Evaluating Robustness of Consensus Algorithms Under Measurement Error over Digraphs

Chengda Ji, Enrique Mallada and Dennice F. Gayme

***Abstract*— Consensus algorithms constitute a powerful tool for computing average values or coordinating agents in many distributed applications. Unfortunately, the same property that allows this computation (i.e., the nontrivial nullspace of the state matrix) leads to unbounded state variance in the presence of measurement errors. In this work, we explore the trade-off between relative and absolute communication (feedback) in the presence of measurement errors. We evaluate the robustness of first and second order integrator systems under a parameterized family of controllers (homotopy), that continuously trade between relative and absolute feedback interconnections, in terms of the $\mathcal{H}_2$ norm of an appropriately defined input-output system. Our approach extends the previous $\mathcal{H}_2$ norm based analysis to systems with directed feedback interconnections whose underlying weighted graph Laplacians are diagonalizable. Our results indicate that any level of absolute communication is sufficient to achieve a finite $\mathcal{H}_2$ norm, but purely relative feedback can only achieve finite norms when the measurement error is not exciting the subspace associated with the consensus state. Numerical examples demonstrate that smoothly reducing the proportion of absolute feedback in double integrator systems smoothly decreases the system performance (increases the $\mathcal{H}_2$ norm) and that this performance degradation is more rapid in systems with relative feedback in only the first state (position).**

I. INTRODUCTION

Consensus problems arise in a wide range of applications including coordination of vehicular or robotic networks [1], [2], synchronization in power systems [3]–[6] and biological networks [7], as well as clock synchronization in computer networks [8]. There have been a number of algorithms proposed to attain consensus, most typically employ local or distributed information sharing to compute an average value or coordinated state, see e.g [1], [2]. The majority of these algorithms exploit the structure of an underlying Laplacian matrix, which provides the system with a nontrivial nullspace in the system matrix that corresponds to the state average [9].

However, in many applications where consensus algorithms are used, agents are required to take measurements, which are inherently imperfect, and thus subject to stochastic disturbances. As a result of these disturbances, consensus algorithms typically result in agents (nodes) fluctuating around the equilibrium state rather than converging to a single value. Interestingly, the same feature that allows consensus algorithms to work (the nontrivial nullspace of the state operator) is the same source of this undesired behavior that leads to poor system robustness to perturbations.

Some works overcome this limitation by employing a common reference value or providing some sort of 'absolute' state information at each node. For example, the authors of [10] show that for vehicular networks under distributed disturbances, at least one vehicle must have access to its global position (absolute state) to achieve closed-loop stability. The underlying interconnection topology has also been shown to strongly affect achievable performance limits in linear consensus networks [11]. Bamieh et al. [12] demonstrated that for the continuous-time version of the consensus algorithm agents communicating over a 1D lattice can maintain coherence subject to stochastic disturbances. Robustness of leader-follower consensus under measurement error has been shown to depend on both the communication topology and scaling factors [13].

This work explores both of these aspects by studying the trade-offs between relative and absolute communication (feedback) in the presence of measurement errors using a parametrized family of controllers (homotopy) that continuously adjust a convex combination of absolute feedback (global feedback) and relative feedback (local feedback) in the control law. More precisely, we compute the system performance in terms of the input-output $\mathcal{H}_2$ norm of first and second order integrator systems subject to distributed disturbances, see e.g. [12]. We extend previous analysis which assumed symmetric feedback interconnections [14] and specific output structure to systems with directed feedback interconnection. In particular, we analyze systems whose underlying graph structure emits a diagonalizable weighted graph Laplacian. This class of system is slightly more general than systems whose interconnections are described by normal weighted graph Laplacians, which were analyzed in [15], [16].

Using this novel framework for $\mathcal{H}_2$ norm computations, we evaluate the norm of single and double integrator systems connected over strongly connected digraphs with measurement error. The effects of measurement errors have previously been studied in the context of symmetric feedback interconnections [17]. Mallada et al. studied the clock synchronization problem and explored how the graph structure affects the consensus under measurement error [8]. Our results are consistent with previous studies of systems with symmetric feedback which indicated that absolute feedback can improve system robustness, e.g. [7]. In fact, we find that any level of absolute communication is sufficient to achieve

C. Ji and D. F. Gayme are with Dept. of Mechanical Engineering, E. Mallada is with the Dept. of Electrical and Computer Engineering at the Johns Hopkins University, 3400 N. Charles St., Baltimore, MD 21218 USA chengda@jhu.edu, mallada@jhu.edu dennice@jhu.edu Partial support from the NSF through grant numbers CNS 1544771 (D.F.G, E.M and C.J) and CAREER 1752362 (E.M), and the ARO through contract W911NF-17-1-0092 (E.M) is gratefully acknowledged

finite $\mathcal{H}_2$ norm. In contrast, purely relative feedback can only achieve finite norms when the measurement error is not exciting the unobservable subspace associated with the consensus state.

We present numerical examples exploring the system performance as we trade between relative and absolute feedback for second order systems. The results demonstrate that smoothly reducing the proportion of absolute feedback smoothly decreases the system performance and that this performance degradation is more rapid in a system with relative feedback only in the first order state (position).

