Distributed optimization decomposition for joint economic dispatch and frequency regulation

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Abstract—**Economic dispatch and frequency regulation are typically viewed as fundamentally different problems in power systems and, hence, are typically studied separately. In this paper, we frame and study a joint problem that co-optimizes both slow timescale economic dispatch resources and fast timescale frequency regulation resources. We show how the joint problem can be decomposed without loss of optimality into slow and fast timescale sub-problems that have appealing interpretations as the economic dispatch and frequency regulation problems respectively. We solve the fast timescale sub-problem using a distributed frequency control algorithm that preserves network stability during transients. We solve the slow timescale sub-problem using an efficient market mechanism that coordinates with the fast timescale sub-problem. We investigate the performance of our approach on the IEEE 24-bus reliability test system.**

Index Terms—**Economic dispatch, frequency regulation, optimization decomposition, markets.**

I. Introduction

ONE of the major objectives of every Independent System Operator (ISO) is to schedule generation to meet demand at every time instant [2]–[4]. This is a challenging task – it involves responding rapidly to supply-demand imbalances, minimizing generation costs, and respecting operating limitations (such as ramp constraints, capacity constraints, and line constraints). Due to the complexity of this global system operation problem, it is typically divided into two separate problems: *economic dispatch*, which focuses on control of slower timescale resources and is solved using market mechanisms, and *frequency regulation*, which focuses on control of faster timescale resources and is solved using engineered controllers. Economic dispatch and frequency regulation are typically studied independently of each other.

Economic dispatch operates at the timescale of 5 minutes or longer and focuses on cost efficiency. In particular, the economic dispatch problem seeks to optimally schedule generators to minimize total generation costs. Economic dispatch has a long history [2], [5]–[9]. It is currently implemented using a market mechanism known as supply function bidding. In this mechanism, generators submit supply functions to the ISO which specify (as a function of price) the quantity a generator is willing to produce. The ISO solves a centralized optimization problem (over single or multiple time periods) to schedule generators to minimize system costs while satisfying demand and slow timescale operating constraints (such as line constraints, capacity constraints, ramping constraints, security constraints, etc.). Each generator is compensated at the locational marginal price (LMP) which reflect the system cost of serving an incremental unit of demand at its node.

Frequency regulation operates at a faster timescale (from a few minutes to 30 seconds) and focuses on stability rather than efficiency. In particular, the ISO seeks to restore the nominal frequency in the system by rescheduling fast ramping generators. Frequency regulation has a long history [3], [10], [11]. It is currently implemented by a mechanism known as Automatic Generation Control (AGC). In this mechanism, the ISO computes the aggregate generation that would rebalance power within each independent control area (and hence restore nominal frequency) and allocates the imbalance generation among generators based on the solution of the previous economic dispatch run [2]. These allocations determine the setpoints in a distributed control algorithm that drives the power system to a stable operating point using local information on frequency deviations. Similar to dispatch resources, regulation resources are compensated at the applicable LMP. Note that, since the economic dispatch mechanism runs every 5 minutes, the applicable LMPs would be those from the most recent economic dispatch run.

A. Contributions of this paper

While economic dispatch and frequency regulation each have large and active literatures, these literatures are typically disparate, with the exception of studies on the design of hierarchical control in power systems [12]–[17]. The latter studies typically propose solutions to efficiently integrate primary, secondary (frequency regulation), and tertiary (economic dispatch) control. Another related stream of literature involves the analysis and design of frequency regulation controllers (including AGC) that converge to the solution of a cost minimization problem [18], [19]. However, these studies do not explain how to integrate the controllers with slower timescale dispatch mechanisms. To date, we are not aware of

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any analysis of whether the existing combination of economic dispatch and frequency regulation solves the global system operator's goal of dispatching generation resources efficiently across both timescales. The goal of this paper is to study this as well as present one framework for a principled top-down approach for the design of economic dispatch and frequency regulation.

Our main result provides an initial answer. In the context of a DC power flow model and two classes of generators (dispatch and regulation), we show that the global system operator's problem can be decomposed into two sub-problems that correspond to the economic dispatch and frequency regulation timescales, without loss of optimality, as long as the ISO is able to estimate the difference between the average LMP in the frequency regulation periods and the LMP in the economic dispatch period (Theorem 1). This result can be viewed as a first-principles justification for the existing separation of power systems control into economic dispatch and frequency regulation problems. Moreover, this result provides a guide to modify the existing architecture to *optimally* control power systems across timescales. In particular, using this result, we design an optimal control policy for frequency regulation and an optimal market mechanism for economic dispatch, in a way such that the control and market mechanisms jointly solve the global system operator's problem. Our mechanims differ from existing economic dispatch and frequency regulation mechanisms in important ways.

In the case of frequency regulation (Section IV), our mechanism has a key advantage over the AGC mechanism in that our mechanism is efficient. The frequency regulation controller proposed in this paper is built on the distributed controller in [20], [21] and controls generation based on information about generators' costs in a way such that the power system converges to an operating point that minimizes system costs. On the other hand, AGC allocates generation based on participation factors, which might not reflect actual costs, and hence the resulting allocation might not be efficient. In [18], the authors proposed a modification of the participation factors so that the AGC mechanism is cost efficient. However, unlike our mechanism, the mechanism in [18] does not respect line constraints. Another related work is [19], in which the authors showed that droop controllers can be designed to converge asymptotically to the solution of a cost minimization problem. However, their mechanism does not respect line constraints or capacity constraints.

In the case of economic dispatch (Section V), our mechanism has a key advantage over the existing economic dispatch operations in that it coordinates efficiently with the frequency regulation timescale. This coordination does not require additional communication in the market beyond the existing mechanism used in practice. This coordination involves two main components. First, our economic dispatch mechanism communicates the supply function bids from the generators to the frequency regulation mechanism, which uses them in the distributed controllers to allocate frequency regulation resources efficiently. In contrast, the AGC mechanism allocates frequency regulation resources without regard to generation costs. Second, our economic dispatch mechanism accounts for the value that economic dispatch resources provide to frequency regulation. It does so by adjusting the resource costs in the economic dispatch objective based on the difference between the LMP in the frequency regulation periods and that in the economic dispatch period. In contrast, the existing economic dispatch objective does not perform this adjustment and hence might allocate economic dispatch resources inefficiently.

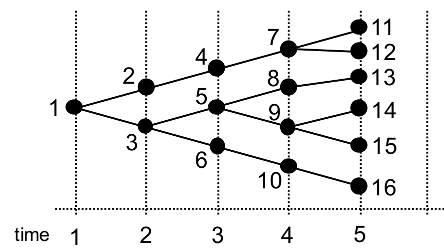


Fig. 1: Example of a scenario tree with $S = 16$ outcomes over $K = 5$ periods. The outcomes are numbered $1, \dots, S$.

In practice, the ISO is unlikely to be able to estimate exactly the adjustment it should make to the economic dispatch objective. In Section VI, we investigate numerically the potential benefits of our proposed mechanism on the IEEE 24-bus reliability test system.

II. System Model

Our aim is to understand how the combination of economic dispatch and frequency regulation can dispatch generation resources efficiently across both timescales. To this end, we formulate a model of the global objective that includes balancing supply and demand at both timescales. We use a DC power flow model and consider two generation types – dispatch and regulation – which differ in responsiveness.

Consider a connected network consisting of a set of nodes N and a set of links L. We focus on a single economic dispatch interval of the real-time market which is typically 5 minutes in existing markets. We partition this time interval into K discrete periods numbered $1, \dots, K$. In general, the length of each period may range from as little as seconds to as long as minutes. However, in this work, we focus on the case where each period is on the order tens of seconds.

A. Stochastic demand

We use a stochastic demand model motivated by the frameworks in [22]–[24]. Assume that there is a set of possible demand outcomes S that can be described by a scenario tree (an example is given in Fig. 1). For each outcome $s \in S$, let $d_{s,n} \in \mathbb{R}$ denote the real power demand at node $n \in N$ and $\mathbf{d}_s := (d_{s,n}, n \in N) \in \mathbb{R}^N$ denote the vector of demands at all nodes. In addition, let $\kappa(s) \in \{1, \dots, K\}$ denote the period of this outcome and p_s denote the probability of this outcome conditioned on the information that the period is $\kappa(s)$. Hence, $\sum_{\{s|\kappa(s)=k\}} p_s = 1$ for each $k \in \{1, \dots, K\}$. Without loss of generality, we assume that $\kappa(1) = 1$ and $p_1 = 1$. That is, there exists an outcome labeled $1 \in S$ associated with period 1 and the demand in that period is deterministic.

B. Generation

We assume that each node $n \in N$ has two generators – a dispatch generator and a regulation generator – where the regulation generator is more responsive than the dispatch generator.[1] To model the differing responsiveness, we assume that the dispatch generator produces at a constant level over the entire economic dispatch interval while the regulation generator may change its production level every period after uncertain demand is realized [25]. Our results extend to the setting where the dispatch generator has ramp constraints; the latter can be modelled by linearly prorating its allocation over the entire economic dispatch interval. Since this feature does not provide new insights, and yet introduces significant complexity to the notations, we assume in this work that the dispatch generator is only subject to instantaneous capacity constraints. Formally, we assume that the dispatch generator produces $q_n^d \in \mathbb{R}$ in all outcomes, and the regulation generator produces $q_n^r \in \mathbb{R}$ in period 1 and $q_n^r + r_{s,n}^r \in \mathbb{R}$ in each subsequent outcome $s \in S \setminus \{1\}$. Hence, q_n^r and $r_{s,n}^r$ can be interpreted as the regulation generator's setpoint and recourse respectively. To simplify notations, we define a dummy variable $r_{1,n}^r := 0$ so that we may write the regulation generator's production in period 1 as $q_n^r + r_{1,n}^r$. We assume that the regulation and dispatch generators have capacity constraints $[\underline{q}_n^r, \bar{q}_n^r]$ and $[\underline{q}_n^d, \bar{q}_n^d]$ respectively, and incur costs $c_n^r(q_n^r + r_{s,n}^r)$ and $c_n^d(q_n^d)$ respectively in period $\kappa(s)$, where the functions $c_n^r : [\underline{q}_n^r, \bar{q}_n^r] \to \mathbb{R}_+$ and $c_n^d : [\underline{q}_n^d, \bar{q}_n^d] \to \mathbb{R}_+$ are strictly convex and continuously differentiable.

Define vectors $\mathbf{q}^r := (q_n^r, n \in N)$, $\mathbf{r}_s^r := (r_{s,n}^r, n \in N)$, $\mathbf{q}^d := (q_n^d, n \in N)$, $\underline{\mathbf{q}}^r := (\underline{q}_n^r, n \in N)$, $\underline{\mathbf{q}}^d := (\underline{q}_n^d, n \in N)$, $\bar{\mathbf{q}}^r := (\bar{q}_n^r, n \in N)$, $\bar{\mathbf{q}}^d := (\bar{q}_n^d, n \in N)$. Then the generation constraints in outcome $s \in S$ are given by:

$$\underline{\mathbf{q}}^d \le \mathbf{q}^d \le \bar{\mathbf{q}}^d, \tag{1}$$

$$\underline{\mathbf{q}}^r \le \mathbf{q}^r + \mathbf{r}_s^r \le \bar{\mathbf{q}}^r. \tag{2}$$

We also let the vector $\mathbf{r}^r := (\mathbf{r}_s^r, s \in S)$.

C. Network constraints

Note that $\mathbf{q}^d + \mathbf{q}^r + \mathbf{r}_s^d - \mathbf{d}_s$ is the vector of nodal injections for $s \in S$. Thus, the supply-demand balance constraint is:

$$\mathbf{1}^\top (\mathbf{q}^d + \mathbf{q}^r + \mathbf{r}_s^r - \mathbf{d}_s) = 0, \tag{3}$$

where $\mathbf{1} \in \mathbb{R}^N$ denotes the vector of all ones.

We adopt the DC power flow model for line flows. Let $\theta_{s,n}$ denote the phase angle of node n. Without loss of generality, assign each link l an arbitrary orientation and let $i(l)$ and $j(l)$ denote the tail and head of the link respectively. Let B_l denote the sensitivity of the flow with respect to changes in the phase difference $\theta_{s,i(l)} - \theta_{s,j(l)}$ and let $v_{s,l}$ denote its power flow. Define the vectors $\boldsymbol{\theta}_s := (\theta_{s,n}, n \in N)$ and $\mathbf{v}_s := (v_{s,l}, l \in L)$ and the matrix $\mathbf{B} := \mathrm{diag}(B_l, l \in L)$. Then, the line flows are given by $\mathbf{v}_s = \mathbf{B}\mathbf{C}^\top \boldsymbol{\theta}_s$ where $\mathbf{C} \in \mathbb{R}^{N \times L}$ is the incidence matrix of the directed graph. And the injections are:

$$\mathbf{q}^d + \mathbf{q}^r + \mathbf{r}_s^r - \mathbf{d}_s = \mathbf{C}\mathbf{v}_s = \mathbf{L}\boldsymbol{\theta}_s, \tag{4}$$

where $\mathbf{L} := \mathbf{C}\mathbf{B}\mathbf{C}^\top$.

