &\quad + \beta_1 \nabla_x F(x, z)^\top \nabla_{xx} F(x, z) u_x \\ &\quad + \beta_1 \nabla_x F(x, z)^\top \nabla_{xz} F(x, z) u_z \\ &\quad + \beta_1 \nabla_z F(x, z)^\top \nabla_{zx} F(x, z) u_x \\ &\quad + \beta_2 (x - x_*)^\top u_x + \beta_2 (z - z_*)^\top u_z. \end{aligned}$$

Due to the particular form of  $F$ , we have

$$\begin{aligned} \nabla_x F(x, z) &= \nabla f(x) + A^\top z, & \nabla_z F(x, z) &= Ax - b, \\ \nabla_{xx} F(x, z) &= \nabla^2 f(x), & \nabla_{xz} F(x, z) &= A^\top, \\ \nabla_{zx} F(x, z) &= A, & \nabla_{zz} F(x, z) &= 0. \end{aligned}$$

Also,  $\nabla_x F(x_*, z_*) = \nabla_x f(x_*) + A^\top z_* = 0$  and  $\nabla_z F(x_*, z_*) = Ax_* - b = 0$ . Substituting these values in the expression of  $\mathcal{L}_{X_{\text{sp}}^p} V_3$ , replacing  $\nabla_x F(x, z) = \nabla_x F(x, z) - \nabla_x F(x_*, z_*) = \nabla f(x) - \nabla f(x_*) + A^\top(z - z_*) = U(x_*, x)(x - x_*) + A^\top(z - z_*)$ , and simplifying

$$\begin{aligned} \mathcal{L}_{X_{\text{sp}}^p} V_3(x, z) &= \\ &- \beta_1 (U(x_*, x)(x - x_*) + A^\top(z - z_*))^\top \nabla^2 f(x) (U(x_*, x)(x - x_*)) \\ &- \beta_1 (z - z_*)^\top A \nabla^2 f(x) A^\top (z - z_*) \\ &- \beta_1 (U(x_*, x)(x - x_*) + A^\top(z - z_*))^\top \nabla^2 f(x) A^\top (z - z_*) \\ &- \beta_1 (z - z_*)^\top A \nabla^2 f(x) (U(x_*, x)(x - x_*)) \\ &- (x - x_*)^\top U(x_*, x)(x - x_*) \\ &+ \beta_1 (U(x_*, x)(x - x_*) + A^\top(z - z_*))^\top \nabla^2 f(x) u_x \\ &+ \beta_1 (U(x_*, x)(x - x_*) + A^\top(z - z_*))^\top A^\top u_z \\ &+ \beta_2 (x - x_*)^\top u_x + \beta_1 (A(x - x_*))^\top A u_x + \beta_2 (z - z_*)^\top u_z. \end{aligned}$$

Upper bounding now the terms using  $\|\nabla^2 f(x)\|, \|U(x_*, x)\| \leq M$  for all  $x \in \mathbb{R}^n$  yields

$$\begin{aligned} \mathcal{L}_{X_{\text{sp}}^p} V_3(x, z) &\leq -[x - x_*; A^\top(z - z_*)]^\top \bar{U}(x_*, x)[x - x_*; A^\top(z - z_*)] \\ &\quad + C_x(x, z)\|u_x\| + C_z(x, z)\|u_z\| \end{aligned} \quad (28)$$

where

$$\begin{aligned} C_x(x, z) &= \left( \beta_1 M^2 \|x - x_*\| + \beta_1 M \|A\| \|z - z_*\| \right. \\ &\quad \left. + \beta_2 \|x - x_*\| + \beta_1 \|A\|^2 \|x - x_*\| \right), \\ C_z(x, z) &= \left( \beta_1 M \|A\| \|x - x_*\| + \beta_1 \|A\|^2 \|z - z_*\| \right. \\ &\quad \left. + \beta_2 \|z - z_*\| \right), \end{aligned}$$

and  $\bar{U}(x_*, x)$  is

$$\begin{bmatrix} \beta_1 U \nabla^2 f(x) U + \beta_2 U & \beta_1 U \nabla^2 f(x) \\ \beta_1 \nabla^2 f(x) U & \beta_1 \nabla^2 f(x) \end{bmatrix}.$$

where  $U = U(x_*, x)$ . Note that  $C_x(x, z) \leq \tilde{C}_x \|x - x_*; z - z_*\| = \tilde{C}_x \|(x, z)\|_{\text{Saddle}(F)}$  and  $C_z(x, z) \leq \tilde{C}_z \|x - x_*; z - z_*\| = \tilde{C}_z \|(x, z)\|_{\text{Saddle}(F)}$ , where

$$\begin{aligned} \tilde{C}_x &= \beta_1 M^2 + \beta_1 M \|A\| + \beta_2 + \beta_1 \|A\|^2, \\ \tilde{C}_z &= \beta_1 M \|A\| + \beta_1 \|A\|^2 + \beta_2. \end{aligned}$$

From Lemma A.1, we have  $\bar{U}(x_*, x) \succeq \lambda_m I$ , where  $\lambda_m > 0$ . Employing these facts in (28), we obtain

$$\begin{aligned} \mathcal{L}_{X_{\text{sp}}^p} V_3(x, z) &\leq -\lambda_m (\|x - x_*\|^2 + \|A^\top(z - z_*)\|^2) \\ &\quad + (\tilde{C}_x + \tilde{C}_z) \|(x, z)\|_{\text{Saddle}(F)} \|u\|. \end{aligned}$$