This paper is organized as follows. In Section II, we define the notation and provide mathematical preliminaries related to our analysis. In Section III, we introduce the dynamics of the first and second order integrator systems subject to measurement errors. In Section IV, we present our main results, which utilize a novel framework to compute the $\mathcal{H}_2$ norm for systems whose communication graphs are represented by diagonalizable weighted graph Laplacian matrices. Section V provides a numerical study investigating the trade-off between absolute and relative feedback in each of the states. Section VI concludes the paper.

II. Preliminaries

For a complex number $x = a + bi$, $Re(x) = a$ is the real part of x and $Im(x) = b$ is the imaginary part of x. $\bar{x} = a - bi$ denotes the conjugate of x. Given a set of complex numbers $S = \{s_1 \; s_2 \; \cdots \; s_n\}$, $diag(S) \in \mathbb{C}^{n\times n}$ denotes a diagonal matrix with the ordered elements of S along its main diagonal. $\mathbf{0}_{n\times n} \in \mathbb{R}^{n\times n}$ denotes an matrix with all elements equal to zero, and $I_{n\times n} \in \mathbb{R}^{n\times n}$ indicates an $n \times n$ identity matrix. $\mathbf{1}_n \in \mathbb{R}^n$ is a column vector with all elements equal to 1, and $\mathbf{0}_n \in \mathbb{R}^n$ is a column vector of zeros.

Given a matrix $A \in \mathbb{C}^{n\times n}$, A^{-1} is the inverse of A, i.e., $A^{-1}A = AA^{-1} = I_{n\times n}$. $\bar{A}$ and A^T respectively denote the conjugate and transpose of A. A^* denotes the conjugate transpose of A. $\mathrm{tr}(A)$ denotes the trace of A. $[A]_{pq}$ denotes the element in the p^{th} row and q^{th} column of A.

A *weighted digraph* is a triplet $\mathcal{G} = (N, \mathcal{E}, W)$, where N is the set of nodes, and $\mathcal{E}$ is a set of ordered pairs (i, j) of nodes $i, j \in N$ called edges. W is a set of nonnegative weights $w_{(i,j)}$ associated with each ordered node pair (i, j). When $(i, j) \in \mathcal{E}$ then $w_{(i,j)} > 0$ is the associated edge weight, $w_{(i,j)} = 0$ for node pairs $(i, j) \notin \mathcal{E}$. A directed path is an ordered sequence of nodes $i \in N$ such that any pair of consecutive nodes in this sequence is an edge of the graph. A directed graph $\mathcal{G}$ is called *strongly connected* if there exists directed path from any node to any other node.

Given a weighted digraph $\mathcal{G} = (N, \mathcal{E}, W)$, with $N = \{1, 2, \cdots, n\}$, the elements of its weighted Laplacian matrix, L, are defined as

$$[L]_{ij} = \begin{cases} -w_{(i,j)}, & if \; i \neq j \\ \sum_{h=1,h\neq i}^{n} w_{(i,h)} & otherwise. \end{cases}$$

Given such a Laplacian matrix, if we denote the i^{th} eignevalue of L as λ_i and sort the eigenvalues as $Re(\lambda_1) \leq Re(\lambda_2) \leq \cdots \leq Re(\lambda_n)$, then $\lambda_1 = 0$. Given the associated matrix of eigenvectors $T = \begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix}$ where t_i is the eigenvector associated with λ_i, $t_1 = \frac{1}{n}\mathbf{1}_n \in \mathbb{R}^n$. The following remark comes from [9].

Remark 1: The weighted graph Laplacian matrix associated with strongly connected directed graph has only one zero eigenvalue.

III. Problem setup

In this section, we first introduce the dynamics for first and second-order systems subject to distributed state measurement errors. These systems have feedback interconnections defined over directed graphs $\mathcal{G} = (N, \mathcal{E}, W)$ wherein a nonzero edge weight $w_{(i,j)}$ indicates that node i passes information to node j. We thus refer to the graphs describing the feedback interconnection structures as communication graphs.

We consider linear systems, G, of the form

$$\dot{z} = Az + Bw \qquad (1a)$$
$$y = Cz, \qquad (1b)$$

where $z \in \mathbb{R}^n$ is the system state variable, $w \in \mathbb{R}^m$ is the disturbance and $y \in \mathbb{R}^p$ is the output. We next define the A, B and C matrices for the types of first and second order systems analyzed in this work.

A. First order systems

Given a set of coupled first order systems connected over a weighted digraph $\mathcal{G} = (N, \mathcal{E}, W)$, each node $i \in N$ has the following dynamics:

$$\dot{z}_i = u_i, \qquad (2)$$

where

$$u_i = -\alpha \sum_{(i,j)\in\mathcal{E}} a_{ij}(\hat{z}_i - \hat{z}_j) - (1-\alpha)\hat{z}_i \qquad (3)$$

is the control input at node $i \in N$ defined in terms of a weighting parameter $\alpha \in (0, 1]$. The state measurement at node $i \in N$ is given by

$$\hat{z}_i = z_i + e_i, \qquad (4)$$

where z_i is the actual value of the state and e_i is measurement error. Substituting (4) into (3) leads to

$$\begin{aligned} \dot{z}_i = &-\alpha \sum_{(i,j)\in\mathcal{E}} a_{ij}(z_i - z_j) - (1-\alpha)z_i \\ &-\alpha \sum_{(i,j)\in\mathcal{E}} a_{ij}(e_i - e_j) - (1-\alpha)e_i, \end{aligned} \qquad (5)$$

which can be rewritten in the matrix form of (1) as

$$\dot{z} = -(\alpha L + (1-\alpha))z - (\alpha L + (1-\alpha))e \qquad (6a)$$
$$y = Cz. \qquad (6b)$$