Note that (3) and (4) are equivalent. For any set of injections that satisfy (3), we can always find $\boldsymbol{\theta}_s$ that satisfies (4). Conversely, since $\mathbf{1}^\top \mathbf{C} = 0$, any injections that satisfy (4) also satisfy (3). Hence, the line flows can be written in terms of the power injections:

$$\mathbf{v}_s = \mathbf{B}\mathbf{C}^\top \mathbf{L}^\dagger (\mathbf{q}^d + \mathbf{q}^r + \mathbf{r}_s^r - \mathbf{d}_s),$$

where $\mathbf{L}^\dagger$ denotes the pseudo-inverse of $\mathbf{L}$. Let $\mathbf{H} := \mathbf{B}\mathbf{C}^\top \mathbf{L}^\dagger$. Let f_l denote the capacity of line l and define the vector $\mathbf{f} := (f_l, l \in L)$. Then the line flow constraints are:

$$-\mathbf{f} \le \mathbf{H}\left(\mathbf{q}^d + \mathbf{q}^r + \mathbf{r}_s^r - \mathbf{d}_s\right) \le \mathbf{f}. \tag{5}$$

To simplify notations, we define the set $\Omega(\mathbf{d}_s)$ of feasible generation for a given demand vector $\mathbf{d}_s$ as:

$$\Omega(\mathbf{d}_s) := \left\{ (\mathbf{q}^d, \mathbf{q}^r, \mathbf{r}_s^r) : (1), (2), (3), (5) \text{ holds} \right\}.$$

D. System operator's objective

The global system operator's objective is to allocate the dispatch and regulation generations $(\mathbf{q}^d, \mathbf{q}^r, \mathbf{r}^r)$ to minimize the expected cost of satisfying demand and operating constraints. This is formalized as follows.

$$\begin{aligned} SYSTEM: \min_{\mathbf{q}^d, \mathbf{q}^r, \mathbf{r}^r} \quad & \sum_{s \in S} p_s \sum_{n \in N} \left(c_n^d(q_n^d) + c_n^r(q_n^r + r_{s,n}^r) \right) \\ \text{s.t.} \quad & (\mathbf{q}^d, \mathbf{q}^r, \mathbf{r}_s^r) \in \Omega(\mathbf{d}_s), \quad \forall s \in S, \\ & \mathbf{r}_1^r = \mathbf{0}. \end{aligned}$$

This optimization is solved at the beginning of the economic dispatch interval. We assume that this optimization is feasible. Note that $SYSTEM$ differs from the existing economic dispatch mechanism, which minimizes the costs of satisfying the forecasted demand at the end of the economic dispatch interval. Observe that $SYSTEM$ is a stochastic optimization problem. Although it is in general computationally challenging to solve, the design of algorithms for such problems is an active research area in the power systems and optimization communities [26], [27]. The goals of this work, however, are to formulate a global system operation problem and decompose it into subproblems in a way that provide insights into optimal design of economic dispatch and frequency regulation mechanisms.

Let λ_s and $(\underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s)$ be the Lagrange multipliers associated with constraints (3) and (5) respectively in $SYSTEM$. Then, the function $\boldsymbol{\pi} : \mathbb{R} \times \mathbb{R}_+^{2L} \to \mathbb{R}^N$, defined by:

$$\boldsymbol{\pi}(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) := \lambda_s \mathbf{1} + \mathbf{H}^\top (\underline{\boldsymbol{\mu}}_s - \bar{\boldsymbol{\mu}}_s), \tag{6}$$

gives the nodal prices in outcome $s \in S$.

[1] Our results may be extended to settings where a node has more than one of each type of generator, or there is only one type of generator at a node, or there are no generators at a node. The assumption that each node has exactly one of both types of generators is made to simplify the notations and derivations. In our case study, we will validate our approach via simulations on the IEEE 24-bus reliability test system, in which certain nodes have only one type of generator or have no generator.

III. Architectural Decomposition

Our main result is a decomposition of $SYSTEM$ into setpoint and recourse sub-problems. Importantly, our decomposition identifies a rigorous connection between the setpoint and recourse sub-problems that ensures that the combination solves $SYSTEM$. In particular, our decomposition divides $SYSTEM$ into sub-problems ED and FR defined by:

$$\begin{aligned} ED(\mathbf{d}_1): \quad & \min_{\mathbf{q}^d, \mathbf{q}^r} \quad \sum_{n \in N} \left(Kc_n^d(q_n^d) + Kc_n^r(q_n^r) - \delta_n q_n^d \right) \\ & \quad \text{s.t.} \quad (\mathbf{q}^d, \mathbf{q}^r, \mathbf{0}) \in \Omega(\mathbf{d}_1), \\ FR(\mathbf{q}^d, \mathbf{q}^r, \mathbf{d}_s): \quad & \min_{\mathbf{r}_s^r} \quad \sum_{n \in N} c_n^r(q_n^r + r_{s,n}^r) \\ & \quad \text{s.t.} \quad (\mathbf{q}^d, \mathbf{q}^r, \mathbf{r}_s^r) \in \Omega(\mathbf{d}_s), \end{aligned}$$

where $\boldsymbol{\delta} \in \mathbb{R}^N$ is a constant. $ED(\mathbf{d}_1)$ is implemented in time period 1 and $FR(\mathbf{q}^d, \mathbf{q}^r, \mathbf{d}_s)$ is implemented in subsequent time periods $\kappa(s) > 1$.

We denote the first optimization problem by ED, since it optimizes only generation setpoints $(\mathbf{q}^d, \mathbf{q}^r)$ assuming constant demand $\mathbf{d}_1$ over the K time periods, and hence it is on the same timescale as the existing economic dispatch mechanism. We denote the second optimization problem by FR, since it optimizes regulation generators' recourse production $\mathbf{r}_s^r$ in subsequent time periods, and hence it is on the same timescale as the existing frequency regulation mechanism.

Definition 1. *We say that $SYSTEM$ can be optimally decomposed into ED-FR if $(\mathbf{q}^d, \mathbf{q}^r, \mathbf{r}^r)$ is an optimal solution to $SYSTEM$ if and only if $\mathbf{r}_1^r = \mathbf{0}$, $(\mathbf{q}^d, \mathbf{q}^r)$ is an optimal solution to $ED(\mathbf{d}_1)$, and $\mathbf{r}_s^r$ is an optimal solution to $FR(\mathbf{q}^d, \mathbf{q}^r, \mathbf{d}_s)$ for all $s \in S$.*

Theorem 1 (Decomposition). *Let λ_s and $(\underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s)$ be any Lagrange multipliers associated with constraints* (3) *and* (5) *respectively in $SYSTEM$.*

(a) *If $\boldsymbol{\delta}$ is the average, over all time periods, of the difference between the expected nodal prices in each period and that in period* 1*, that is, for each $n \in N$,*

$$\delta_n = \sum_{s \in S} p_s \left(\pi_n(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) - \pi_n(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1) \right), \quad (7)$$

then $SYSTEM$ can be optimally decomposed into ED-FR.

(b) *If $SYSTEM$ can be optimally decomposed into ED-FR, then for all n such that $\underline{q}_n^d < q_n^d < \bar{q}_n^d$ and $\underline{q}_n^r < q_{1,n}^r < \bar{q}_n^r$,* (7) *holds.*

The proof of Theorem 1 is given in the Appendix. The result follows from analyzing the Karush-Kuhn-Tucker (KKT) conditions of the system operator's problem and those of ED and FR. As $SYSTEM$, ED, and FR are all convex, the KKT conditions are necessary and sufficient for optimality. Upon substituting (7) into the KKT conditions, one can show that any solution to the KKT conditions of $SYSTEM$ is also a solution to the KKT conditions of ED-FR, and vice versa. As mentioned, we denote the two sub-problems by ED and FR because they focus on the economic dispatch and frequency regulation timescales respectively. Hence, these sub-problems can serve as guides for the optimal design of economic dispatch and frequency regulation mechanisms. The insights are immediate in the case of economic dispatch and we show how ED leads to an improved market mechanism in Section V. However, the insights may not be as clear in the case of frequency regulation. We show in Section IV that FR can in fact be solved via distributed frequency control algorithms, although these algorithms deviate from current practice that do not optimize generation costs.

The most important feature of Theorem 1 is that, one way to choose generation setpoints optimally at the economic dispatch timescale, is to include, in the optimization objective, an offset of the dispatch generators' marginal costs by the expected changes in nodal prices during the frequency regulation timescale. The latter can be interpreted as the expected changes in the marginal value of dispatch generation. Hence, if the latter is zero, then generation setpoints can be chosen optimally at the economic dispatch timescale without regard to the behavior of the system in the frequency regulation timescale [1].

A byproduct of our decomposition is the insight that the stochastic optimization problem $SYSTEM$ may be solved by solving a sequence of deterministic subproblems $ED(\mathbf{d}_1)$ and $FR(\mathbf{q}^d, \mathbf{q}^r, \mathbf{d}_s)$ if the system operator is able to predict the RHS of (7). Note that $ED(\mathbf{d}_1)$ has the same complexity as the existing economic dispatch mechanism; and we will show in the next section that $FR(\mathbf{q}^d, \mathbf{q}^r, \mathbf{d}_s)$ can be solved using a distributed frequency control algorithm. Therefore, the computations of the subproblems have the same complexity as existing operations.

An important extension of this work is to design algorithms to iteratively estimate the RHS of (7) online. Such approaches resemble value function iterations in dynamic programming. Also important is to understand the suboptimality of the solutions under estimation errors in the RHS of (7). Note that negative estimation errors cause $ED(\mathbf{d}_1)$ to use less than optimal dispatch resources (and more than optimal regulation resources) and vice versa. In such situations, the dispatch generation $\mathbf{q}^d$ might not be optimal, and therefore $FR(\mathbf{q}^d, \mathbf{q}^r, \mathbf{d}_s)$ might not be feasible. To ensure that $FR(\mathbf{q}^d, \mathbf{q}^r, \mathbf{d}^s)$ is feasible, we may modify $ED(\mathbf{d}_1)$ into a robust optimization problem by adding constraints $(\mathbf{q}^d, \mathbf{q}^r, \mathbf{r}_s^r) \in \Omega(\mathbf{d}_s)$ for all $s \in S \setminus \{1\}$. The size of such a problem is exponential in S but can be reduced using the technique in [28]. Note that this should not be viewed as a drawback of our decomposition, as the current practice based on AGC might also not be feasible. In practice, the risks of infeasibility are mitigated using reserves. Moreover, our decomposition has the advantage that it coordinates the economic dispatch and frequency regulation resources efficiently, and hence, may reduce reserve requirements.

Theorem 1 provides a rigorous way to think about architectural design of power networks. Theorem 1 is close in spirit to work in communication networks that use optimization decomposition to justify and optimize protocol layering [29]–[31]. In the latter, different protocol layers coordinate by communicating primal and dual variables between sub-optimization problems. It is an interesting open direction as to whether these mechanisms can be applied to coordinate between *ED* and *FR*, since the sub-optimizations in protocol layering use instantaneous primal and dual variables while *ED* uses expected prices.

IV. Distributed Frequency Regulation

This section illustrates how to implement the solution to FR using distributed frequency regulation controllers. Besides achieving optimality, a practical implementation should preserve network stability, be robust to unexpected system events, aggregate network information in a distributed manner, and satisfy constraints (2), (3) and (5). The distributed algorithm that we provide in this section satisfies all the above characteristics. It can be interpreted as performing distributed frequency regulation by sending different regulation signals to each bus.

A. Dynamic model

Before introducing our algorithm we add dynamics to our system model to describe the system behavior within a single time period. Let t denote the time evolution within the time period of outcome s, and assume without loss of generality that $t \in (k, k+1]$ where $k = \kappa(s)$. Let $\mathbf{r}^r_s(t) := (r^r_{s,n}(t), n \in N)$ denote the recourse quantities generated by the regulation generators at time t. For the purpose of the analysis, we assume that dispatch generation and demand do not change within the time period. And we will use simulations to study the performance of the proposed mechanism in a setting where demand is changing continuously.