From Lemma A.2, we get

$$\begin{aligned} \mathcal{L}_{X_{\text{sp}}^p} V_3(x, z) &\leq -\lambda_m (\|x - x_*\|^2 + \lambda_s(AA^\top) \|z - z_*\|^2) \\ &\quad + (\tilde{C}_x + \tilde{C}_z) \|(x, z)\|_{\text{Saddle}(F)} \|u\| \\ &\leq -\tilde{\lambda}_m \|(x, z)\|_{\text{Saddle}(F)}^2 \\ &\quad + (\tilde{C}_x + \tilde{C}_z) \|(x, z)\|_{\text{Saddle}(F)} \|u\|, \end{aligned}$$

where  $\tilde{\lambda}_m = \lambda_m \min\{1, \lambda_s(AA^\top)\}$ . Now pick any  $\theta \in (0, 1)$ . Then

$$\begin{aligned} \mathcal{L}_{X_{\text{sp}}^p} V_3(x, z) &\leq - (1 - \theta) \tilde{\lambda}_m \|(x, z)\|_{\text{Saddle}(F)}^2 \\ &\quad - \theta \tilde{\lambda}_m \|(x, z)\|_{\text{Saddle}(F)}^2 \\ &\quad + (\tilde{C}_x + \tilde{C}_z) \|(x, z)\|_{\text{Saddle}(F)} \|u\| \\ &\leq - (1 - \theta) \tilde{\lambda}_m \|(x, z)\|_{\text{Saddle}(F)}^2, \end{aligned}$$

whenever  $\|(x, z)\|_{\text{Saddle}(F)} \geq \frac{\tilde{C}_x + \tilde{C}_z}{\theta \tilde{\lambda}_m} \|u\|$ , which proves the ISS property.  $\blacksquare$

*Remark 6.2 (Relaxing global bounds on Hessian of  $f$ ):* The assumption on the Hessian of  $f$  in Theorem 6.1 is restrictive, but there are functions other than quadratic that satisfy it, see, e.g., [31, Sec. 6]. We conjecture that the global upper bound on the Hessian can be relaxed by resorting to the notion of semiglobal ISS, and we will explore this in the future.  $\bullet$

The above result has the following consequence.

*Corollary 6.3 (Lyapunov function for saddle-point dynamics):* Let the saddle function  $F$  be of the form (21), with  $f$

strongly convex, twice continuously differentiable, and satisfying  $mI \preceq \nabla^2 f(x) \preceq MI$  for all  $x \in \mathbb{R}^n$  and some constants  $0 < m \leq M < \infty$ . Then, the function  $V_3$  (24) is a Lyapunov function with respect to the set  $\text{Saddle}(F)$  for the saddle-point dynamics (22).

*Remark 6.4 (ISS with respect to  $\text{Saddle}(F)$  does not imply bounded trajectories):* Note that Theorem 6.1 bounds only the distance of the trajectories of (23) to  $\text{Saddle}(F)$ . Thus, if  $\text{Saddle}(F)$  is unbounded, the trajectories of (23) can be unbounded under arbitrarily small constant disturbances. However, if the matrix  $A$  has full row rank, then  $\text{Saddle}(F)$  is a singleton and the ISS property implies that the trajectory of (23) remains bounded under bounded disturbances. •

As pointed out in the previous remark, if  $\text{Saddle}(F)$  is not unique, then the trajectories of the dynamics might not be bounded. We next look at a particular type of disturbance input that guarantees bounded trajectories even when  $\text{Saddle}(F)$  is unbounded. Pick any  $(x_*, z_*) \in \text{Saddle}(F)$  and define the function  $\tilde{V}_3 : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}$  as

$$\tilde{V}_3(x, z) = \frac{\beta_1}{2} \|X_{\text{sp}}(x, z)\|^2 + \frac{\beta_2}{2} (\|x - x_*\|^2 + \|z - z_*\|^2)$$

with  $\beta_1 > 0$ ,  $\beta_2 = \frac{4\beta_1 M^4}{m^2}$ . One can show, following similar steps as those of proof of Theorem 6.1, that the function  $\tilde{V}_3$  is an ISS Lyapunov function with respect to the point  $(x_*, z_*)$  for the dynamics  $X_{\text{sp}}^p$  when the disturbance input to  $z$ -dynamics has the special structure  $u_z = A\tilde{u}_z$ ,  $\tilde{u}_z \in \mathbb{R}^n$ . This type of disturbance is motivated by scenarios with measurement errors in the values of  $x$  and  $z$  used in (22) and without any computation error of the gradient term in the  $z$ -dynamics. The following statement makes precise the ISS property for this particular disturbance.

*Corollary 6.5 (ISS of saddle-point dynamics):* Let the saddle function  $F$  be of the form (21), with  $f$  strongly convex, twice continuously differentiable, and satisfying  $mI \preceq \nabla^2 f(x) \preceq MI$  for all  $x \in \mathbb{R}^n$  and some constants  $0 < m \leq M < \infty$ . Then, the dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\nabla_x F(x, z) \\ \nabla_z F(x, z) \end{bmatrix} + \begin{bmatrix} u_x \\ A\tilde{u}_z \end{bmatrix}, \quad (29)$$

where  $(u_x, \tilde{u}_z) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{2n}$  is measurable and locally essentially bounded input, is ISS with respect to every point of  $\text{Saddle}(F)$ .

The proof is analogous to that of Theorem 6.1 with the key difference that the terms  $C_x(x, z)$  and  $C_z(x, z)$  appearing in (28) need to be upper bounded in terms of  $\|x - x_*\|$  and  $\|A^\top(z - z_*)\|$ . This can be done due to the special structure of  $u_z$ . With these bounds, one arrives at the condition (4) for Lyapunov  $\tilde{V}_3$  and dynamics (29). One can deduce from Corollary 6.5 that the trajectory of (29) remains bounded for bounded input even when  $\text{Saddle}(F)$  is unbounded.