B. Second order systems

Given a set of coupled second order systems, the dynamics at each node $i \in N$ are given by

$$\ddot{z}_i = u_i, \tag{7}$$

where $z = x$ and $\dot{z} = v$ are the states of the system whose measurements are defined analogously to (4) as

$$\hat{z}_i = \hat{x}_i = x_i + e_i^x \tag{8a}$$
$$\dot{\hat{z}}_i = \hat{v}_i = v_i + e_i^v. \tag{8b}$$

The control input is given by

$$u_i = -\alpha \sum_{(i,j)\in\mathcal{E}_\alpha} a_{ij}(\hat{x}_i - \hat{x}_j) - (1-\alpha)a\hat{x}_i - \beta \sum_{(i,j)\in\mathcal{E}_\beta} b_{ij}(\hat{v}_i - \hat{v}_j) - (1-\beta)b\hat{v}_i. \tag{9}$$

This control structure leads to two different communication graphs; $\mathcal{G}_\alpha = (N, \mathcal{E}_\alpha, W_\alpha)$ with $w_{\alpha_{(i,j)}} = a_{ij}$ for $(i,j) \in \mathcal{E}_\alpha$, and $\mathcal{G}_\beta(N, \mathcal{E}_\beta, W_\beta)$ with $w_{\beta_{(i,j)}} = b_{ij}$ for $(i,j) \in \mathcal{E}_\alpha$, which are respectively associated with the states $x = [x_1, \ldots, x_n]^T$ and $v = [v_1, \ldots, v_n]^T$. When both $\alpha, \beta \neq 0$ the two communication graphs respectively correspond to relative position and relative velocity feedback, whereas nonzero coefficients $(1-\alpha)$ and $(1-\beta)$ respectively correspond to absolute position and velocity feedback.

Rewriting the system (7) in terms of the state and measurement errors leads to

$$\begin{aligned}\ddot{z}_i = &-\alpha \sum_{(i,j)\in\mathcal{E}_\alpha} a_{ij}(x_i - x_j) - (1-\alpha)ax_i \\ &- \beta \sum_{(i,j)\in\mathcal{E}_\beta} b_{ij}(v_i - v_j) - (1-\beta)bv_i \\ &- \alpha \sum_{(i,j)\in\mathcal{E}_\alpha} a_{ij}(e_i^x - e_j^x) - (1-\alpha)ae_i^x \\ &- \beta \sum_{(i,j)\in\mathcal{E}_\beta} b_{ij}(e_i^v - e_j^v) - (1-\beta)be_i^v.\end{aligned}$$

The corresponding state space form is

$$\begin{bmatrix}\dot{x}\\ \dot{v}\end{bmatrix} = \begin{bmatrix}\mathbf{0}_{n\times n} & I_{n\times n}\\ -(\alpha L_a + (1-\alpha)aI) & -(\beta L_b + (1-\beta)bI)\end{bmatrix}\begin{bmatrix}x\\ v\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{n\times n} & \mathbf{0}_{n\times n}\\ -(\alpha L_a + (1-\alpha)aI) & -(\beta L_b + (1-\beta)bI)\end{bmatrix}\begin{bmatrix}e^x\\ e^v\end{bmatrix}, \tag{10a}$$

$$y = C\begin{bmatrix}x\\ v\end{bmatrix}, \tag{10b}$$

where α, $\beta \in [0,1]$, $a, b > 0$ and$C = [H\ \mathbf{0}_{p\times n}]$.

In this work we focus on two control schemes: (1) **Position and velocity control (PV)** control, where we set $a = b = 1$ so that there is both relative and absolute feedback for x and v, and (2) **Position and absolute velocity control (PAV)**, with $a = 1$, $\beta = 0$, which leads to relative feedback in position, x and absolute feedback in both x and velocity, v. These strategies are closely related to the relative position and absolute velocity control (RPAV) and relative position relative velocity (RPRV) control strategies introduced in [12]. More precisely, RPAV control corresponds to our PAV strategy with $\alpha = 1$, and RPRV corresponds to our PV strategy with both $\alpha = 1$ and $\beta = 1$.

IV. $\mathcal{H}_2$ NORM COMPUTATIONS

In this section, we provide a novel framework to compute the $\mathcal{H}_2$ norm for first and second order systems with measurement errors whose communication graphs are represented by diagonalizable weighted graph Laplacian matrices.

We invoke the following interpretation of the $\mathcal{H}_2$ norm of a linear system G of the form (1) as the sum of responses to impulses at all inputs, see e.g. [18]. This quantity can computed as

$$\|G\|_{H_2}^2 = \operatorname{tr}\left(\int_0^\infty \hat{g}^*(t)\hat{g}(t)dt\right), \tag{11}$$

where $\hat{g}(t) = Ce^{At}B$, $\hat{g}(t) \in \mathbb{R}^{m\times p}$ is the impulse response of system G and therefore

$$\|G\|_{H_2}^2 = \operatorname{tr}\left(B^* \int_0^\infty e^{A^*t}C^*Ce^{At}dtB\right). \tag{12}$$

The following proposition specializes (12) to a system with a diagonalizable state matrix.