Then, the system changes within the time period are governed by the swing equations which we assume to be:

$$\dot{\boldsymbol{\theta}}_s(t) = \boldsymbol{\omega}_s(t); \tag{8a}$$

$$\mathbf{M}\dot{\boldsymbol{\omega}}_s(t) = \mathbf{q}^d + \mathbf{q}^r + \mathbf{r}^r_s(t) - \mathbf{d}_s - \mathbf{D}\boldsymbol{\omega}_s(t) - \mathbf{L}\boldsymbol{\theta}_s(t), \tag{8b}$$

where $\boldsymbol{\omega}_s(t) := (\omega_{s,n}(t), n \in N)$ are the frequency deviations from the nominal value at time t, $\boldsymbol{\theta}_s(t) := (\theta_{s,n}(t), n \in N)$ are the phase angles at time t, $\mathbf{M} := \text{diag}(M_1, \dots, M_N)$ where M_n is the aggregate inertia of the generators at node n, and $\mathbf{D} := \text{diag}(D_1, \dots, D_N)$ where D_n is the aggregate damping of the generators at node n. The notation $\dot{x}$ denotes the time derivative, i.e. $\dot{x} = dx/dt$. Equation (8) is a linearized version of the nonlinear network dynamics [3], [32], and has been widely used in the design of frequency regulation controllers. See, e.g., [11], [33].

B. Distributed frequency regulation

We now introduce a distributed, continuous-time algorithm that provably solves FR *while preserving system stability*. Our solution is based on a novel reverse and forward engineering approach for distributed control design in power systems [18], [21], [34]–[37]. The algorithm operates as follows. Each regulation generator n updates its power generation using

$$r^r_{s,n}(t) = [c^{r\prime-1}_n(-\omega_{s,n}(t) - \pi^r_{s,n}(t))]^{\bar{q}^r_n - q^r_n}_{\underline{q}^r_n - q^r_n}, \tag{9}$$

where $c^{r\prime}_n(x) = \frac{\partial}{\partial x} c^r_n(x)$ and $c^{r\prime-1}_n$ denotes its inverse. The projection $[r]^{\bar{q}^r_n - q^r_n}_{\underline{q}^r_n - q^r_n}$ ensures that $\underline{q}^r_n - q^r_n \le r \le \bar{q}^r_n - q^r_n$ (or equivalently $\underline{q}^r_n \le r + q^r_n \le \bar{q}^r_n$) and $\pi^r_{s,n}(t)$ is a control signal generated using:

$$DFR: \ \dot{\boldsymbol{\pi}}^r_s(t) = \boldsymbol{\zeta}^\pi \left(\mathbf{q}^d + \mathbf{q}^r + \mathbf{r}^r_s(t) - \mathbf{d}_s - \boldsymbol{L}\boldsymbol{\phi}_s(t)\right); \tag{10a}$$

$$\dot{\bar{\boldsymbol{\mu}}}_s(t) = \boldsymbol{\zeta}^{\bar{\mu}} \left[\mathbf{B}\mathbf{C}^\top \boldsymbol{\phi}_s(t) - \mathbf{f}\right]^+_{\bar{\boldsymbol{\mu}}_s}; \tag{10b}$$

$$\dot{\underline{\boldsymbol{\mu}}}_s(t) = \boldsymbol{\zeta}^{\underline{\mu}} \left[-\mathbf{f} - \mathbf{B}\mathbf{C}^\top \boldsymbol{\phi}_s(t)\right]^+_{\underline{\boldsymbol{\mu}}_s}; \tag{10c}$$

$$\dot{\boldsymbol{\phi}}_s(t) = \boldsymbol{\chi}^\phi \left(\mathbf{L}\boldsymbol{\pi}^p_s(t) - \mathbf{C}\mathbf{B}(\bar{\boldsymbol{\mu}}_s(t) - \underline{\boldsymbol{\mu}}_s(t))\right), \tag{10d}$$

where $\boldsymbol{\zeta}^\pi := \text{diag}(\zeta^\pi_1, \dots, \zeta^\pi_N)$, $\boldsymbol{\zeta}^{\bar{\mu}} := \text{diag}(\zeta^{\bar{\mu}}_1, \dots, \zeta^{\bar{\mu}}_L)$, $\boldsymbol{\zeta}^{\underline{\mu}} := \text{diag}(\zeta^{\underline{\mu}}_1, \dots, \zeta^{\underline{\mu}}_L)$, $\boldsymbol{\chi}^\phi := \text{diag}(\chi^\phi_1, \dots, \chi^\phi_N)$ denote the respective control gains. The element-wise projection $[\mathbf{y}]^+_{\mathbf{x}} := ([y_n]^+_{x_n}, n \in N)$ ensures that the dynamics $\dot{\mathbf{x}} = [\mathbf{y}]^+_{\mathbf{x}}$ have a solution $\mathbf{x}(t)$ that remains in the positive orthant, that is, $[y_n]^+_{x_n} = 0$ if $x_n = 0$ and $y_n < 0$, and $[y_n]^+_{x_n} = y_n$ otherwise.

The proposed solution (9) – (10) can be interpreted as a frequency regulation algorithm in which each regulation generator receives a different regulation signal (9) depending on its location in the network. The key step in the design of DFR is reformulating FR into the following equivalent optimization problem:

$$FR'(\boldsymbol{q}^d, \boldsymbol{q}^r, \boldsymbol{d}_s):$$

$$\min_{\mathbf{r}^r_s, \boldsymbol{\omega}_s, \mathbf{v}_s, \boldsymbol{\phi}_s} \quad \sum_{n \in N} \left(c^r_n(q^r_n + r^r_{s,n}) + D_n \omega^2_{s,n}/2\right)$$

$$\text{s.t.} \quad \mathbf{q}^d + \mathbf{q}^r + \mathbf{r}^r_s - \mathbf{d}_s - \mathbf{D}\boldsymbol{\omega}_s = \mathbf{C}\mathbf{v}_s; \tag{11a}$$

$$\mathbf{q}^d + \mathbf{q}^r + \mathbf{r}^r_s - \mathbf{d}_s = \mathbf{L}\boldsymbol{\phi}_s; \tag{11b}$$

$$-\mathbf{f} \le \mathbf{B}\mathbf{C}^\top \boldsymbol{\phi}_s \le \mathbf{f}; \tag{11c}$$

$$\underline{\mathbf{q}}^r \le \mathbf{q}^r \le \bar{\mathbf{q}}^r. \tag{11d}$$

Recall from Section II-C that $\mathbf{v}_s$ denote line flows. Constraint (11a) is reformulated from the per node supply-demand balance constraint (4), and makes explicit the fact that, whenever supply and demand do not match, the mismatch is compensated by a change in the frequency. Constraint (11b) ensures that $\boldsymbol{\omega}_s = 0$ at the optimal solution so that supply and demand are balanced. Constraint (11c) imposes line flow limits. However, instead of using actual line flows $\mathbf{v}_s$, these limits are imposed on *virtual flows* $\mathbf{B}\mathbf{C}^\top \boldsymbol{\phi}_s$, which are identical to line flows at the optimal solution [21].

It can be shown that FR' has a primal-dual algorithm that contains the component (8) resembling power network dynamics and the components (9) – (10) that can be implemented via distributed communication and computation. This new problem FR' also makes explicit the role of frequency in maintaining supply-demand balance.

The next proposition formally relates the optimal solutions of FR and FR' and guarantees the optimality of (9) – (10).

Proposition 1 (Optimality). *Let $\mathbf{r}^r_s$ and $(\mathbf{r}^{r\prime}_s, \boldsymbol{\omega}'_s, \mathbf{v}'_s, \boldsymbol{\phi}'_s)$ be optimal solutions of FR and FR' respectively. Then, the following statements are true: (i) Frequency restoration: $\boldsymbol{\omega}'_s = 0$; (ii) Generation equivalence: $\mathbf{r}^r_s = \mathbf{r}^{r\prime}_s$; (iii) Line flow equivalence: $\mathbf{H}\left(\mathbf{q}^d + \mathbf{q}^r + \mathbf{r}^r_s - \mathbf{d}_s\right) = \mathbf{B}\mathbf{C}^\top \boldsymbol{\phi}'_s$. Moreover, there exists $\boldsymbol{\theta}'_s \in \mathbb{R}^N$ and $\mathbf{y}'_s \in \mathbb{R}^L$, satisfying $\mathbf{C}\mathbf{y}'_s = 0$, such that $\mathbf{v}'_s = \mathbf{B}\mathbf{C}^\top \boldsymbol{\theta}'_s + \mathbf{y}'_s$ and $\mathbf{B}\mathbf{C}^\top \boldsymbol{\phi}'_s = \mathbf{B}\mathbf{C}^\top \boldsymbol{\theta}'_s$. And $(\mathbf{r}^{r\prime}_s, \boldsymbol{\omega}'_s, \boldsymbol{\theta}'_s, \boldsymbol{\phi}'_s, \boldsymbol{\pi}^{r\prime}_s, \underline{\boldsymbol{\mu}}'_s, \bar{\boldsymbol{\mu}}'_s)$ is an equilibrium point of (8) –*

(10) *if and only if* $(\mathbf{r}_s^{r\prime}, \boldsymbol{\omega}_s', \mathbf{v}_s', \boldsymbol{\phi}_s', \boldsymbol{\pi}_s^{r\prime}, \underline{\boldsymbol{\mu}}_s', \bar{\boldsymbol{\mu}}_s')$ *is a primal-dual optimal solution of* FR'*, where* $\boldsymbol{\omega}_s'$*,* $\boldsymbol{\pi}_s^{r\prime}$*, and* $(\underline{\boldsymbol{\mu}}_s', \bar{\boldsymbol{\mu}}_s')$ *are the Lagrange multipliers associated with constraints* (11a)*,* (11b)*, and* (11c)*, respectively.*

The proof of Proposition 1 is given in the Appendix. What remains is to guarantee the convergence of the distributed frequency regulation algorithm.

Proposition 2 (Convergence)**.** *If* c_n^r *is twice continuous differentiable with* $c_n^{r\prime\prime} \geq \alpha > 0$ *(i.e.,* α*-strictly convex) and* $c_n^r(q_n^r + r_{s,n}^r) \to +\infty$ *as* $q_n^r + r_{s,n}^r \to \{\underline{q}_n^r, \bar{q}_n^r\}$*, then* $\mathbf{r}_s^r(t)$ *in* (8) – (10) *converge globally to an optimal solution of* FR*.*

The proof of Proposition 2 follows from [21] and uses the machinery developed in [38] to handle projections (10b) – (10c). By substituting the line flows $\mathbf{v}_s(t) = \mathbf{B}\mathbf{C}^\top \boldsymbol{\theta}_s(t)$ into (8) and eliminating $\boldsymbol{\theta}_s(t)$, we can show that the entire system (8) – (10) is a primal-dual algorithm of FR' (see [21, Theorem 5]). Therefore, Theorem 10 in [21] guarantees global asymptotic convergence to an equilibrium point which by Proposition 1 is an optimal solution of both FR' and FR. Our setup is simpler than the controllers in [21], which had additional states, but the same proof technique applies. Although Proposition 2 requires costs to blow up as regulation generations approach minimum and maximum capacities, this assumption is not restrictive, as it can be achieved by adding a barrier function to the actual cost before implementing in the controllers. Moreover, as our mechanism is distributed, it can be implemented on large scale systems with minimal computational requirements and guaranteed convergence. However, further studies have to be performed on the convergence properties of the algorithm as the system size increases, and how the speed of convergence is affected by the cost functions of the generators and the design of the control gains.

V. Market Mechanism for Economic Dispatch

This section illustrates how to implement the solution to ED through a market mechanism for economic dispatch. The mechanism works in the following manner. In the first time period, the ISO collects supply function bids from generators (both dispatch and regulation) and uses those bids to solve ED. Then, in subsequent time periods, the ISO uses the regulation generators' supply function bids to implement the controller in (9). This mechanism is efficient if $SYSTEM$ can be decomposed into ED-FR and does not require any more communication than the existing market mechanisms used in practice.