*Example 6.6 (ISS property of saddle-point dynamics):* Consider  $F : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$  of the form (21) with

$$f(x) = x_1^2 + (x_2 - 2)^2, \quad (30)$$

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then,  $\text{Saddle}(F) = \{(x, z) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x = (1, 1), z = (0, 2) + \lambda(1, 1), \lambda \in \mathbb{R}\}$  is a continuum of points. Note that  $\nabla^2 f(x) =$

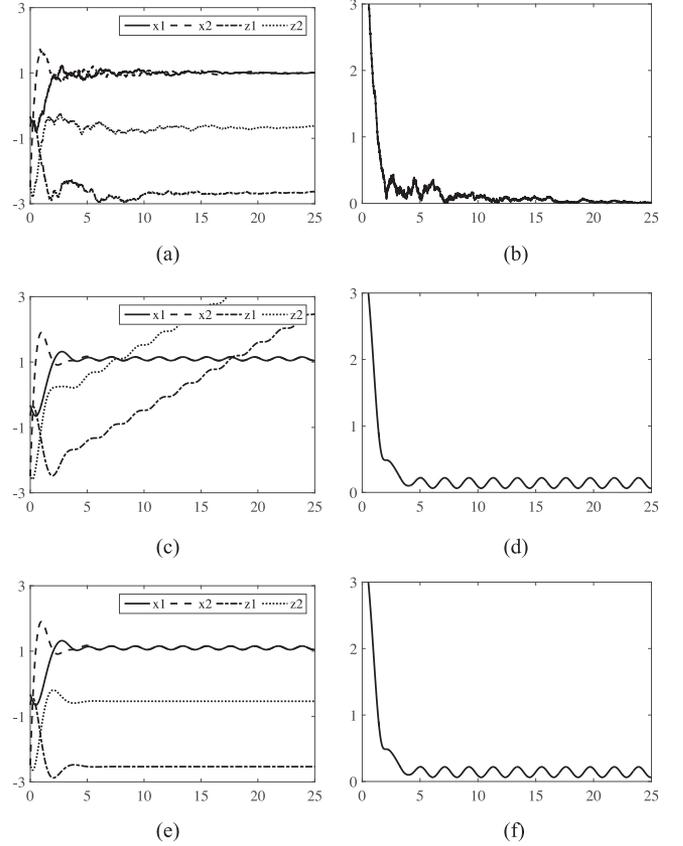


Fig. 2. Plots (a) and (b) show the ISS property, cf., Theorem 6.1, of the dynamics (23) for the saddle function  $F$  defined by (30). The initial condition is  $x(0) = (-0.3254, -2.4925)$  and  $z(0) = (-0.6435, -2.4234)$  and the input  $u$  is exponentially decaying in magnitude. Panels (a), (c), and (e) depict the evolution of  $(x, z)$ . Panels (b), (d), and (f) depict the distance of the state trajectories to the set of saddle points. As shown in (a) and (b), the trajectory converges asymptotically to a saddle point as the input is vanishing. Plots (c) and (d) have the same initial condition but the disturbance input consists of a constant plus a sinusoid. The trajectory is unbounded under bounded input while the distance to the set of saddle points remains bounded, cf., Remark 6.4. Plots (e) and (f) have the same initial condition but the disturbance input to the  $z$ -dynamics is of the form (29). In this case, the trajectory remains bounded as the dynamics is ISS with respect to each saddle point, cf., Corollary 6.5.

$2I$ , thus, satisfying the assumption of bounds on the Hessian of  $f$ . By Theorem 6.1, the saddle-point dynamics for this saddle function  $F$  is input-to-state stable with respect to the set  $\text{Saddle}(F)$ . This fact is illustrated in Fig. 2, which also depicts how the specific structure of the disturbance input in (29) affects the boundedness of the trajectories. •

*Remark 6.7 (Quadratic ISS-Lyapunov function):* For the saddle-point dynamics (22), the ISS property stated in Theorem 6.1 and Corollary 6.5 can also be shown using a quadratic Lyapunov function. Let  $V_4 : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}$  be

$$V_4(x, z) = \frac{1}{2} \|(x, z)\|_{\text{Saddle}(F)}^2 + \epsilon(x - x_p)^\top A^\top(z - z_p)$$

where  $(x_p, z_p) = \text{proj}_{\text{Saddle}(F)}(x, z)$  and  $\epsilon > 0$ . Then, one can show that there exists  $\epsilon_{\text{max}} > 0$  such that  $V_4$  for any  $\epsilon \in (0, \epsilon_{\text{max}})$  is an ISS-Lyapunov function for the dynamics (22). For space reasons, we omit the complete analysis of this fact here. •

## B. Self-Triggered Implementation

In this section, we develop an opportunistic state-triggered implementation of the (continuous-time) saddle-point dynamics. Our aim is to provide a discrete-time execution of the algorithm, either on a physical system or as an optimization strategy, that do not require the continuous evaluation of the vector field and instead adjust the stepsize based on the current state of the system. Formally, given a sequence of triggering time instants  $\{t_k\}_{k=0}^{\infty}$ , with  $t_0 = 0$ , we consider the following implementation of the saddle-point dynamics

$$\dot{x}(t) = -\nabla_x F(x(t_k), z(t_k)), \quad (31a)$$

$$\dot{z}(t) = \nabla_z F(x(t_k), z(t_k)). \quad (31b)$$

for  $t \in [t_k, t_{k+1})$  and  $k \in \mathbb{Z}_{\geq 0}$ . The objective is then to design a criterium to opportunistically select the sequence of triggering instants, guaranteeing at the same time, the feasibility of the execution and global asymptotic convergence, see, e.g., [32]. Toward this goal, we look at the evolution of the Lyapunov function  $V_3$  in (24) along (31) as

$$\begin{aligned} & \nabla V_3(x(t), z(t))^\top X_{\text{sp}}(x(t_k), z(t_k)) \\ &= \mathcal{L}_{X_{\text{sp}}} V_3(x(t_k), z(t_k)) \\ &+ \left( \nabla V_3(x(t), z(t)) - \nabla V_3(x(t_k), z(t_k)) \right)^\top \\ & \quad X_{\text{sp}}(x(t_k), z(t_k)). \end{aligned} \quad (32)$$