Proposition 4.1: Given a linear system G of the form (1). If the state matrix A is diagonalizable and $A = T\Lambda T^{-1}$, then the $\mathcal{H}_2$ norm of this system is,

$$\|G\|_{H_2}^2 = \operatorname{tr}\left((T^{-1}B)^*\hat{X}T^{-1}B\right), \tag{13}$$

where $\hat{X} = \int_0^\infty e^{\Lambda^*t}(CT)^*CTe^{\Lambda t}dt$. In particular

$$\left[e^{\Lambda^*t}(CT)^*CTe^{\Lambda t}\right]_{ls} = e^{(\bar{\lambda}_l + \lambda_s)t}\left[(CT)^*CT\right]_{ls}. \tag{14}$$

Proof: Since A is diagonalizable it can be written as $A = T\Lambda T^{-1}$. Then directly applying (12), we have

$$\begin{aligned}\|G\|_{H_2}^2 &= \operatorname{tr}\left(B^*\int_0^\infty e^{A^*t}C^*Ce^{At}dtB\right) \\ &= \operatorname{tr}\left((T^{-1}B)^*\int_0^\infty e^{\Lambda^*t}(CT)^*(CT)e^{\Lambda t}dtT^{-1}B\right).\end{aligned}$$

Since $e^{\Lambda t}$ is a diagonal matrix, (14) can be proved by matrix multiplication. ■

A. First order systems

We now present the main results for first order systems beginning with a discussion of the conditions under which the state matrix is diagonalizable.

Lemma 4.2: If a Laplacian matrix $L = T\Lambda T^{-1}$ is diagonalizable, where $[\Lambda]_{ii} = \lambda_i$ and $T = \begin{bmatrix}t_1 & t_2 & \cdots & t_n\end{bmatrix}$, $t_i \in \mathbb{C}^n$ is the eigenvector associated with eigenvalue λ_i. Then $-(\alpha L + (1-\alpha)I_{n\times n}) = T(-\alpha\Lambda - (1-\alpha)I)T^{-1}$ is also diagonalizable. All of the eigenvalues of $-(\alpha L + (1-\alpha)I_{n\times n})$ have non-positive real part if $\alpha \in [0,1]$.

Lemma 4.2 shows that for a first order system of the form (6), the state matrix is diagonalizable if and only if the Laplacian matrix is diagonalizable. Given a fixed communication graph and $\alpha \in [0,1]$, the eigenvectors of the state matrix are exactly the same as those of the weighted Laplacian matrix, L. When $\alpha \in [0,1)$, all of the eigenvalues of $\alpha L + (1-\alpha)I_{n\times n}$ are strictly positive.

Theorem 4.3: Given a linear system (1) with state matrix $A = -(\alpha L + (1-\alpha)I_{n\times n})$. If the system has a directed communication graph associated with a diagonalizable Laplacian matrix L and $0 \le \alpha < 1$, then the $\mathcal{H}_2$ norm can be represented in the form (13) with

$$\left[\hat{X}\right]_{ls} = \frac{[T^*C^*CT]_{ls}}{(\alpha\bar{\lambda}_l + (1-\alpha)) + (\alpha\lambda_s + (1-\alpha))}. \tag{15}$$

Here T is a invertible matrix that diagonalizes L, i.e. $L = T\Lambda T^{-1}$ and $[\Lambda]_{ii} = \lambda_i$ is the i^{th} eigenvalue of L. λ_i is sorted as $0 = \lambda_1 \le Re(\lambda_2) \le \cdots \le Re(\lambda_n)$.

Proof: From lemma 4.2, we find that all of the eigenvalues of the state matrix are strictly negative for $0 \le \alpha < 1$. Applying (14), we obtain

$$\left[\int_0^\infty e^{\Lambda^* t}(CT)^* CT e^{\Lambda t} dt\right]_{ls} = 0 + \frac{[T^*C^*CT]_{ls}}{(\alpha\bar{\lambda}_l + (1-\alpha)) + (\alpha\lambda_s + (1-\alpha))}.$$

■

Note that Theorem 4.3 only applies for the case $0 \le \alpha < 1$ (Hurwitz systems). A notable observation is that as $\alpha \to 1$, the terms in $\hat{X}$ associated with $\lambda_1 = 0$ tend to infinity. This observation indicates that standard consensus algorithms are not in general robust to noise. Theorem 4.3 implies that for general systems, some amount of absolute feedback is needed.

Theorem 4.4: Given a linear system (1) with state matrix as $A = -(\alpha L + (1-\alpha)I_{n\times n})$, $0 \le \alpha \le 1$. If the system has a strongly connected communication graph associated with a diagonalizable Laplacian matrix L and $C\mathbf{1}_n = \mathbf{0}_n \in \mathbb{R}^n$, then the $\mathcal{H}_2$ norm can be represented in the form (13) with

$$\left[\hat{X}\right]_{ls} = \begin{cases} 0, & l \ or \ s = 1 \\ \dfrac{[T^*C^*CT]_{ls}}{(\alpha\bar{\lambda}_l + (1-\alpha)) + (\alpha\lambda_s + (1-\alpha))}, & otherwise. \end{cases}$$

T is a invertible matrix that diagonalizes Laplacian matrix $L = T\Lambda T^{-1}$. $[\Lambda]_{ii} = \lambda_i$ is the i^{th} eigenvalue of L. λ_i is sorted as $0 = \lambda_1 < Re(\lambda_2) \le \cdots \le Re(\lambda_n)$.