A. Market model

We assume that generators are price-takers. Let π_n^d denote the price paid to dispatch generator n in each period and $\pi_{s,n}^r$ denote the price paid to regulation generator n in outcome s. Then, the expected profit of the dispatch and regulation generators at node n are: Note that the regulation generator's profit is a function of its total production $q_n^r + r_{s,n}^r$ in each outcome $s \in S$. The supply function bids indicate the quantities the generators are willing to produce at every price.[2] We assume that these bids are chosen from a parameterized family of functions. In particular, for node n, we represent the dispatch and regulation generators' supply functions by parameters $\alpha_n^d > 0$ and $\alpha_n^r > 0$ respectively, and these bids indicate that the dispatch generator is willing to supply the quantity $q_n^d = [\alpha_n^d s_n^d(\pi_n^d)]_{\underline{q}_n^d}^{\bar{q}_n^d}$ in the first time period and the regulation generator is willing to supply the quantity $q_n^r + r_{s,n}^r = [\alpha_n^r s_n^r(\pi_{s,n}^r)]_{\underline{q}_n^r}^{\bar{q}_n^r}$ in outcome s, for some fixed functions $s_n^d : [\underline{q}_n^d, \bar{q}_n^d] \to \mathbb{R}_+$ and $s_n^r : [\underline{q}_n^r, \bar{q}_n^r] \to \mathbb{R}_+$.[3] We also assume that $s_n^d(\pi_n^d) \neq 0$ for all $\pi_n^d \in \mathbb{R}$ and $s_n^r(\pi_{s,n}^r) \neq 0$ for all $\pi_{s,n}^r \in \mathbb{R}$.[4] The generators choose their bids to maximize their profits subject to their capacity constraints. Note that the regulation generator submits only one supply function for all possible outcomes. Hence, its bid in the economic dispatch timescale is also used as its bid in the frequency regulation timescale.

The system operator interprets bids α_n^d and α_n^r as signals that the dispatch and regulation generators at node n have marginal costs π_n^d and $\pi_{s,n}^r$ respectively when supplying quantities $\alpha_n^d s_n^d(\pi_n^d)$ and $\alpha_n^r s_n^r(\pi_{s,n}^r)$ respectively. Hence, it associates with the generators the following bid cost functions:

$$\hat{c}_n^d(q_n^d) := \int_{\underline{q}_n^d}^{q_n^d} (s_n^d)^{-1}(w/\alpha_n^d)\, dw, \tag{12}$$

$$\hat{c}_n^r(q_n^r) := \int_{\underline{q}_n^r}^{q_n^r} (s_n^r)^{-1}(w/\alpha_n^r)\, dw. \tag{13}$$

Let $\boldsymbol{\alpha}^d := (\alpha_n^d, n \in N)$ and $\boldsymbol{\alpha}^r := (\alpha_n^r, n \in N)$ denote the vectors of bids. Given bids $(\boldsymbol{\alpha}^d, \boldsymbol{\alpha}^r)$, the system operator solves ED to minimize expected bid costs. The prices for the regulation generator in the first time period are the nodal prices in ED while the prices for the dispatch generator are the nodal prices offset by $\boldsymbol{\delta}$. Then, in each subsequent outcome $s \in S$, the system operator implements the controller in (9) using regulation generators' bid costs. The prices are the nodal prices in FR (which are computed by DFR).

B. Market equilibrium

Our focus is on understanding the efficiency of the mechanism. Formally, we consider the following notion of a competitive equilibrium.

[2]In practice, supply function bids are, in fact, functions from quantity to minimum acceptable price. This is not captured by our model because it would involved multi-valued maps instead of functions. Nevertheless, in line with previous work [7], [39]–[42], we will use supply functions that map price to quantities in this paper.

[3]Numerous studies have explored different functional forms of the supply functions and their impact on market efficiency, e.g., see [7], [39]–[42]. The focus of this work is on illustrating that ED can be implemented using a simple market mechanism. Hence, we restrict ourselves to linearly parameterized supply functions and leave the analyses of other more sophisticated supply functions to future work. We refer the reader to [41] for some appealing properties of linearly parameterized supply functions.

[4]This assumption is a technical condition to avoid the degenerate situation where a generator's supply quantity is not sensitive to its bid parameter which would occur if $s_n^d(\pi_n^d) = 0$ or $s_n^r(\pi_{s,n}^r) = 0$.

Definition 2. *We say that bids* $(\boldsymbol{\alpha}^d, \boldsymbol{\alpha}^r)$ *are a competitive equilibrium if there exists prices* $\boldsymbol{\pi}^d \in \mathbb{R}^N$ *and* $\boldsymbol{\pi}^r = (\boldsymbol{\pi}^r_s, s \in S) \in \mathbb{R}^{NS}$ *such that:*

(a) *For all* n*,* α^d_n *is an optimal solution to:*

$$\max_{\hat{\alpha}^d_n > 0} \quad \mathrm{PF}^d_n \left([\hat{\alpha}^d_n s^d_n(\pi^d_n)]^{\bar{q}^d_n}_{\underline{q}^d_n}, \pi^d_n \right).$$

(b) *For all* n*,* α^r_n *is an optimal solution to:*

$$\max_{\hat{\alpha}^r_n > 0} \quad \mathrm{PF}^r_n \left(([\hat{\alpha}^r_n s^r_n(\pi^r_{s,n})]^{\bar{q}^r_n}_{\underline{q}^r_n}, \pi^r_{s,n}), s \in S \right).$$

(c) $\boldsymbol{\pi}^d = (1/K)\left(\boldsymbol{\pi}(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1) + \boldsymbol{\delta}\right)$ *and* $\boldsymbol{\pi}^r_1 = (1/K)\boldsymbol{\pi}(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1)$ *where* λ_1 *and* $(\underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1)$ *are the Lagrange multipliers associated with constraints* (3) *and* (5) *respectively in:*

$$\begin{aligned} \hat{ED}(\mathbf{d}_1): \quad \min_{\mathbf{q}^d, \mathbf{q}^r} \quad & \sum_{n \in N} \left(K\hat{c}^d_n(q^d_n) + K\hat{c}^r_n(q^r_n) - \delta_n q^d_n \right) \\ \text{s.t.} \quad & (\mathbf{q}^d, \mathbf{q}^r, \mathbf{0}) \in \Omega(\mathbf{d}_1). \end{aligned}$$

(d) *For all* $s \in S$*,* $\boldsymbol{\pi}^r_s = \boldsymbol{\pi}(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s)$ *where* λ_s *and* $(\underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s)$ *are the Lagrange multipliers associated with constraints* (3) *and* (5) *respectively in:*

$$\begin{aligned} \hat{FR}(\mathbf{q}^d, \mathbf{q}^r, \mathbf{d}_s): \quad \min_{\mathbf{r}^r_s} \quad & \sum_{n \in N} \hat{c}^r_n(q^r_n + r^r_{s,n}) \\ \text{s.t.} \quad & (\mathbf{q}^d, \mathbf{q}^r, \mathbf{r}^r_s) \in \Omega(\mathbf{d}_s), \end{aligned}$$

where $\mathbf{q}^d = \left([\alpha^d_n s^d_n(\pi^d_n)]^{\bar{q}^d_n}_{\underline{q}^d_n}, n \in N \right)$ *and* $\mathbf{q}^r = \left([\alpha^r_n s^r_n(\pi^r_{1,n})]^{\bar{q}^r_n}_{\underline{q}^r_n}, n \in N \right)$.

At each node $n \in N$*, the dispatch and regulation generators produce at setpoints* $[\alpha^d_n s^d_n(\pi^d_n)]^{\bar{q}^d_n}_{\underline{q}^d_n}$ *and* $[\alpha^r_n s^r_n(\pi^r_{1,n})]^{\bar{q}^r_n}_{\underline{q}^r_n}$ *respectively in period* 1*, and the regulation generator produces an additional quantity* $[\alpha^r_n s^r_n(\pi^r_{s,n})]^{\bar{q}^r_n}_{\underline{q}^r_n} - [\alpha^r_n s^r_n(\pi^r_{1,n})]^{\bar{q}^r_n}_{\underline{q}^r_n}$ *in outcome* $s \in S$.

The following is our main result for this section. It highlights that, as a consequence of Theorem 1, any competitive equilibrium is efficient.

Proposition 3 (Efficiency)**.** *Suppose that, for each* $n \in N$*, the functions* $s^d_n(\cdot) = c^{d\prime -1}_n(\cdot)/\gamma^d_n$ *and* $s^r_n(\cdot) = c^{r\prime -1}_n(\cdot)/\gamma^p_n$ *for some constants* $\gamma^d_n, \gamma^r_n > 0$*. Let* λ_s *and* $(\underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s)$ *be the Lagrange multipliers associated with constraints* (3) *and* (5) *respectively in* $SYSTEM$*. Suppose that* (7) *holds. Then:*

(a) *Any competitive equilibrium has a production schedule that solves* $SYSTEM$*.*

(b) *Any production schedule that solves* $SYSTEM$ *can be sustained by a competitive equilibrium.*

Proposition 3 resembles classical welfare theorems, e.g., [41], [43]–[45]. However, it differs from typical competitive equilibria frameworks because each regulation generator is restricted to bidding a single supply function over the entire economic dispatch interval even though there are multiple fast timescale instances. The latter creates challenges in guaranteeing existence and efficiency of equilibria that do not arise in typical competitive equilibria frameworks. In particular, the space of bid functions needs to be sufficiently expressive for generators to convey their costs over multiple fast timescale instances via a single bid function. Proposition 3 circumvented this challenge by restricting supply functions to be in the linear space of regulation generators' true cost functions. An important extension is to understand the existence and efficiency of equilibria under less restrictive bid spaces. Proposition 3 also highlights that nodal pricing is not always efficient and that the pricing mechanism needs to be jointly designed and analyzed with decomposition principles in order to achieve efficiency.

VI. Case Study

In this section, we compare the proposed mechanism to the current practice using a case study on the IEEE 24-bus reliability test system [46]. For each demand node, we use the values from the data as the demand at time $t = 0$, and we generate 100 samples of a zero-mean random process to obtain the demands over the 5-minute interval. Fig. 3 shows how the total system demand evolves for the 100 samples. Therefore, system demand increases/decreases by up to 20 MW over the 5-minute interval which is consistent with practice. We construct the scenario tree for the economic dispatch problem in the following manner. We assume that the 5-minute interval is partitioned into $K = 20$ time periods; therefore, each time period lasts 15 seconds. We subsample the demand trajectories at 15-second intervals and assign equal probabilities to all subsampled trajectories. Therefore, the scenario tree is a tall tree, where the root node has 100 children, and all other nodes either have one child or is a leaf node.

Table I summarizes the properties of the generators on the system. We assume that hydro and combustion turbine (CT) generators are regulation resources while all other generators are dispatch resources.[5] There are 6 hydro units that each generate between 10 to 50 MW and 4 CT units that each generate between 16 to 20 MW. To satisfy the convergence conditions in Proposition 2, we assume that the distributed controllers for the hydro and CT resources are operated with cost functions as shown in Fig. 2. These are obtained by adding barrier functions to the original linear cost functions in the test system data. We assume a damping of 2.0 p.u. for all generators.

Recall that the demand evolution has zero mean. Therefore, in the current practice, the economic dispatch mechanism will

[5]Notice that certain nodes have only one type of generator or no generator. It is straightforward to extend Theorem 1 and Proposition 3 to such a setting. However, to extend Propositions 1 and 2 to such a setting requires the following modification to the DFR algorithm:

$$\dot{\boldsymbol{\pi}}^r_s(t) = \boldsymbol{\zeta}^\pi \left(\mathbf{q}^d + \mathbf{q}^r + \mathbf{r}^r_s(t) - \mathbf{d}_s - \boldsymbol{L}\boldsymbol{\phi}_s(t) \right); \tag{14a}$$

$$\dot{\bar{\boldsymbol{\mu}}}_s(t) = \boldsymbol{\zeta}^{\bar{\mu}} \left[\mathbf{B}\mathbf{C}^\top \boldsymbol{\phi}_s(t) - \mathbf{f} \right]^+_{\bar{\boldsymbol{\mu}}_s}; \tag{14b}$$

$$\dot{\underline{\boldsymbol{\mu}}}_s(t) = \boldsymbol{\zeta}^{\underline{\mu}} \left[-\mathbf{f} - \mathbf{B}\mathbf{C}^\top \boldsymbol{\phi}_s(t) \right]^+_{\underline{\boldsymbol{\mu}}_s}; \tag{14c}$$

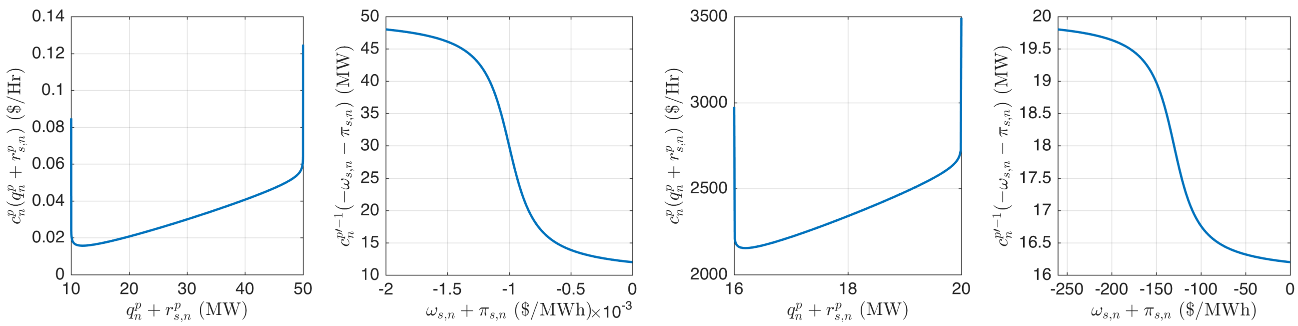
$$\dot{\boldsymbol{\phi}}_s(t) = \boldsymbol{\chi}^\phi \left(\mathbf{L}\boldsymbol{\pi}^p_s(t) - \mathbf{C}\mathbf{B} \left(\bar{\boldsymbol{\mu}}_s(t) - \underline{\boldsymbol{\mu}}_s(t) + \mathbf{B}\mathbf{C}^\top \boldsymbol{\phi}_s(t) - \boldsymbol{\rho}_s(t) \right) \right), \tag{14d}$$

$$\dot{\boldsymbol{\rho}}_s(t) = \boldsymbol{\chi}^\rho \left(\mathbf{B}\mathbf{C}^\top \boldsymbol{\phi}_s(t) - \boldsymbol{\rho}_s(t) \right), \tag{14e}$$

where $\rho_{s,e}$ is a new state variable associated with each line. The new algorithm (14) is equivalent to modifying FR' by adding $\frac{1}{2}||\boldsymbol{\rho} - \mathbf{B}\mathbf{C}^\top \boldsymbol{\phi}||^2$ in the objective and adding $\boldsymbol{\rho}_s$ as a new optimization variable. While this change does not modify the optimal solution, it provides additional convexity that ensures convergence of the primal-dual algorithm when generators are not present at every bus.