We know from Corollary 6.3 that the first summand is negative outside  $\text{Saddle}(F)$ . Clearly, for  $t = t_k$ , the second summand vanishes, and by continuity, for  $t$  sufficiently close to  $t_k$ , this summand remains smaller in magnitude than the first, ensuring the decrease of  $V_3$ . To make this argument precise, we employ Proposition A.3 in (32) and obtain

$$\begin{aligned} & \nabla V_3(x(t), z(t))^\top X_{\text{sp}}(x(t_k), z(t_k)) \\ & \leq \mathcal{L}_{X_{\text{sp}}} V_3(x(t_k), z(t_k)) + \xi(x(t_k), z(t_k)) \\ & \quad \| (x(t) - x(t_k)); (z(t) - z(t_k)) \| \| X_{\text{sp}}(x(t_k), z(t_k)) \| \\ &= \mathcal{L}_{X_{\text{sp}}} V_3(x(t_k), z(t_k)) \\ & \quad + (t - t_k) \xi(x(t_k), z(t_k)) \| X_{\text{sp}}(x(t_k), z(t_k)) \|^2, \end{aligned}$$

where the equality follows from writing  $(x(t), z(t))$  in terms of  $(x(t_k), z(t_k))$  by integrating (31). Therefore, in order to ensure the monotonic decrease of  $V_3$ , we require the above expression to be nonpositive. That is

$$t_{k+1} \leq t_k - \frac{\mathcal{L}_{X_{\text{sp}}} V_3(x(t_k), z(t_k))}{\xi(x(t_k), z(t_k)) \| X_{\text{sp}}(x(t_k), z(t_k)) \|^2}. \quad (33)$$

Note that to set  $t_{k+1}$  equal to the right-hand side of the above expression, one needs to compute the Lie derivative at  $(x(t_k), z(t_k))$ . We then distinguish between two possibilities. If the self-triggered saddle-point dynamics acts as a closed-loop physical system and its equilibrium points are known, then computing the Lie derivative is feasible and one can use (33) to determine the triggering times. If, however, the dynamics is employed to seek the primal-dual optimizers of an optimization problem, then computing the Lie derivative is infeasible as it requires knowledge of the optimizer. To overcome this limitation, we propose the following alternative triggering criterium

that satisfies (33) as shown later in our convergence analysis:

$$t_{k+1} = t_k + \frac{\tilde{\lambda}_m}{3(M^2 + \|A\|^2)\xi(x(t_k), z(t_k))} \quad (34)$$

where  $\tilde{\lambda}_m = \lambda_m \min\{1, \lambda_s(AA^\top)\}$ ,  $\lambda_m$  is given in Lemma A.1, and  $\lambda_s(AA^\top)$  is the smallest nonzero eigenvalue of  $AA^\top$ . In either (33) or (34), the right-hand side depends only on the state  $(x(t_k), z(t_k))$ . These triggering times for the dynamics (31) define a first-order Euler discretization of the saddle-point dynamics with step-size selection based on the current state of the system. It is for this reason that we refer to (31) together with either the triggering criterium (33) or (34) as the *self-triggered saddle-point dynamics*. In integral form, this dynamics results in a discrete-time implementation of (22) given as

$$\begin{bmatrix} x(t_{k+1}) \\ z(t_{k+1}) \end{bmatrix} = \begin{bmatrix} x(t_k) \\ z(t_k) \end{bmatrix} + (t_{k+1} - t_k) X_{\text{sp}}(x(t_k), z(t_k)).$$

Note that this dynamics can also be regarded as a state-dependent switched system with a single continuous mode and a reset map that updates the sampled state at the switching times, cf., [33]. We understand the solution of (31) in the Caratheodory sense (note that this dynamics has a discontinuous right-hand side). The existence of such solutions, possibly defined only on a finite-time interval, is guaranteed from the fact that along any trajectory of the dynamics there are only countable number of discontinuities encountered in the vector field. The next result however shows that solutions of (31) exist over the entire domain  $[0, \infty)$  as the difference between consecutive triggering times of the solution is lower bounded by a positive constant. Also, it establishes the asymptotic convergence of solutions to the set of saddle points.

*Theorem 6.8 (Convergence of the self-triggered saddle-point dynamics):* Let the saddle function  $F$  be of the form (21), with  $A$  having full row rank,  $f$  strongly convex, twice differentiable, and satisfying  $mI \preceq \nabla^2 f(x) \preceq MI$  for all  $x \in \mathbb{R}^n$  and some constants  $0 < m \leq M < \infty$ . Let the map  $x \mapsto \nabla^2 f(x)$  be Lipschitz with some constant  $L > 0$ . Then,  $\text{Saddle}(F)$  is singleton. Let  $\text{Saddle}(F) = \{(x_*, z_*)\}$ . Then, for any initial condition  $(x(0), z(0)) \in \mathbb{R}^n \times \mathbb{R}^m$ , we have

$$\lim_{k \rightarrow \infty} (x(t_k), z(t_k)) = (x_*, z_*)$$

for the solution of the self-triggered saddle-point dynamics, defined by (31) and (34), starting at  $(x(0), z(0))$ . Further, there exists  $\mu_{(x(0), z(0))} > 0$  such that the triggering times of this solution satisfy

$$t_{k+1} - t_k \geq \mu_{(x(0), z(0))} \quad \text{for all } k \in \mathbb{N}.$$