Proof: As the communication graph is strongly connected, there is only one zero eigenvalue in Laplacian matrix L. As $-(\alpha L + (1-\alpha)I_{n\times n}) = TDT^{-1}$, $[D]_{ii} = \hat{\lambda}_i$, the eigenvalues can be sorted as $-(1-\alpha) = \hat{\lambda}_1 > Re(\hat{\lambda}_2) \ge \cdots \ge Re(\hat{\lambda}_n)$. For $T = \begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix}$, as t_1 associated with 0 eigenvalue, $t_1 = \frac{1}{n}\mathbf{1}_n$. As $Ct_1 = \mathbf{0}_n$, we have

$$T^*C^*CT = \begin{bmatrix} 0 & \mathbf{0}_{n-1}^T \\ \mathbf{0}_{n-1} & (T^*C^TCT)_* \end{bmatrix}.$$

$(T^*C^TCT)_*$ is the principal submatix of T^*C^*CT. From form (14), we have the first row and first column of $\hat{X}$ as zero vector. Thus even $\alpha = 1$, $\hat{\lambda}_1 = \lambda_1 = 0$, the system is non Hurwitz, the $\mathcal{H}_2$ norm still exists. ■

Theorem 4.4 provide the condition when $\alpha = 1$ the $\mathcal{H}_2$ norm still exist and derive the expression of $\mathcal{H}_2$ norm.

Remark 2: The condition $C\mathbf{1}_n = \mathbf{0}_n$ of Theorem 4.4 means that in the absence of absolute control the system performance is unbounded in any direction, except for the directions orthogonal to vector $\mathbf{1}_n$. Therefore the same property that allows a system to achieve consensus, i.e., a zero eigenvalue on the direction spanned by $\mathbf{1}_n$, makes the system non-robust to noise.

Theorems 4.3 and 4.4 apply to general input and output matrices B and C. The following corollary addresses the special case in (6) where the inputs are defined through measurement errors of the form (4).

Corollary 1: Given a first order system under measurement error of the form (6). If $0 \le \alpha < 1$, then $\mathcal{H}_2$ norm can be represented as

$$\|G\|_{H_2}^2 = \text{tr}\left((T^{-1})^* \hat{X} T^{-1}\right), \tag{16a}$$

$$\left[\hat{X}\right]_{ls} = \frac{(\alpha\bar{\lambda}_l + (1-\alpha))(\alpha\lambda_s + (1-\alpha))\,[T^*C^*CT]_{ls}}{(\alpha\bar{\lambda}_l + (1-\alpha)) + (\alpha\lambda_s + (1-\alpha))}, \tag{16b}$$

where $L = T\Lambda T^{-1}$. The matrix Λ is a diagonal matrix with eigenvalues of L on its main diagonal, $[\Lambda]_{ii} = \lambda_i$, $0 \le Re(\lambda_2) \le \cdots \le Re(\lambda_n)$. If $0 \le \alpha \le 1$, the communication graph is strongly connected, and $C\mathbf{1}_n = \mathbf{0}_n$, the $\mathcal{H}_2$ norm can also be represented in (16) with (16b) replaced by

$$\left[\hat{X}\right]_{ls} = \begin{cases} 0, & l \ or \ s = 1 \\ \dfrac{(\alpha\bar{\lambda}_l + (1-\alpha))(\alpha\lambda_s + (1-\alpha))[T^*C^*CT]_{ls}}{(\alpha\bar{\lambda}_l + (1-\alpha)) + (\alpha\lambda_s + (1-\alpha))}, & otherwise. \end{cases}$$

Proof: The proof of this lemma is based on Theorem 4.3 and 4.4. Notice that with $B = A$, A and B are simultaneously diagonalizable. ■

B. Second order systems

We now present the $\mathcal{H}_2$ norms computations for second order systems with PV and PAV control laws. We first present a method for diagonalizing the system state matrices

Lemma 4.5: Given a matrix

$$A = \begin{bmatrix} 0 & I \\ G & F \end{bmatrix} \in \mathbb{R}^{2n\times 2n}.$$

If G and F are simultaneously diagonalizable and G is not singular, then matrix A is diagonalizable

$$A = \hat{T}\Delta\hat{T}^{-1}. \tag{17}$$

Proof: As G and F are simultaneously diagonalizable and G is not singular, there is no zeros in the eigenvalues of matrix G ($\lambda^G \ne 0$) and there exist T such that $G = T\Lambda^G T^{-1}$ and $F = T\Lambda^F T^{-1}$, where $\left[\Lambda^G\right]_{ii} = \lambda_i^G$ and $\left[\Lambda^F\right]_{ii} = \lambda_i^F$. We can define a invertible block diagonal matrix

$$T_{diag} = \begin{bmatrix} T & \\ & T \end{bmatrix}. \tag{18}$$

We denote a permutation matrix

$$E = \begin{bmatrix} e_1 & e_{n+1} & \cdots & e_i & e_{i+n} & \cdots & e_n & e_{2n} \end{bmatrix}. \quad (19)$$

where $\forall i \leq 2n$, e_i is a column vector with 1 on its i^{th} position and 0s on the other positions. $e_i = \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}^T \in \mathbb{R}^{2n}$. A block diagonal matrix

$$K = \begin{bmatrix} K_1 & & \\ & \ddots & \\ & & K_n \end{bmatrix}, \quad K_i = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda_i^G} \end{bmatrix}. \quad (20)$$