TABLE I: Generators on test system.

Unit Group	Unit Type	Number	Production Range (MW)	Marginal Cost Range ($/MWh)	Assignment
U12	Oil/Steam	5	[2.4, 12]	[58.14, 64.446]	Dispatch
U20	Oil/CT	4	[16, 20]	See Fig. 2	Regulation
U50	Hydro	6	[10, 50]	See Fig. 2	Regulation
U76	Coal/Steam	4	[15.2, 76]	[16.511, 18.231]	Dispatch
U100	Oil/Steam	3	[25, 100]	[46.295, 54.196]	Dispatch
U155	Coal/Steam	4	[54.3, 155]	[13.294, 14.974]	Dispatch
U197	Oil/Steam	3	[69, 197]	[49.57, 51.405]	Dispatch
U350	Coal/Steam	1	[140, 350]	[13.22, 15.276]	Dispatch
U400	Nuclear	2	[100, 400]	[4.466, 4.594]	Dispatch



(a) Hydro cost (b) Hydro control (c) CT cost (d) CT control

Fig. 2: Regulation cost and control functions.

be cleared based on the demand at time $t = 0$. Our simulations will reveal that the current practice is suboptimal. To simulate the current practice, we run $ED(\mathbf{d}_1)$ with $\delta_n = 0$, and assume all regulation resources run the standard automatic generation control (AGC) [3]:

$$ACE_s(t) = \frac{1}{N}\mathbf{1}^\top \boldsymbol{\omega}_s(t), \tag{15a}$$

$$\dot{q}_s^{\mathsf{imb}}(t) = -ACE(t), \tag{15b}$$

$$\mathbf{r}_s^r(t) = \frac{q_s^{\mathsf{imb}}(t)}{\mathbf{1}^\top \mathbf{q}^r} \cdot \mathbf{q}^r. \tag{15c}$$

Therefore, we assume that the entire network is one area and there is zero net inter-area flow. In practice, AGC signals are sent to the generators every few seconds. However, in our simulations, we assume that these signals are sent continuously as it is not our focus to study the impact of control delays. We assume that each regulation resource reserves 10% of its capacity for regulation service. Therefore, the hydro dispatch ranges from 12.5 to 47.5 MW and the CT dispatch ranges from 17 to 19 MW. This provides a total regulation capacity of 19 MW in both directions (up and down). Since the maximum change in demand over the 5-minute interval is about 20 MW, in the worst case scenario, all regulation capacity will be used.

To focus the simulation on the gains due to efficient use of regulation resources, we assume that at time $t = 0$, the generators are operating at the solution of the economic dispatch problem (this is implemented by starting the simulation at a large negative time $t' < 0$, but using the demand at time $t = 0$, where t' is sufficiently negative such that the dynamics have converged by time $t = 0$. Fig. 4 and 5 show, for one demand trajectory, the evolution of the frequency at the bus where the hydro resources are located, using the AGC and DFR mechanisms proposed in Section IV, respectively. Observe that both mechanisms are able to rebalance power and maintain the nominal frequency. In fact, for this example, AGC regulates frequency more successfully than DFR.

Fig. 6 shows the evolution of the prices in DFR. Unlike AGC, which compensates frequency regulation based on the LMP in the most recent economic dispatch run, the prices in DFR adjusts dynamically to reflect real-time and local conditions in the power system. Fig. 7 and 8 show an example of hydro and CT production. These figures illustrate the inefficiency of AGC – it is constrained to the usage of static participation factors that do not take into account generators' capacity constraints and line congestion. Therefore, AGC is unable to utilize the regulation reserves efficiently. Although hydro is significantly cheaper than CT, the system under AGC is unable to substitute hydro for CT due to the static participation factors. On the other hand, under DFR, the system substitutes hydro for CT dynamically to reduce costs.

Next, we illustrate the potential monetary savings that can be obtained under DFR compared to AGC. Fig. 9 shows a histogram of the percentage reduction in the costs of hydro and CT generation under DFR; Fig. 10 shows a histogram of the percentage reduction in the costs of non-hydro and non-CT generation under DFR; and Fig. 11 shows a histogram of the percentage reduction in total generation costs under DFR. Observe that DFR reduces hydro and CT costs by an average of 2.5% due to more efficient usage of regulation resources in real-time. Moreover, DFR also reduces dispatch costs of non-hydro and non-CT resources by 0.7% due to more efficient dispatch of those resources and avoiding the need to reserve

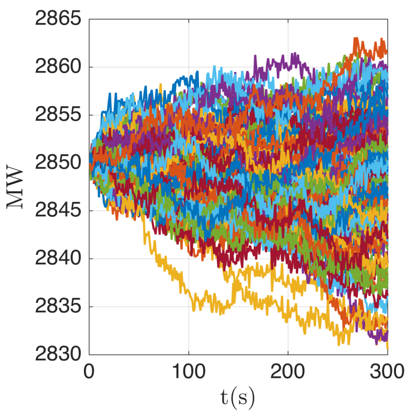


Fig. 3: Demand Processes

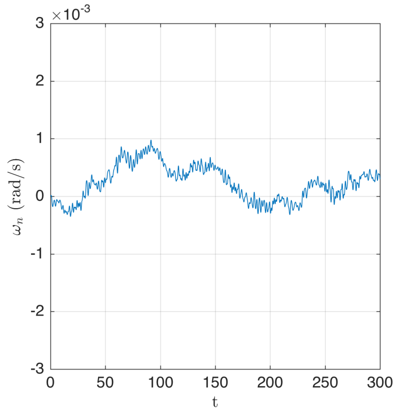


Fig. 4: Frequency evolution: AGC

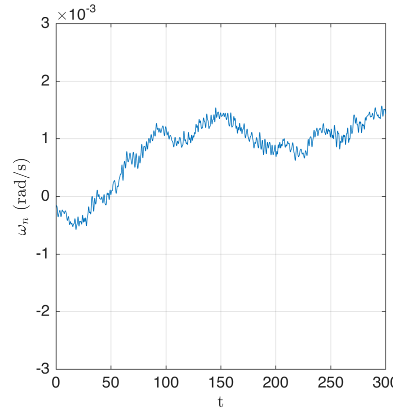


Fig. 5: Frequency evolution: DFR

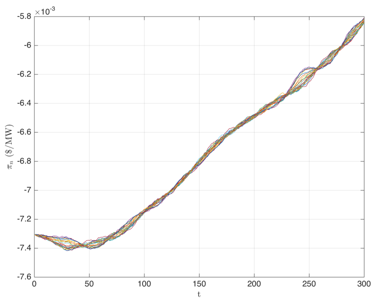


Fig. 6: Price evolution: DFR

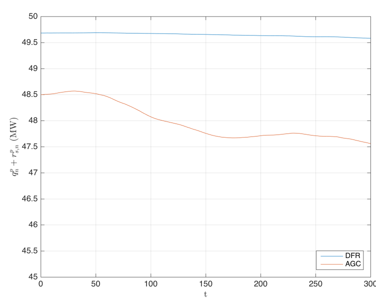


Fig. 7: An example of hydro production

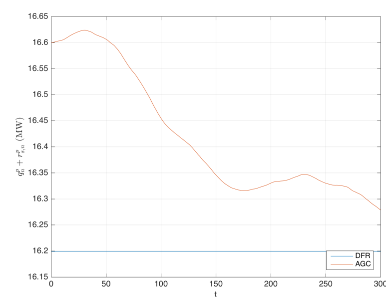


Fig. 8: An example of CT production

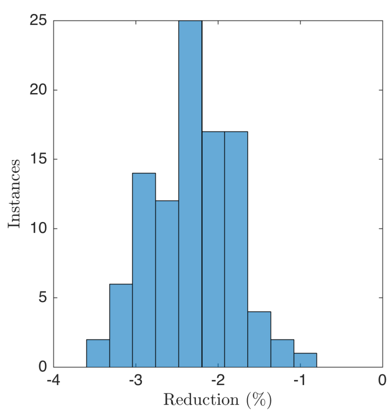


Fig. 9: Histogram of reduction in costs of hydro and CT generation under DFR

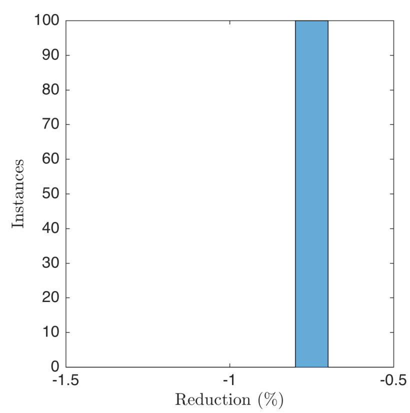


Fig. 10: Histogram of reduction in costs of non-hydro and non-CT generation under DFR

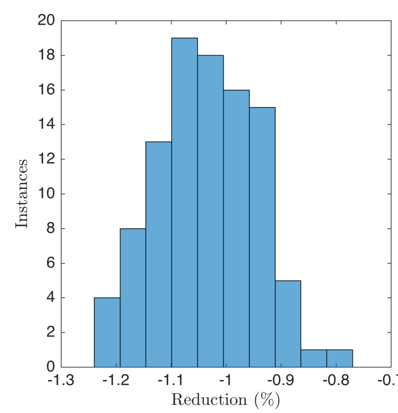


Fig. 11: Histogram of reduction in total costs under DFR

capacity for regulation. Since hydro and CT costs comprise on average 17.5% of total costs, the net savings on all generation costs is an average of 1%. There are also further savings in capacity costs that may be estimated at about 0.35% (based on the fact that CAISO's ancillary costs in 2015 is 0.7% of energy costs and about half of ancillary costs is attributable to regulation service [47]). Further studies should be performed on other systems with different mix of generation resources. In addition, recall that DFR has the added benefit of converging to operating points that respect line limits, while AGC does not guarantee this.

VII. CONCLUSION

This paper proposes an optimization decomposition approach for co-optimizing economic dispatch and frequency regulation resources. It demonstrates that optimization decomposition provides a rigorous way to design power system operations to allocate resources efficiently across timescales. Our main result, in Theorem 1, shows one way to choose generation setpoints optimally at the economic dispatch timescale, and provides a guide on how to design a principled architecture for power system operations. In particular, using this result, we design an optimal frequency control scheme and an optimal economic dispatch mechanism, both of which differ from existing approaches in crucial ways and reveal potential inefficiencies in the latter. Hence, this paper underscores the need to jointly analyze economic dispatch and frequency regulation mechanisms when investigating the efficiency of the overall system.

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NOMENCLATURE

A. Sets and Indices

S	Set of outcomes ($s \in S$).
N	Set of nodes in the network ($n \in N$).
L	Set of links in the network ($l \in L$).
K	Number of discrete time periods in one economic dispatch inteval ($k = 1, \ldots, K$).