*Proof:* Note that there is a unique equilibrium point to the saddle-point dynamics (22) for  $F$  satisfying the stated hypotheses. Therefore, the set of saddle point is singleton for this  $F$ . Now, given  $(x(0), z(0)) \in \mathbb{R}^n \times \mathbb{R}^m$ , let  $V_3^0 = V_3(x(0), z(0))$  and define

$$G = \max\{\|\nabla_x F(x, z)\| \mid (x, z) \in V_3^{-1}(\leq V_3^0)\},$$

where, we use the notation for the sublevel set of  $V_3$  as

$$V_3^{-1}(\leq \alpha) = \{(x, z) \in \mathbb{R}^n \times \mathbb{R}^m \mid V_3(x, z) \leq \alpha\}$$

for any  $\alpha \geq 0$ . Since  $V_3$  is radially unbounded, the set  $V_3^{-1}(\leq V_3^0)$  is compact and so,  $G$  is well-defined and finite.

If the trajectory of the self-triggered saddle-point dynamics is contained in  $V_3^{-1}(\leq V_3^0)$ , then we can bound the difference between triggering times in the following way. From Proposition A.3 for all  $(x, z) \in V_3^{-1}(\leq V_3^0)$ , we have  $\xi_1(x, z) = M\xi_2 + L\|\nabla_x F(x, z)\| \leq M\xi_2 + LG =: T_1$ . Hence, for all  $(x, z) \in V_3^{-1}(\leq V_3^0)$ , we get

$$\begin{aligned} \xi(x, z) &= \left( \beta_1^2 (\xi_1(x, z))^2 + \|A\|^4 + \|A\|^2 \xi_2^2 + \beta_2^2 \right)^{\frac{1}{2}} \\ &\leq \left( \beta_1^2 (T_1^2 + \|A\|^4 + \|A\|^2 + \xi_2^2) + \beta_2^2 \right)^{\frac{1}{2}} \\ &=: T_2. \end{aligned}$$

Using the above bound in (34), we get for all  $k \in \mathbb{N}$

$$\begin{aligned} t_{k+1} - t_k &= \frac{\tilde{\lambda}_m}{3(M^2 + \|A\|^2)\xi(x(t_k), z(t_k))} \\ &\geq \frac{\tilde{\lambda}_m}{3(M^2 + \|A\|^2)T_2} > 0. \end{aligned}$$

This implies that as long as the trajectory is contained in  $V_3^{-1}(\leq V_3^0)$ , the intertrigger times are lower bounded by a positive quantity. Our next step is to show that the trajectory is contained in  $V_3^{-1}(\leq V_3^0)$ . Note that if (33) is satisfied for the triggering condition (34), then the sequence  $\{V_3(x(t_k), z(t_k))\}_{k \in \mathbb{N}}$  is strictly decreasing. Since  $V_3$  is non-negative, this implies that  $\lim_{k \rightarrow \infty} V_3(x(t_k), z(t_k)) = 0$  and so, by continuity,  $\lim_{k \rightarrow \infty} (x(t_k), z(t_k)) = (x_*, z_*)$ . Thus, it remains to show that (34) implies (33). To this end, first note the following inequalities shown in the proof of Theorem 6.1

$$\frac{\|X_{\text{sp}}(x, z)\|^2}{3(M^2 + \|A\|^2)} \leq \|(x - x_*); (z - z_*)\|^2, \quad (35a)$$

$$|\mathcal{L}_{X_{\text{sp}}} V_3(x, z)| \geq \tilde{\lambda}_m \|(x - x_*); (z - z_*)\|^2. \quad (35b)$$

Using these bounds, we get from (34)

$$\begin{aligned} &t_{k+1} - t_k \\ &= \frac{\tilde{\lambda}_m}{3(M^2 + \|A\|^2)\xi(x(t_k), z(t_k))} \\ &\stackrel{(a)}{=} \frac{\tilde{\lambda}_m \|X_{\text{sp}}(x(t_k), z(t_k))\|^2}{3(M^2 + \|A\|^2)\xi(x(t_k), z(t_k))\|X_{\text{sp}}(x(t_k), z(t_k))\|^2} \\ &\stackrel{(b)}{\leq} \frac{\tilde{\lambda}_m \|(x(t_k) - x_*); (z(t_k) - z_*)\|^2}{\xi(x(t_k), z(t_k))\|X_{\text{sp}}(x(t_k), z(t_k))\|^2} \\ &\stackrel{(c)}{\leq} \frac{|\mathcal{L}_{X_{\text{sp}}} V_3(x(t_k), z(t_k))|}{\xi(x(t_k), z(t_k))\|X_{\text{sp}}(x(t_k), z(t_k))\|^2} \\ &= \frac{\mathcal{L}_{X_{\text{sp}}} V_3(x(t_k), z(t_k))}{\xi(x(t_k), z(t_k))\|X_{\text{sp}}(x(t_k), z(t_k))\|^2}, \end{aligned}$$

where (a) is valid as  $\|X_{\text{sp}}(x(t_k), z(t_k))\| \neq 0$ , (b) follows from (35a), and (c) follows from (35b). Thus, (34) implies (33), which completes the proof.  $\blacksquare$

Note from the above proof that the convergence implication of Theorem 6.8 is also valid when the triggering criterium is given by (33) with the inequality replaced by the equality.