As all of the eigenvalues of K are non-zero, K is invertible. Now let R be another block diagonal matrix

$$R = \begin{bmatrix} R_1 & & \\ & \ddots & \\ & & R_n \end{bmatrix}, \quad (21a)$$

where

$$R_i = \begin{bmatrix} -\frac{\lambda_i^F + \sqrt{(\lambda_i^F)^2 + 4\lambda_i^G}}{2\sqrt{\lambda_i^G}} & -\frac{\lambda_i^F - \sqrt{(\lambda_i^F)^2 + 4\lambda_i^G}}{2\sqrt{\lambda_i^G}} \\ 1 & 1 \end{bmatrix}. \quad (21b)$$

Since all of the eigenvalues of G are non-zero, R exists and is invertible. Then we define diagonal matrix

$$\Delta = \begin{bmatrix} \Delta_1 & & \\ & \ddots & \\ & & \Delta_n \end{bmatrix}, \quad (22a)$$

$$\Delta_i = \begin{bmatrix} \frac{\lambda_i^F - \sqrt{(\lambda_i^F)^2 + 4\lambda_i^G}}{2} & \\ & \frac{\lambda_i^F + \sqrt{(\lambda_i^F)^2 + 4\lambda_i^G}}{2} \end{bmatrix}. \quad (22b)$$

It is easy to verify that $A = \hat{T}\Delta\hat{T}^{-1}$ with $\hat{T} = T_{diag}EKR$. ■

Proposition 4.6: Consider a linear system (1), and

$$A = \begin{bmatrix} 0 & I \\ G & F \end{bmatrix}.$$

If G and F are simultaneous diagonalizable with G and F are Hurwitz, the system is Hurwitz.

Proof: The proof of Proposition 4.6 is based on the proof of Lemma 4.5. The eigenvalues are $\lambda_i(A) = \frac{\lambda_i^F \pm \sqrt{(\lambda_i^F)^2 + 4\lambda_i^G}}{2}$. With $Re(\lambda_i^F) < 0$, $\lambda_i(A) < 0$ if and only if $\lambda_i^G < 0$. If G and F are Hurwitz, all of the eigenvalues of A have negative real part. ■

Based on Lemma 4.5, we can diagonalize the state matrix of second order system with our controller. Then we can use Proposition 4.1 to calculate the $\mathcal{H}_2$ norm.

First we deal with the cases where matrices B and C do not have special structures.

Theorem 4.7: Given a second order system with state matrix

$$A = \begin{bmatrix} \mathbf{0}_{n\times n} & I_{n\times n} \\ -(\alpha L_a + (1-\alpha)aI_{n\times n}) & -(\beta L_b + (1-\beta)bI_{n\times n}) \end{bmatrix}.$$

If the Laplacian matrices L_a and L_b are simultaneously diagonalizable, $\alpha, \beta \in [0, 1)$, and $a, b > 0$, the $\mathcal{H}_2$ norm of such system can be represented as

$$\|G_a\|_{H_2}^2 = \operatorname{tr}\left(P^{-1}T_{diag}^{-1}B)^*\hat{X}(P^{-1}T_{diag}^{-1}B)\right), \quad (23a)$$

where $P = EKR$.

$$[X]_{ls} = \frac{-1}{\bar{\delta}_l + \delta_s}\left[(CT_{diag}P)^* CT_{diag}P\right]_{kn}, \quad (23b)$$

where $T_{diag} = \begin{bmatrix} T & \\ & T \end{bmatrix}$, T is the matrix that diagonalizes $L_a = T\Lambda^a T^{-1}$ and $L_b = T\Lambda^b T^{-1}$, $[\Lambda^a]_{ii} = \lambda_i^a$. E is defined in (19), K, R and Δ are obtained from (20), (21) and (22) respectively, with

$$\lambda_i^G = (-\alpha\lambda_i^a - (1-\alpha)a)$$
$$\lambda_i^F = (-\beta\lambda_i^b - (1-\beta)b),$$

and δ_i is the eigenvalue of A with form

$$\delta_i = \begin{cases} \dfrac{\lambda_i^F - \sqrt{(\lambda_i^F)^2 + 4\lambda_i^G}}{2}, & (i \; mod \; 2) = 1 \\ \dfrac{\lambda_i^F + \sqrt{(\lambda_i^F)^2 + 4\lambda_i^G}}{2}, & otherwise. \end{cases}$$

Proof: The proof of Theorem 4.7 is directly obtained from Proposition 4.1 and Lemma 4.5. ■

Notice that in Theorem 4.7, when $\alpha, \beta \to 1$, $\|G_a\|_{H_2}^2 \to \infty$ as some $\delta_i \to 0$. Similarly to the analysis of the previous section, we now introduce a technique analogous to Theorem 4.4 to deal with the cases when α and β may be equal to one.