B. Parameters

$\kappa(s)$	Period associated with outcome s.
p_s	Probability of outcome s given that period is $\kappa(s)$.
$d_{s,n}$	Real power demand at node n in outcome s.
c_n^d	Cost function of dispatch generator n.
c_n^r	Cost function of regulation generator n.
B_l	Sensitivity of flow on line l with respect to phase difference between its buses.
$\mathbf{C}$	Incidence matrix of network.
$\mathbf{H}$	Matrix of shift factors.
f_l	Capacity of line l.
$\underline{q}_n^d, \bar{q}_n^d$	Minimum and maximum generation limits of dispatch generator n.
$\underline{q}_n^r, \bar{q}_n^r$	Minimum and maximum generation limits of regulation generator n.
M_n	Aggregate inertia of generators at node n.
D_n	Aggregate damping of generators at node n.
$\zeta_n^\pi, \zeta_l^{\bar{\mu}}, \zeta_l^{\underline{\mu}}, \chi_n^\phi$	Control gains in distributed frequency regulation algorithm.
s_n^d	Basis supply function of dispatch generator n. Specifies quantity as a function of price.
s_n^r	Basis supply function of regulation generator n. Specifies quantity as a function of price.

C. Variables

q_n^d	Setpoint of dispatch generator n.
q_n^r	Setpoint of regulation generator n.
$r_{n,s}^r$	Recourse of regulation generator n in outcome s.
$\theta_{s,i}$	Phase at bus i in outcome s.
λ_s	Lagrange multiplier associated with demand-supply constraint in outcome s.
$\underline{\mu}_{s,l}, \bar{\mu}_{s,l}$	Lagrange multipliers associated with line-flow constraint in outcome s.
$\omega_{s,n}$	Frequency deviations from nominal.
$\pi_{s,n}$	Locational marginal price at node n in outcome s.
α_n^d	Bid of dispatch generator n. Indicates generator is willing to supply $\left[\alpha_n^d s_n^d(\pi_n^d)\right]_{\underline{q}_n^d}^{\bar{q}_n^d}$ at price π_n^d.
α_n^r	Bid of dispatch generator n. Indicates generator is willing to supply $[\alpha_n^r s_n^r(\pi_n^r)]_{\underline{q}_n^r}^{\bar{q}_n^r}$ at price π_n^r.

APPENDIX

Proof of Theorem 1. The result follows from analyzing the Karush-Kuhn-Tucker (KKT) conditions of $SYSTEM$, ED, and FR. However, we first reformulate the problems as the notations are simpler with the reformulations. Define $\mathbf{q}_s^r := \mathbf{q}^r + \mathbf{r}_s^r$. Note that, due to the constraint that $\mathbf{r}_1^r = \mathbf{0}$, there is a bijection between the set of feasible $(\mathbf{q}^d, \mathbf{q}^r, \mathbf{r}^r)$ and the set of feasible $(\mathbf{q}^d, \mathbf{q}_1^r, \ldots, \mathbf{q}_S^r)$. Hence, $SYSTEM$ can be reformulated as:

$$\begin{aligned} \min_{\mathbf{q}^d, \mathbf{q}_1^r, \ldots, \mathbf{q}_S^r} \quad & \sum_{s \in S} p_s \sum_{n \in N} \left(c_n^d(q_n^d) + c_n^r(q_{s,n}^r) \right) \\ \text{s.t.} \quad & (\mathbf{q}^d, \mathbf{q}_1^r, \mathbf{q}_s^r - \mathbf{q}_1^r) \in \Omega(\mathbf{d}_s), \quad \forall s \in S. \end{aligned} \tag{16}$$

Also, $ED(\mathbf{d}_1)$ can be reformulated as:

$$\begin{aligned} \min_{\mathbf{q}^d, \mathbf{q}_1^r} \quad & \sum_{n \in N} \left(K c_n^d(q_n^d) + K c_n^r(q_{1,n}^r) - \delta_n q_n^d \right) \\ \text{s.t.} \quad & (\mathbf{q}^d, \mathbf{q}_1^r, \mathbf{0}) \in \Omega(\mathbf{d}_1). \end{aligned} \tag{17}$$

And, $FR(\mathbf{q}^d, \mathbf{q}^r, \mathbf{d}_s)$ can be reformulated as:

$$\begin{aligned} \min_{\mathbf{q}_s^r} \quad & \sum_{n \in N} c_n^r(q_{s,n}^r) \\ \text{s.t.} \quad & (\mathbf{q}^d, \mathbf{q}_1^r, \mathbf{q}_s^r - \mathbf{q}_1^r) \in \Omega(\mathbf{d}_s). \end{aligned} \tag{18}$$

Hence, $SYSTEM$ can be optimally decomposed into ED-FR if $(\mathbf{q}^d, \mathbf{q}_1^r, \ldots, \mathbf{q}_S^r)$ is an optimal solution to (16) if and only if $(\mathbf{q}^d, \mathbf{q}_1^r)$ is an optimal solution to (17) and $\mathbf{q}_s^r$ is an optimal solution to (18) for all $s \in S$.

Next, we prove (a). It is easy to see that (16) has compact sub-level sets. Moreover, its objective function is strictly convex. Hence, (16) has a unique optimal solution. By similar arguments, we conclude that (17) has a unique optimal solution, and that (18) has a unique optimal solution if the set $\left\{\mathbf{q}_s^r \in \mathbb{R}^N : (\mathbf{q}^d, \mathbf{q}_1^r, \mathbf{q}_s^r - \mathbf{q}_1^r) \in \Omega(\mathbf{d}_s)\right\}$ is non-empty. Hence, to prove (a), it suffices to show the forward implication, that is, if (7) holds, then $(\mathbf{q}^d, \mathbf{q}_1^r, \ldots, \mathbf{q}_S^r)$ is an optimal solution to (16) implies that $(\mathbf{q}^d, \mathbf{q}_1^r)$ is an optimal solution to (17) and $\mathbf{q}^r_s$ is an optimal solution to (18) for all $s \in S$. The reverse implication follows from the existence and uniqueness of the optimal solutions.

Let the Lagrangian of (16) be denoted by:

$$\begin{aligned} &L(\mathbf{q}^d, \mathbf{q}^r_1, \dots, \mathbf{q}^r_S, \underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}, \underline{\boldsymbol{\nu}}, \bar{\boldsymbol{\nu}}, \underline{\boldsymbol{\mu}}, \bar{\boldsymbol{\mu}}, \boldsymbol{\lambda}) \\ &:= \sum_{s\in S} p_s \sum_{n\in N} \left(c^d_n(q^d_n) + c^r_n(q^r_{s,n})\right) + L^d(\mathbf{q}^d, \underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}) \\ &\quad + \sum_{s\in S} p_s L^r(\mathbf{q}^r_s, \underline{\boldsymbol{\nu}}_s, \bar{\boldsymbol{\nu}}_s) + \sum_{s\in S} p_s L^f(\mathbf{q}^d, \mathbf{q}^r_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) \\ &\quad - \sum_{s\in S} p_s \lambda_s \mathbf{1}^\top \left(\mathbf{q}^d + \mathbf{q}^r_s - \mathbf{d}_s\right), \end{aligned}$$

where:

$$\begin{aligned} L^d(\mathbf{q}^d, \underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}) &:= \underline{\boldsymbol{\xi}}^\top \left(\underline{\mathbf{q}}^d - \mathbf{q}^d\right) + \bar{\boldsymbol{\xi}}^\top \left(\mathbf{q}^d - \bar{\mathbf{q}}^d\right) \\ L^r(\mathbf{q}^r_s, \underline{\boldsymbol{\nu}}_s, \bar{\boldsymbol{\nu}}_s) &:= \underline{\boldsymbol{\nu}}_s^\top \left(\underline{\mathbf{q}}^r - \mathbf{q}^r_s\right) + \bar{\boldsymbol{\nu}}_s^\top \left(\mathbf{q}^r_s - \bar{\mathbf{q}}^r\right) \\ L^f(\mathbf{q}^d, \mathbf{q}^r_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) &:= \underline{\boldsymbol{\mu}}_s^\top \left(-\mathbf{f} - \mathbf{H}\left(\mathbf{q}^d + \mathbf{q}^r_s - \mathbf{d}_s\right)\right) \\ &\quad + \bar{\boldsymbol{\mu}}_s^\top \left(\mathbf{H}\left(\mathbf{q}^d + \mathbf{q}^r_s - \mathbf{d}_s\right) - \mathbf{f}\right). \end{aligned}$$

Note that we scaled the constraints by their probabilities, and $\underline{\boldsymbol{\xi}} \in \mathbb{R}^N_+$, $\bar{\boldsymbol{\xi}} \in \mathbb{R}^N_+$, $\underline{\boldsymbol{\nu}} = (\underline{\boldsymbol{\nu}}_s, s \in S) \in \mathbb{R}^{NS}_+$, $\bar{\boldsymbol{\nu}} = (\bar{\boldsymbol{\nu}}_s, s \in S) \in \mathbb{R}^{NS}_+$, $\underline{\boldsymbol{\mu}} = (\underline{\boldsymbol{\mu}}_s, s \in S) \in \mathbb{R}^{LS}_+$, $\bar{\boldsymbol{\mu}} = (\bar{\boldsymbol{\mu}}_s, s \in S) \in \mathbb{R}^{LS}_+$, $\boldsymbol{\lambda} = (\lambda_s, s \in S) \in \mathbb{R}^S$ are appropriate Lagrange multipliers.

Since (16) has a convex objective and linear constraints, from the KKT conditions, we infer that $(\mathbf{q}^d, \mathbf{q}^r_1, \dots, \mathbf{q}^r_S)$ is an optimal solution to (16) if and only if $(\mathbf{q}^d, \mathbf{q}^r_1, \mathbf{q}^r_s - \mathbf{q}^r_1) \in \Omega(\mathbf{d}_s)$ for all $s \in S$ and there exists $\underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}} \in \mathbb{R}^N_+, \underline{\boldsymbol{\nu}}, \bar{\boldsymbol{\nu}} \in \mathbb{R}^{NS}_+, \underline{\boldsymbol{\mu}}, \bar{\boldsymbol{\mu}} \in \mathbb{R}^{LS}_+, \boldsymbol{\lambda} \in \mathbb{R}^S$ such that:

$$\left(K c^{d\prime}_n(q^d_n), n \in N\right) + \bar{\boldsymbol{\xi}} - \underline{\boldsymbol{\xi}} - \sum_{s\in S} p_s \boldsymbol{\pi}(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) = 0; \quad \text{(19a)}$$

$$L^d(\mathbf{q}^d, \underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}) = 0; \quad \text{(19b)}$$

$$\left(c^{r\prime}_n(q^r_{s,n}), n \in N\right) + \bar{\boldsymbol{\nu}}_s - \underline{\boldsymbol{\nu}}_s - \boldsymbol{\pi}(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) = 0; \quad \text{(19c)}$$

$$L^r(\mathbf{q}^r_s, \underline{\boldsymbol{\nu}}_s, \bar{\boldsymbol{\nu}}_s) = 0; \quad \text{(19d)}$$

$$L^f(\mathbf{q}^d, \mathbf{q}^r_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) = 0, \quad \text{(19e)}$$

for all $s \in S$.