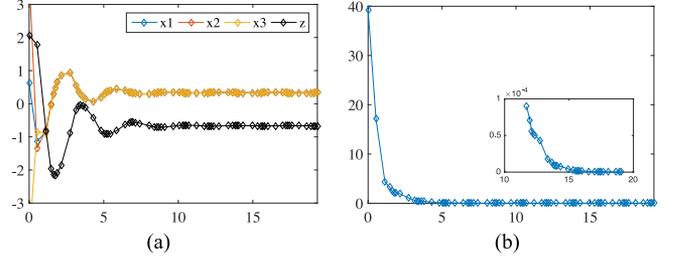


Fig. 3. Illustration of the self-triggered saddle-point dynamics defined by (31) with the triggering criterium (33). Panel (a) and (b) show the evolution of  $(x, z)$  and  $V_3$ , respectively. The saddle function  $F$  is defined in (36). With respect to the notation of Theorem 6.8, we have  $m = M = 2$  and  $\|A\| = \sqrt{3}$ . We select  $\beta_1 = 0.1$ , then  $\beta_2 = 1.6$ , and from (A.39),  $\xi_1 = 2$ . These constants define functions  $V_3$  [cf., (24)],  $\xi$ , and  $\xi_2$  [cf., (A.39)] and also, the triggering times (34). In plot (a), the initial condition is  $x(0) = (0.6210, 3.9201, -4.0817)$ ,  $z(0) = 2.0675$ . The trajectory converges to the unique saddle point and the intertrigger times are lower bounded by a positive quantity.

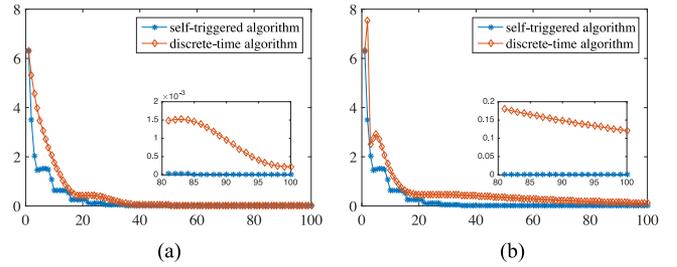


Fig. 4. Comparison between the self-triggered saddle-point dynamics and a first-order Euler discretization of the saddle-point dynamics with two different stepsize rules. The initial condition and implementation details are the same as in Fig. 3. Both plots show the evolution of the distance to the saddle point, compared in (a) against a constant-stepsize implementation with value 0.1 and in (b) against a decaying-stepsize implementation with value  $1/k$  at the  $k$ th iteration. The self-triggered dynamics converges faster in both cases.

*Example 6.9 (Self-triggered saddle-point dynamics):* Consider the function  $F : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$ ,

$$F(x, z) = \|x\|^2 + z(x_1 + x_2 + x_3 - 1). \quad (36)$$

Then, with the notation of (21), we have  $f(x) = \|x\|^2$ ,  $A = [1, 1, 1]$ , and  $b = 1$ . The set of saddle points is a singleton,  $\text{Saddle}(F) = \{((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), -\frac{2}{3})\}$ . Note that  $\nabla^2 f(x) = 2I$  and  $A$  has full row-rank, thus, the hypotheses of Theorem 6.8 are met. Hence, for this  $F$ , the self-triggered saddle-point dynamics (31) with triggering times (34) converges asymptotically to the saddle point of  $F$ . Moreover, the difference between two consecutive triggering times is lower bounded by a finite quantity. Fig. 3 illustrates a simulation of dynamics (31) with triggering criteria (33) (replacing inequality with equality), showing that this triggering criteria also ensures convergence as commented previously. Finally, Fig. 4 compares the self-triggered implementation of the saddle-point dynamics with a constant-stepsize and a decaying-stepsize first-order Euler discretization. In both cases, the self-triggered dynamics achieves convergence faster,

and this may be attributed to the fact that it tunes the stepsize in a state-dependent way. •

## VII. CONCLUSION

This paper has studied the global convergence and robustness properties of the projected saddle-point dynamics. We have provided a characterization of the omega-limit set in terms of the Hessian blocks of the saddle function. Building on this result, we have established global asymptotic convergence assuming only local strong convexity–concavity of the saddle function. For the case when this strong convexity–concavity property is global, we have identified a Lyapunov function for the dynamics. In addition, when the saddle function takes the form of a Lagrangian of an equality constrained optimization problem, we have established the ISS of the saddle-point dynamics by identifying an ISS Lyapunov function, which we have used to design a self-triggered discrete-time implementation. In the future, we aim to generalize the ISS results to more general classes of saddle functions. In particular, we wish to define a “semiglobal” ISS property that we conjecture will hold for the saddle-point dynamics when we relax the global upper bound on the Hessian block of the saddle function. Further, to extend the ISS results to the projected saddle-point dynamics, we plan to develop the theory of ISS for general projected dynamical systems. Finally, we intend to apply these theoretical guarantees to determine robustness margins and design opportunistic state-triggered implementations for frequency regulation controllers in power networks.

## APPENDIX

Here, we collect a couple of auxiliary results used in the proof of Theorem 6.1.