Theorem 4.8: Given a second order system with state matrix

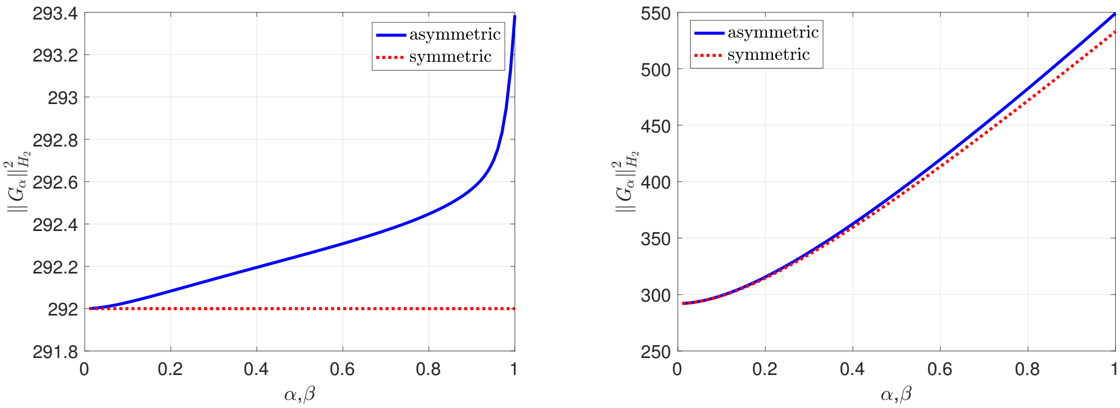
$$A = \begin{bmatrix} \mathbf{0}_{n\times n} & I_{n\times n} \\ -(\alpha L_a + (1-\alpha)aI_{n\times n}) & -(\beta L_b + (1-\beta)bI_{n\times n}) \end{bmatrix}$$

where $\alpha, \beta \in [0, 1]$ and $a, b > 0$. The Laplacian matrices L_a and L_b are simultaneously diagonalizable and the corresponding communication graphs of L_a and L_b are strongly connected, with $L_b = T\Lambda_a T^{-1}$ in which $[\Lambda_b]_{ii} = \lambda_i^b$, λ_i^b are sorted as $0 = \lambda_1^b < Re(\lambda_2^b) \leq \cdots \leq Re(\lambda_n^b)$ and $T = \begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix}$. $C\mathbf{1}_n = \mathbf{0}_n$, the $\mathcal{H}_2$ norm can be represented in (23a) with

$$\left[\hat{X}\right]_{ls} = \begin{cases} 0, & l \; or \; s \leq 2, \\ \dfrac{-\left[(CT_{diag}EFR)^* CT_{diag}EFR\right]_{ls}}{\bar{\delta}_l + \delta_s}, & otherwise, \end{cases}$$

in which T_{diag}, E are the same as they are in Theorem 4.7. For K, R and Δ , they have similar structure as they are in Theorem 4.7 except for the first 2 by 2 diagonal blocks of them.

$$K_1 = I_{2\times 2}, \quad R_1 = I_{2\times 2}, \quad \Delta_1 = \begin{bmatrix} 0 & 1 \\ (\alpha-1)a & (\beta-1)b \end{bmatrix}.$$



(a) PV control laws

(b) PAV control laws

Fig. 1: The performance ($\mathcal{H}_2$ norm) as a function of α and β for the case where $\alpha = \beta$ for the two different second order control laws. Smoothly reducing the proportion of absolute feedback in double integrator systems smoothly decreases the system performance except for systems with symmetric PV control laws. In that case, the absolute feedback and relative feedback equally contribute to the performance, and increasing the proportion of absolute feedback does not change the performance.

Proof: It is easy to show that $A = T_{diag}EKR\Delta(T_{diag}EKR)^{-*}$. Notice that different from Therorem 4.7, in this case, the first 2 by 2 block of Δ is not diagonal matrix. As

$$CT_{diag}E = \begin{bmatrix}\mathbf{0}_n & \mathbf{0}_n & CT_{diag}E^*\end{bmatrix}.$$

and K, R are all block diagonal matrix with block size 2,

$$\begin{aligned}(CT_{diag}P)^*(CT_{diag}P) &= (CT_{diag}EKR)^*(CT_{diag}EKR)\\ &= \begin{bmatrix} 0 & 0 & \mathbf{0}_{2n-2}^T \\ 0 & 0 & \mathbf{0}_{2n-2}^T \\ \mathbf{0}_{2n-2} & \mathbf{0}_{2n-2} & \mathbf{Y}\end{bmatrix} \quad (24)\end{aligned}$$

with first two rows and columns are all zeros and $\mathbf{Y}$ is a principal submatrix of $(CT_{diag}P)^*(CT_{diag}P)$. ■

Remark 3: Under the condition of Theorem 4.8 (the corresponding communication graph of L_b is strongly connected), if the system is under PV control ($a = b = 1$) and both $\alpha = \beta = 1$, the first two by two block of Δ is acturally a Jordan block associated with zero eigenvalues. which means the system at most has two zero eigenvalues. If the system is under PAV control ($a = 1$, $\beta = 0$), however, the system at most has one zero eigenvalue when $\alpha = 1$.