Similarly, $(\mathbf{q}^d, \mathbf{q}^r_1)$ is an optimal solution to (17) if and only if $(\mathbf{q}^d, \mathbf{q}^r_1, \mathbf{0}) \in \Omega(\mathbf{d}_1)$ and there exists $\underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}} \in \mathbb{R}^N_+, \underline{\boldsymbol{\nu}}_1, \bar{\boldsymbol{\nu}}_1 \in \mathbb{R}^N_+, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1 \in \mathbb{R}^L_+, \lambda_1 \in \mathbb{R}$ such that:

$$\left(K c^{d\prime}_n(q^d_n), n \in N\right) + \bar{\boldsymbol{\xi}} - \underline{\boldsymbol{\xi}} - \boldsymbol{\pi}(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1) - \boldsymbol{\delta} = 0; \quad \text{(20a)}$$

$$L^d(\mathbf{q}^d, \underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}) = 0; \quad \text{(20b)}$$

$$\left(K c^{r\prime}_n(q^r_{1,n}), n \in N\right) + \bar{\boldsymbol{\nu}}_1 - \underline{\boldsymbol{\nu}}_1 - \boldsymbol{\pi}(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1) = 0; \quad \text{(20c)}$$

$$L^r(\mathbf{q}^r_1, \underline{\boldsymbol{\nu}}_1, \bar{\boldsymbol{\nu}}_1) = 0; \quad \text{(20d)}$$

$$L^f(\mathbf{q}^d, \mathbf{q}^r_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1) = 0. \quad \text{(20e)}$$

And $\mathbf{q}^r_s$ is an optimal solution to (18) if and only if $(\mathbf{q}^d, \mathbf{q}^r_1, \mathbf{q}^r_s - \mathbf{q}^r_1) \in \Omega(\mathbf{d}_s)$ and there exists $\underline{\boldsymbol{\nu}}_s, \bar{\boldsymbol{\nu}}_s \in \mathbb{R}^N_+, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s \in \mathbb{R}^L_+, \lambda_s \in \mathbb{R}$ such that:

$$\left(c^{r\prime}_n(q^r_{s,n}), n \in N\right) + \bar{\boldsymbol{\nu}}_s - \underline{\boldsymbol{\nu}}_s - \boldsymbol{\pi}(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) = 0; \quad \text{(21a)}$$

$$L^p(\mathbf{q}^r_s, \underline{\boldsymbol{\nu}}_s, \bar{\boldsymbol{\nu}}_s) = 0; \quad \text{(21b)}$$

$$L^f(\mathbf{q}^d, \mathbf{q}^r_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) = 0. \quad \text{(21c)}$$

Suppose $(\mathbf{q}^d, \mathbf{q}^r_1, \dots, \mathbf{q}^r_S)$ is an optimal solution to (16) with associated Lagrange multipliers $(\underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}, \underline{\boldsymbol{\nu}}, \bar{\boldsymbol{\nu}}, \underline{\boldsymbol{\mu}}, \bar{\boldsymbol{\mu}}, \boldsymbol{\lambda})$. Note that $(\mathbf{q}^d, \mathbf{q}^r_1, \mathbf{0}) \in \Omega(\mathbf{d}_1)$. From the fact that the variables $(\mathbf{q}^d, \underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}, \underline{\boldsymbol{\mu}}, \bar{\boldsymbol{\mu}}, \lambda)$ satisfy (19a) and (7) and the fact that $\sum_{s\in S} p_s = K$, we infer that the variables $(\mathbf{q}^d, \underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}, K\underline{\boldsymbol{\mu}}_1, K\bar{\boldsymbol{\mu}}_1, K\lambda_1)$ satisfy (20a). From the fact that $(\mathbf{q}^d, \mathbf{q}^r_s, \bar{\boldsymbol{\xi}}, \underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\nu}}_s, \underline{\boldsymbol{\nu}}_s, \bar{\boldsymbol{\mu}}_s, \underline{\boldsymbol{\mu}}_s, \lambda_s)$ satisfy (19b) – (19e), we infer that the variables $(\mathbf{q}^d, \mathbf{q}^r_1, \bar{\boldsymbol{\xi}}, \underline{\boldsymbol{\xi}}, K\bar{\boldsymbol{\nu}}_1, K\underline{\boldsymbol{\nu}}_1, K\bar{\boldsymbol{\mu}}_1, K\underline{\boldsymbol{\mu}}_1, K\lambda_1)$ satisfy (20b) – (20e). Hence, $(\mathbf{q}^d, \mathbf{q}^r_1)$ is an optimal solution to (17). Note also that $(\mathbf{q}^d, \mathbf{q}^r_1, \mathbf{q}^r_s - \mathbf{q}^r_1) \in \Omega(\mathbf{d}_s)$ for all $s \in S$. From the fact that the variables $(\mathbf{q}^d, \mathbf{q}^r_s, \underline{\boldsymbol{\nu}}_s, \bar{\boldsymbol{\nu}}_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s, \lambda_s)$ satisfy (19c) – (19e), we infer that those variables satisfy (21). Hence, $\mathbf{q}^r_s$ is an optimal solution to (18) for all $s \in S$.

Next, we prove (b). Let $(\mathbf{q}^d, \mathbf{q}^r_1, \dots, \mathbf{q}^r_S)$ be a solution to (16) such that $(\mathbf{q}^d, \mathbf{q}^r_1)$ is a solution to (17). If $\underline{q}^d_n < q^d_n < \bar{q}^d_n$ and $\underline{q}^r_n < q^r_{1,n} < \bar{q}^r_n$, then the complementary slackness conditions imply that $\underline{\xi}_n = \bar{\xi}_n = 0$ and $\underline{\nu}_{1,n} = \bar{\nu}_{1,n} = 0$. From the KKT conditions of (16), which are given by (19), we infer that:

$$K c^{d\prime}_n(q^d_n) - \sum_{s\in S} p_s \pi_n(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) = 0; \quad \text{(22)}$$

$$c^{r\prime}_n(q^r_{1,n}) - \pi_n(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1) = 0, \quad \text{(23)}$$

where $(\underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s, \boldsymbol{\lambda})$ are the associated Lagrange multipliers. From the KKT conditions of (17), which are given by (20), we infer that:

$$K c^{d\prime}_n(q^d_n) - \pi_n(\lambda'_1, \underline{\boldsymbol{\mu}}'_1, \bar{\boldsymbol{\mu}}'_1) - \delta_n = 0; \quad \text{(24)}$$

$$K c^{r\prime}_n(q^r_{1,n}) - \pi_n(\lambda'_1, \underline{\boldsymbol{\mu}}'_1, \bar{\boldsymbol{\mu}}'_1) = 0, \quad \text{(25)}$$

where $(\underline{\boldsymbol{\mu}}'_s, \bar{\boldsymbol{\mu}}'_s, \boldsymbol{\lambda}')$ are the associated Lagrange multipliers. It follows that:

$$\begin{aligned} \delta_n &= \sum_{s\in S} p_s \pi_n(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) - \pi_n(\lambda'_1, \underline{\boldsymbol{\mu}}'_1, \bar{\boldsymbol{\mu}}'_1) \\ &= \sum_{s\in S} p_s \pi_n(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) - K\pi_n(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1) \\ &= \sum_{s\in S} p_s \left(\pi_n(\lambda_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) - \pi_n(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1)\right). \end{aligned}$$

The first equality follows from comparing (22) and (24). The second equality follows from comparing (23) and (25). The last equality follows from the fact that $\sum_{s\in S} p_s = K$. □

Proof of Proposition 1. We provide a proof sketch of this result. The skipped details can be found in [21]. (i) follows from the KKT conditions of $FR'(\mathbf{q}^d, \mathbf{q}^r, \mathbf{d}_s)$ and is shown in [21, Lemma 2]. Since $\boldsymbol{\omega}'_s = 0$, it follows from constraints (11a) and (11b) of $FR'(\mathbf{q}^d, \mathbf{q}^r, \mathbf{d}_s)$ that $\mathbf{L}\boldsymbol{\theta}'_s = \mathbf{L}\boldsymbol{\phi}'_s$, which, since the null space of $\mathbf{L}$ is span$\{\mathbf{1}\}$, implies that $\boldsymbol{\theta}'_s = \boldsymbol{\phi}'_s + \alpha\mathbf{1}$ for some $\alpha \in \mathbb{R}$. This implies that $\mathbf{B}\mathbf{C}^\top\boldsymbol{\phi}'_s = \mathbf{B}\mathbf{C}^\top\boldsymbol{\theta}'_s$. Therefore, without loss of generality, we can substitute constraint (11a) in $FR'(\mathbf{q}^d, \mathbf{q}^r, \mathbf{d}_s)$ by the constraint $\boldsymbol{\omega}_s = 0$. Then, using the definition of $\mathbf{H}$ and the equivalence between (3) and (4), we infer that the feasible sets of $FR(\mathbf{q}^d, \mathbf{q}^r, \mathbf{d}_s)$ and $FR'(\mathbf{q}^d, \mathbf{q}^r, \mathbf{d}_s)$ are equivalent. Finally, since $c^r_n(\cdot)$ is strictly convex, by uniqueness of the optimal solutions, we get (ii). Lastly, (iii) follows from the definition of $\mathbf{H}$ and $\mathbf{B}\mathbf{C}^\top\boldsymbol{\phi}'_s = \mathbf{B}\mathbf{C}^\top\boldsymbol{\theta}'_s$. The final statement of the proposition follows directly from [21, Theorem 8]. □

Proof of Proposition 3. Our proof proceeds in 6 steps: (1) Characterizing regulation generators' optimal bids $\boldsymbol{\alpha}^r$ given their prices $\boldsymbol{\pi}^r$; (2) Characterizing dispatch generators' optimal bids $\boldsymbol{\alpha}^d$ given their prices $\boldsymbol{\pi}^d$; (3) Characterizing prices $(\boldsymbol{\pi}^d, \boldsymbol{\pi}^r)$ given bids $(\boldsymbol{\alpha}^d, \boldsymbol{\alpha}^r)$ using KKT conditions; (4) Showing that, at an equilibrium, the production schedule is the unique optimal solution to $\hat{ED}\text{-}\hat{FR}$; (5) Showing that any production schedule $(\mathbf{q}^d, \mathbf{q}^r, \mathbf{r}^r)$ that solves $SYSTEM$ can be obtained using bids $(\boldsymbol{\gamma}^d, \boldsymbol{\gamma}^r)$ and the latter satisfy the equilibrium characterizations in steps 1 to 3; and (6) Showing that any bids $(\boldsymbol{\alpha}^d, \boldsymbol{\alpha}^r)$ that satisfy the equilibrium characterizations in steps 1 to 3 give the same production schedule as that under bids $(\boldsymbol{\gamma}^d, \boldsymbol{\gamma}^r)$ (which also solves $SYSTEM$). Note that part (a) follows from step 6 and part (b) follows from step 5.

Step 1: Characterizing regulation generators' optimal bids $\boldsymbol{\alpha}^r$ given their prices $\boldsymbol{\pi}^r$. Since c_n^r is strictly convex and $c_n^r(q_{s,n}^r) \to +\infty$ as $q_{s,n}^r \to \{\underline{q}_n^r, \bar{q}_n^r\}$, $c_n^{r\prime}$ is invertible. Let $\sigma : S \to S$ be any permutation function that satisfies:

$$c_n^{r\prime-1}(\pi_{\sigma(1),n}^r) \le c_n^{r\prime-1}(\pi_{\sigma(2),n}^r) \le \ldots \le c_n^{r\prime-1}(\pi_{\sigma(S),n}^r),$$

and let integers $i, j \in \{0, 1, \ldots, S\}$ be such that:

$$c_n^{r\prime-1}(\pi_{\sigma(s),n}^r) \le \underline{q}_n^r \qquad \forall s = 1, \ldots, i; \tag{26a}$$
$$\underline{q}_n^r < c_n^{r\prime-1}(\pi_{\sigma(s),n}^r) < \bar{q}_n^r \qquad \forall s = i+1, \ldots, j; \tag{26b}$$
$$\bar{q}_n^r \le c_n^{r\prime-1}(\pi_{\sigma(s),n}^r) \qquad \forall s = j+1, \ldots, S. \tag{26c}$$

We now show that $\alpha_n^r \in \mathbb{R}_{++}$ maximizes PF_n^r if and only if:

$$\alpha_n^r s_n^r(\pi_{\sigma(s),n}^r) \le \underline{q}_n^r \qquad \forall s = 1, \ldots, i; \tag{27a}$$
$$\alpha_n^r s_n^r(\pi_{\sigma(s),n}^r) = c_n^{r\prime-1}(\pi_{\sigma(k),n}^r) \qquad \forall s = i+1, \ldots, j; \tag{27b}$$
$$\alpha_n^r s_n^r(\pi_{\sigma(s),n}^r) \ge \bar{q}_n^r \qquad \forall s = j+1, \ldots, S. \tag{27c}$$

For notational brevity, in the rest of this step, we abuse notation and let:

$$q_{s,n}^r(\alpha_n^r) = [\alpha_n^r s_n^r(\pi_{\sigma(s),n}^r)]_{\underline{q}_n^r}^{\bar{q}_n^r}.$$

To prove our characterization, it suffices to show that, given any $\alpha_n^r \in \mathbb{R}_{++}$ that satisfies (27), the vector of per-outcome profits

$$\begin{aligned}&\left(\pi_{\sigma(s),n}^r q_{s,n}^r(\alpha_n^r) - c_n^r\left(q_{s,n}^r(\alpha_n^r)\right), s \in S\right)\\ &\gneq \left(\pi_{\sigma(s),n}^r q_{s,n}^r(\bar{\alpha}_n^r) - c_n^r\left(q_{s,n}^r(\bar{\alpha}_n^r)\right), s \in S\right)\end{aligned} \tag{28}$$

for any $\bar{\alpha}_n^p$ that does not satisfy (27). Since $p_{\sigma(s)} > 0$ for all $s \in S$, it then follows that:

$$\begin{aligned}\text{PF}_n^r|_{\alpha_n^r} &= \sum_s p_{\sigma(s)}\left(\pi_{\sigma(s),n}^r q_{s,n}^r(\alpha_n^r) - c_n^r\left(q_{s,n}^r(\alpha_n^r)\right)\right)\\ &> \sum_s p_{\sigma(s)}\left(\pi_{\sigma(s),n}^r q_{s,n}^r(\bar{\alpha}_n^r) - c_n^r\left(q_{s,n}^r(\bar{\alpha}_n^r)\right)\right)\\ &= \text{PF}_n^r|_{\bar{\alpha}_n^r}.\end{aligned}$$