*Lemma A.1 (Auxiliary result for Theorem 6.1: I):* Let  $B_1, B_2 \in \mathbb{R}^{n \times n}$  be symmetric matrices satisfying  $mI \preceq B_1, B_2 \preceq MI$  for some  $0 < m \leq M < \infty$ . Let  $\beta_1 > 0$ ,  $\beta_2 = \frac{4\beta_1 M^4}{m^2}$ , and  $\lambda_m = \min\{\frac{1}{2}\beta_1 m, \beta_1 m^3\}$ . Then

$$W := \begin{bmatrix} \beta_1 B_1 B_2 B_1 + \beta_2 B_1 & \beta_1 B_1 B_2 \\ \beta_1 B_2 B_1 & \beta_1 B_2 \end{bmatrix} \succ \lambda_m I.$$

*Proof:* Reasoning with the Schur complement [21, Sec. A.5.5], the expression  $W - \lambda_m I \succ 0$  holds if and only if the following hold:

$$\begin{aligned} & \beta_1 B_1 B_2 B_1 + \beta_2 B_1 - \lambda_m I \succ 0, \\ & \beta_1 B_2 - \lambda_m I \\ & - \beta_1 B_2 B_1 (\beta_1 B_1 B_2 B_1 + \beta_2 B_1 - \lambda_m I)^{-1} \beta_1 B_1 B_2 \succ 0. \end{aligned} \quad (\text{A.37})$$

The first of the above inequalities is true since  $\beta_1 B_1 B_2 B_1 + \beta_2 B_1 - \lambda_m I \succeq \beta_1 m^3 I + \beta_2 m I - \lambda_m I \succ 0$  as  $\lambda_m \leq \beta_1 m^3$ . For the second inequality, note that

$$\begin{aligned} & \beta_1 B_2 - \lambda_m I \\ & - \beta_1 B_2 B_1 (\beta_1 B_1 B_2 B_1 + \beta_2 B_1 - \lambda_m I)^{-1} \beta_1 B_1 B_2 \\ & \succeq (\beta_1 m - \lambda_m) I \\ & - \beta_1^2 M^4 \lambda_{\max} \left( (\beta_1 B_1 B_2 B_1 + \beta_2 B_1 - \lambda_m I)^{-1} \right) I \\ & \succeq \left( \frac{1}{2} \beta_1 m - \frac{\beta_1^2 M^4}{\lambda_{\min}(\beta_1 B_1 B_2 B_1 + \beta_2 B_1 - \lambda_m I)} \right) I, \end{aligned}$$

where in the last inequality, we have used the fact that  $\lambda_m \leq \beta_1 m/2$ . Note that  $\lambda_{\min}(\beta_1 B_1 B_2 B_1 + \beta_2 B_1 - \lambda_m I) \geq \beta_1 m^3 + \beta_2 m - \lambda_m \geq \beta_2 m$ . Using this lower bound, the following holds:

$$\begin{aligned} & \frac{1}{2} \beta_1 m - \frac{\beta_1^2 M^4}{\lambda_{\min}(\beta_1 B_1 B_2 B_1 + \beta_2 B_1 - \lambda_m I)} \\ & \geq \frac{1}{2} \beta_1 m - \frac{\beta_1^2 M^4}{\beta_2 m} = \frac{1}{4} \beta_1 m. \end{aligned}$$

The above set of inequalities show that the second inequality in (A.37) holds, which concludes the proof. ■

*Lemma A.2 (Auxiliary result for Theorem 6.1: II):* Let  $F$  be of the form (21) with  $f$  strongly convex. Let  $(x, z) \in \mathbb{R}^n \times \mathbb{R}^m$  and  $(x_*, z_*) = \text{projSaddle}(F)(x, z)$ . Then,  $z - z_*$  is orthogonal to the kernel of  $A^\top$ , and

$$\|A^\top(z - z_*)\|^2 \geq \lambda_s(AA^\top)\|z - z_*\|^2$$

where  $\lambda_s(AA^\top)$  is the smallest nonzero eigenvalue of  $AA^\top$ .

*Proof:* Our first step is to show that there exists  $x_* \in \mathbb{R}^n$  such that if  $(x, z) \in \text{Saddle}(F)$ , then  $x = x_*$ . By contradiction, assume that  $(x_1, z_1), (x_2, z_2) \in \text{Saddle}(F)$  and  $x_1 \neq x_2$ . The saddle point property at  $(x_1, z_1)$  and  $(x_2, z_2)$  yields

$$F(x_1, z_1) \leq F(x_2, z_1) \leq F(x_2, z_2) \leq F(x_1, z_2) \leq F(x_1, z_1).$$

This implies that  $F(x_1, z_1) = F(x_2, z_1)$ , which is a contradiction as  $x \mapsto F(x, z_1)$  is strongly convex and  $x_1$  is a minimizer of this map. Therefore,  $\text{Saddle}(F) = \{x_*\} \times \mathcal{Z}$ ,  $\mathcal{Z} \subset \mathbb{R}^m$ . Further, recall that the set of saddle points of  $F$  are the set of equilibrium points of the saddle point dynamics (22). Hence,  $(x_*, z) \in \text{Saddle}(F)$  if and only if

$$\nabla f(x_*) + A^\top z = 0.$$

We conclude from this that

$$\mathcal{Z} = -(A^\top)^\dagger \nabla f(x_*) + \ker(A^\top) \quad (\text{A.38})$$

where  $(A^\top)^\dagger$  and  $\ker(A^\top)$  are the Moore–Penrose pseudoinverse [21, Sec. A.5.4] and the kernel of  $A^\top$ , respectively. By definition of the projection operator, if  $(x_*, z_*) = \text{projSaddle}(F)(x, z)$ , then  $z_* = \text{proj}_{\mathcal{Z}}(z)$  and so, from (A.38), we deduce that  $(z - z_*)^\top v = 0$  for all  $v \in \ker(A^\top)$ . Using this fact, we conclude the proof by writing

$$\begin{aligned} \|A^\top(z - z_*)\|^2 &= (z - z_*)^\top AA^\top(z - z_*) \\ &\geq \lambda_s(AA^\top)\|z - z_*\|^2, \end{aligned}$$

where the inequality follows by writing the eigenvalue decomposition of  $AA^\top$ , expanding the quadratic expression in  $(z - z_*)$ , and lower bounding the terms. ■