Now we focus on the special case for a linear system (10) with the measurement errors are defined as in (8).

Corollary 2: Given a second order with measurement error of the form (10) where $\alpha, \beta \in [0, 1]$ and $a, b > 0$. The Laplacian matrices L_a and L_b are simultaneously diagonalizable with $L_b = T\Lambda_a T^{-1}$ in which $[\Lambda_b]_{ii} = \lambda_i^b$, λ_i^b are sorted as $0 = \lambda_1^b < Re(\lambda_2^b) \leq \cdots \leq Re(\lambda_n^b)$ and $T = \begin{bmatrix}t_1 & t_2 & \cdots & t_n\end{bmatrix}$. If $\alpha, \beta \in [0, 1)$, the $\mathcal{H}_2$ norm can be represented as

$$\|G\|_{H_2}^2 = tr\Big((R^{-1}K^{-1}\hat{B}E^T T_{diag}^{-1})^* \hat{X} R^{-1}K^{-1}\hat{B}E^T T_{diag}^{-1}\Big) \quad (25a)$$

$$\left[\hat{X}\right]_{ls} = \frac{-\left[(CT_{diag}EKR)^* CT_{diag}EKR\right]_{ls}}{\bar{\delta}_l + \delta_s}, \quad (25b)$$

where T_{diag}, E, K, R and δ are same as in theorem 4.7. $\hat{B}$ is a block diagonal matrix of the form

$$\hat{B}_i = \begin{bmatrix} 0 & 0 \\ -\alpha\lambda_i^a - (1-\alpha)a & -\beta\lambda_i^b - (1-\beta)b \end{bmatrix}.$$

Otherwise, if $\alpha, \beta \in [0, 1]$, the corresponding communication graphs of L_a and L_b are strongly connected, and $Ct_1 = \mathbf{0}_n$ the $\mathcal{H}_2$ norm can still be represented in the form (25) with

$$\left[\hat{X}\right]_{ls} = \begin{cases} 0, & l \ or \ s \leq 2 \\ \dfrac{-\left[(CT_{diag}EKR)^* CT_{diag}EKR\right]_{ls}}{\bar{\delta}_l + \delta_s}, & otherwise. \end{cases}$$

Proof: The proof of this Corollary is based on Theorem 4.7 and Theorem 4.8 with

$$B = \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ -(\alpha L_a + (1-\alpha)aI_{n\times n}) & -(\beta L_b + (1-\beta)bI_{n\times n}) \end{bmatrix}$$

has a special form. It is not hard to prove that

$$B = ET_{diag}\hat{B}T_{diag}^{-1}E^T. \quad (26)$$

Hence we can replace B by form (26) in (23). ■

V. Simulation Results

In this section, we explore the trade off between absolute feedback and relative feedback through numerical simulations of second order systems connected over a line graph. We analyze both symmetric and asymmetric feedback. For the case of asymmetric control gains, the weights of the communication graph in (8) become

$$a_{ij} = b_{ij} = \begin{cases} 1-\epsilon, & j > i; \\ i+\epsilon, & j < i. \end{cases}$$

Using the results in [19], we can show that the Laplacian matrix for all these cases are diagonalizable and therefore we can use the theoretical results provided in the previous section. We further consider the following output matrix

$$C = [H \ \mathbf{0}_{n\times n}], \ H = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}. \quad (27)$$

Since we assign a special form to the output matrix, $C\mathbf{1}_{2n} = 0$, based on remark 2 and corollary 2, the $\mathcal{H}_2$ norm is bounded.

Figure 1 compares the computed performance trade-offs as we move from absolute to relative feedback for a 50 node system with the following parameter values. $\alpha = \beta$, $a = b = 1$ for PV control, and $b = 1$, $\beta = 1$ for PAV control. In both cases we set $\epsilon = 0.1$ to match the degree of asymmetry considered in [10]. The results demonstrate that smoothly reducing the proportion of absolute feedback ($\alpha \to 1$) in double integrator systems smoothly decreases the system performance (larger $\mathcal{H}_2$ norm) in all cases except for those with symmetric PV control laws. In general, the performance degradation is more rapid for PAV control than for PV control.

VI. Conclusion

This paper explores the robustness of first and second order consensus dynamics under a parameterized family of controllers employing a convex combination of relative and absolute feedback structures defined over strongly connected digraphs. We quantify system robustness in terms of the $\mathcal{H}_2$ norm from stochastic disturbance inputs to outputs corresponding to the desired performance measures. We first present a novel method to compute this norm for systems whose communication graphs are described by diagonalizable weighted graph Laplacian matrices. Our results demonstrate that some amount of absolute feedback is required in order to maintain input-output stability for arbitrary output matrices.

We then focus on the special case where the state measurements are noisy and study the robustness of systems with stochastic state measurement errors and find that some amount of absolute feedback is always required to obtain finite $\mathcal{H}_2$ norm unless the output is of a special form. We further explore trade-offs between absolute and relative velocity feedback through numerical simulations of double integrator systems. Our results show that performance degrades smoothly as the relative proportion of relative feedback increases except for one case where the performance remains constant. Future work will explore the effects of changing the degree of the asymmetric feedback coupling and its effect on the importance of absolute position and velocity feedback.

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