**Proposition A.3 (Gradient of  $V_3$  is locally Lipschitz):** Let the saddle function  $F$  be of the form (21), with  $f$  twice differentiable, map  $x \mapsto \nabla^2 f(x)$  Lipschitz with some constant  $L > 0$ , and  $mI \preceq \nabla^2 f(x) \preceq MI$  for all  $x \in \mathbb{R}^n$  and some constants  $0 < m \leq M < \infty$ . Then, for  $V_3$  given in (24), the following holds:

$$\|\nabla V_3(x_2, z_2) - \nabla V_3(x_1, z_1)\| \leq \xi(x_1, z_1) \|x_2 - x_1; z_2 - z_1\|$$

for all  $(x_1, z_1), (x_2, z_2) \in \mathbb{R}^n \times \mathbb{R}^m$ , where

$$\begin{aligned} \xi(x_1, z_1) &= \sqrt{3} \left( \beta_1^2 (\xi_1(x_1, z_1))^2 + \|A\|^4 + \|A\|^2 \xi_2^2 + \beta_2^2 \right)^{\frac{1}{2}}, \\ \xi_1(x_1, z_1) &= M \xi_2 + L \|\nabla_x F(x_1, z_1)\|, \\ \xi_2 &= \max\{M, \|A\|\}. \end{aligned} \quad (\text{A.39})$$

*Proof:* For the map  $(x, z) \mapsto \nabla_x F(x, z)$ , note that

$$\begin{aligned} &\|\nabla_x F(x_2, z_2) - \nabla_x F(x_1, z_1)\| \\ &= \left\| \int_0^1 \nabla_{xx} F(x(s), z(s))(x_2 - x_1) ds \right. \\ &\quad \left. + \int_0^1 \nabla_{zx} F(x(s), z(s))(z_2 - z_1) \right\| \\ &\leq M \|x_2 - x_1\| + \|A\| \|z_2 - z_1\| \\ &\leq \xi_2 \|x_2 - x_1; z_2 - z_1\|, \end{aligned} \quad (\text{A.40})$$

where  $x(s) = x_1 + s(x_2 - x_1)$ ,  $z(s) = z_1 + s(z_2 - z_1)$ , and  $\xi_2 = \max\{M, \|A\|\}$ . In the above inequalities, we have used the fact that  $\|\nabla_{xx} F(x, z)\| = \|\nabla^2 f(x)\| \leq M$  for any  $(x, z)$ . Further, the following Lipschitz condition holds by assumption

$$\|\nabla_{xx} F(x_2, z_2) - \nabla_{xx} F(x_1, z_1)\| \leq L \|x_2 - x_1\|. \quad (\text{A.41})$$

Using (A.40) and (A.41), we get

$$\begin{aligned} &\|\nabla_{xx} F(x_2, z_2) \nabla_x F(x_2, z_2) - \nabla_{xx} F(x_1, z_1) \nabla_x F(x_1, z_1)\| \\ &\leq \|\nabla_{xx} F(x_2, z_2) (\nabla_x F(x_2, z_2) - \nabla_x F(x_1, z_1))\| \\ &\quad + \|(\nabla_{xx} F(x_2, z_2) - \nabla_{xx} F(x_1, z_1)) \nabla_x F(x_1, z_1)\| \\ &\leq \xi_1(x_1, z_1) \|x_2 - x_1; z_2 - z_1\|, \end{aligned} \quad (\text{A.42})$$

where  $\xi_1(x_1, z_1) = M \xi_2 + L \|\nabla_x F(x_1, z_1)\|$ . Also

$$\begin{aligned} &\|\nabla_z F(x_2, z_2) - \nabla_z F(x_1, z_1)\| = \|A(x_2 - x_1)\| \\ &\leq \|A\| \|x_2 - x_1; z_2 - z_1\|. \end{aligned} \quad (\text{A.43})$$

Now note that

$$\begin{aligned} \nabla_x V_3(x, z) &= \beta_1 \left( \nabla_{xx} F(x, z) \nabla_x F(x, z) + A^\top \nabla_z F(x, z) \right) \\ &\quad + \beta_2 (x - x_*), \end{aligned}$$

$$\nabla_z V_3(x, z) = \beta_1 A \nabla_x F(x, z) + \beta_2 (z - z_*).$$

Finally, using (A.40), (A.42), and (A.43), we get

$$\begin{aligned} &\|\nabla V_3(x_2, z_2) - \nabla V_3(x_1, z_1)\|^2 = \|\nabla_x V_3(x_2, z_2) \\ &\quad - \nabla_x V_3(x_1, z_1)\|^2 + \|\nabla_z V_3(x_2, z_2) - \nabla_z V_3(x_1, z_1)\|^2 \\ &\stackrel{(a)}{\leq} 3\beta_1^2 \|\nabla_{xx} F(x_2, z_2) \nabla_x F(x_2, z_2) \\ &\quad - \nabla_{xx} F(x_1, z_1) \nabla_x F(x_1, z_1)\|^2 \\ &\quad + 3\beta_1^2 \|A^\top (\nabla_z F(x_2, z_2) - \nabla_z F(x_1, z_1))\|^2 + 3\beta_2^2 \|x_2 - x_1\|^2 \\ &\quad + 3\beta_1^2 \|A (\nabla_x F(x_2, z_2) - \nabla_x F(x_1, z_1))\|^2 + 3\beta_2^2 \|z_2 - z_1\|^2 \\ &\leq \xi(x_1, z_1)^2 \|x_2 - x_1; z_2 - z_1\|^2, \end{aligned}$$

where in (a), we have used the inequality  $(a + b)^2 \leq 3(a^2 + b^2)$  for any  $a, b \in \mathbb{R}$ . This concludes the proof. ■

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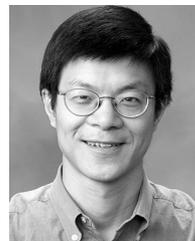
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