Suppose $s \in \{1, \ldots, i\}$. From (26a) and the fact that c_n^r is strictly convex, we infer that $\pi_{\sigma(s),n}^r \le c_n^{r\prime}(\underline{q}_n^r)$. From (27a), we infer that $q_{s,n}^r(\alpha_n^r) = \underline{q}_n^r$. Then:

$$\begin{aligned}&c_n^r(q_{s,n}^r(\bar{\alpha}_n^r))\\ &\ge c_n^r(\underline{q}_n^r) + c_n^{r\prime}(\underline{q}_n^r)\left(q_{s,n}^r(\bar{\alpha}_n^r) - \underline{q}_n^r\right)\\ &\ge c_n^r(\underline{q}_n^r) + \pi_{\sigma(s),n}^r\left(q_{s,n}^r(\bar{\alpha}_n^r) - \underline{q}_n^r\right)\\ &= c_n^r(q_{s,n}^r(\alpha_n^r)) + \pi_{\sigma(s),n}^r\left(q_{s,n}^r(\bar{\alpha}_n^r) - q_{s,n}^r(\alpha_n^r)\right),\end{aligned}$$

where the first inequality follows from the fact that c_n^r is strictly convex, the second inequality follows from $\pi_{\sigma(s),n}^r \le c_n^{r\prime}(\underline{q}_n^r)$ and $q_{s,n}^r(\bar{\alpha}_n^r) \ge \underline{q}_n^r$, and the last equality follows from $q_{s,n}^r(\alpha_n^r) = \underline{q}_n^r$. Furthermore, if $q_{s,n}^r(\bar{\alpha}_n^r) > \underline{q}_n^r$, then the first inequality is strict, and hence:

$$\begin{aligned}&c_n^r(q_{s,n}^r(\bar{\alpha}_n^r))\\ &> c_n^r(q_{s,n}^r(\alpha_n^r)) + \pi_{\sigma(s),n}^r\left(q_{s,n}^r(\bar{\alpha}_n^r) - q_{s,n}^r(\alpha_n^r)\right).\end{aligned}$$

Suppose $s \in \{i+1, \ldots, j\}$. From (26b) and (27b), we infer that $q_{s,n}^r(\alpha_n^r) = c_n^{r\prime-1}(\pi_{\sigma(s),n}^r)$ and $\underline{q}_n^r < q_{s,n}^r(\alpha_n^r) < \bar{q}_n^r$. From $\underline{q}_n^r < q_{s,n}^r(\alpha_n^r) < \bar{q}_n^r$, and the fact that $s_n^r(\pi_{\sigma(s),n}^r) \neq 0$ and $\bar{\alpha}_n^r \neq \alpha_n^r$, we infer that $q_{s,n}^r(\bar{\alpha}_n^r) \neq q_{s,n}^r(\alpha_n^r)$. Then:

$$\begin{aligned}&c_n^r(q_{s,n}^r(\bar{\alpha}_n^r))\\ &> c_n^r(q_{s,n}^r(\alpha_n^r)) + c_n^{r\prime}(q_{s,n}^r(\alpha_n^r))\left(q_{s,n}^r(\bar{\alpha}_n^r) - q_{s,n}^r(\alpha_n^r)\right)\\ &= c_n^r(q_{s,n}^r(\alpha_n^r)) + \pi_{\sigma(s),n}^r\left(q_{s,n}^r(\bar{\alpha}_n^r) - q_{s,n}^r(\alpha_n^r)\right),\end{aligned}$$

where the first inequality follows from the fact that c_n^r is strictly convex and $q_{s,n}^r(\bar{\alpha}_n^r) \neq q_{s,n}^r(\alpha_n^r)$ and the equality follows from $q_{s,n}^r(\alpha_n^r) = c_n^{r\prime-1}(\pi_{\sigma(s),n}^r)$.

Suppose $s \in \{i+1, \ldots, S\}$. From (26c) and the fact that c_n^r is strictly convex, we infer that $\pi_{\sigma(s),n}^r \ge c_n^{r\prime}(\bar{q}_n^r)$. From (27c), we infer that $q_{s,n}^r(\alpha_n^r) = \bar{q}_n^r$. Then:

$$\begin{aligned}&c_n^r(q_{s,n}^r(\bar{\alpha}_n^r))\\ &\ge c_n^r(\bar{q}_n^r) + c_n^{r\prime}(\bar{q}_n^r)\left(q_{s,n}^r(\bar{\alpha}_n^r) - \bar{q}_n^r\right)\\ &\ge c_n^r(\bar{q}_n^r) + \pi_{\sigma(s),n}^r\left(q_{s,n}^r(\bar{\alpha}_n^r) - \bar{q}_n^r\right)\\ &= c_n^r(q_{s,n}^r(\alpha_n^r)) + \pi_{\sigma(s),n}^r\left(q_{s,n}^r(\bar{\alpha}_n^r) - q_{s,n}^r(\alpha_n^r)\right),\end{aligned}$$

where the first inequality follows from the fact that c_n^r is strictly convex, the second inequality follows from $\pi_{\sigma(s),n}^r \ge c_n^{r\prime}(\bar{q}_n^r)$ and $q_{s,n}^r(\bar{\alpha}_n^r) \le \bar{q}_n^r$, and the last equality follows from $q_{s,n}^r(\alpha_n^r) = \bar{q}_n^r$. Furthermore, if $q_{s,n}^r(\bar{\alpha}_n^r) < \bar{q}_n^r$, then the first inequality is strict, and hence:

$$\begin{aligned}&c_n^r(q_{s,n}^r(\bar{\alpha}_n^r))\\ &> c_n^r(q_{s,n}^r(\alpha_n^r)) + \pi_{\sigma(s),n}^r\left(q_{s,n}^r(\bar{\alpha}_n^r) - q_{s,n}^r(\alpha_n^r)\right).\end{aligned}$$

Hence, for all $s \in S$:

$$\begin{aligned}&c_n^r(q_{s,n}^r(\bar{\alpha}_n^r))\\ &\ge c_n^r(q_{s,n}^r(\alpha_n^r)) + \pi_{\sigma(s),n}^r\left(q_{s,n}^r(\bar{\alpha}_n^r - q_{s,n}^r(\alpha_n^r)\right).\end{aligned} \tag{29}$$

Moreover, this inequality is strict for some $s \in S$. If $i < j$, the inequality is strict for $s \in \{i+1, \ldots, j\}$. If $i = j$, then, since $\bar{\alpha}_n^r$ does not satisfy (27), there exists some $s \in \{1, \ldots, i\}$ such that $\alpha_n^r s_n^r(\pi_{\sigma(s),n}^r) > \underline{q}_n^r$ or some $s \in \{i+1, \ldots, S\}$ such that $\alpha_n^r s_n^r(\pi_{\sigma(s),n}^r) < \bar{q}_n^r$, and hence there exists some $s \in \{1, \ldots, i\}$ such that $q_{s,n}^r(\bar{\alpha}_n^r) > \underline{q}_n^r$ or some $s \in \{i+1, \ldots, S\}$

such that $q^r_{s,n}(\bar{\alpha}^r_n) < \bar{q}^r_n$, and the inequality in (29) is strict for that s. Hence, we conclude that:

$$\begin{aligned}&\left(c^r_n(q^r_{s,n}(\bar{\alpha}^r_n)), s \in S\right)\\ &\gneq \left(c^r_n(q^r_{s,n}(\alpha^r_n)) + \pi^r_{\sigma(s),n}\left(q^r_{s,n}(\bar{\alpha}^r_n) - q^r_{s,n}(\alpha^r_n)\right), s \in S\right)\end{aligned}$$

for any $\bar{\alpha}^r_n$ that does not satisfy (27). By rearranging terms, we obtain (28).

Step 2: Characterizing dispatch generators' optimal bids $\boldsymbol{\alpha}^d$ given their prices $\boldsymbol{\pi}^d$. Note that the profit maximization problem for a dispatch generator is a special case of that for a regulation generator with $S = 1$. By applying the characterization in step 1, we infer that $\alpha^d_n \in \mathbb{R}_{++}$ maximizes PF^d_n if and only if:

$$\alpha^d_n s^d_n(\pi^d_n) \le \underline{q}^d_n, \qquad \text{if} \qquad c^{d\prime-1}_n(\pi^d_n) \le \underline{q}^d_n; \tag{30a}$$
$$\alpha^d_n = \gamma^d_n, \qquad \text{if } \underline{q}^d_n < c^{d\prime-1}_n(\pi^d_n) < \bar{q}^d_n; \tag{30b}$$
$$\alpha^d_n s^d_n(\pi^d_n) \ge \bar{q}^d_n, \qquad \text{if } \bar{q}^d_n \le c^{d\prime-1}_n(\pi^d_n). \tag{30c}$$

Step 3: Characterizing prices $(\boldsymbol{\pi}^d, \boldsymbol{\pi}^r)$ given bids $(\boldsymbol{\alpha}^d, \boldsymbol{\alpha}^r)$ using KKT conditions. First, we take the same approach as in the proof of Theorem 1 and reformulate $\hat{ED}$ and $\hat{FR}$ before applying the KKT conditions. Relabeling the variable $\mathbf{q}^r$ to $\mathbf{q}^r_1$ in $\hat{ED}$ gives:

$$\begin{aligned}\min_{\mathbf{q}^d,\mathbf{q}^r_1} \quad & \sum_{n\in N}\left(K\hat{c}^d_n(q^d_n) + K\hat{c}^r_n(q^r_{1,n}) - \delta_n q^d_n\right)\\ \text{s.t.} \quad & (\mathbf{q}^d, \mathbf{q}^r_1, \mathbf{0}) \in \Omega(\mathbf{d}_1).\end{aligned} \tag{31}$$

And substituting $\mathbf{q}^r_s = \mathbf{q}^r + \mathbf{r}^r_s$ in $\hat{FR}$ gives:

$$\begin{aligned}\min_{\mathbf{q}^r_s} \quad & \sum_{n\in N} \hat{c}^r_n(q^r_{s,n})\\ \text{s.t.} \quad & (\mathbf{q}^d, \mathbf{q}^r_1, \mathbf{q}^r_s - \mathbf{q}^r_1) \in \Omega(\mathbf{d}_s).\end{aligned} \tag{32}$$

Substituting $s^d_n = c^{d\prime-1}_n(\cdot)/\gamma^d_n$ and $s^r_n = c^{r\prime-1}_n(\cdot)/\gamma^r_n$ into the definition of $\hat{c}^d_n$ and $\hat{c}^r_n$ implies that:

$$\hat{c}^d_n(q^d_n) = \int_{\underline{q}^d_n}^{q^d_n} c^{d\prime}_n((\gamma^d_n/\alpha^d_n)w)\, dw,$$
$$\hat{c}^r_n(q^r_n) = \int_{\underline{q}^r_n}^{q^r_n} c^{r\prime}_n((\gamma^r_n/\alpha^r_n)w)\, dw.$$

Hence, (31) has a continuous and strictly convex objective and linear constraints. Thus, from the KKT conditions, $(\mathbf{q}^d, \mathbf{q}^r_1)$ is an optimal solution to (31) if and only if $(\mathbf{q}^d, \mathbf{q}^r_1, \mathbf{0}) \in \Omega(\mathbf{d}_1)$ and there exists $\underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}} \in \mathbb{R}^N_+, \underline{\boldsymbol{\nu}}_1, \bar{\boldsymbol{\nu}}_1 \in \mathbb{R}^N_+, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1 \in \mathbb{R}^L_+, \lambda_1 \in \mathbb{R}$ such that:

$$\left(Kc^{d\prime}_n((\gamma^d_n/\alpha^d_n)q^d_n), n \in N\right) + \bar{\boldsymbol{\xi}} - \underline{\boldsymbol{\xi}} - K\boldsymbol{\pi}^d = 0; \tag{33a}$$
$$L^d(\mathbf{q}^d, \underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}) = 0; \tag{33b}$$
$$\left(Kc^{r\prime}_n((\gamma^r_n/\alpha^r_n)q^r_{1,n}), n \in N\right) + \bar{\boldsymbol{\nu}}_1 - \underline{\boldsymbol{\nu}}_1 - K\boldsymbol{\pi}^r_1 = 0; \tag{33c}$$
$$L^r(\mathbf{q}^r_1, \underline{\boldsymbol{\nu}}_1, \bar{\boldsymbol{\nu}}_1) = 0; \tag{33d}$$
$$L^f(\mathbf{q}^d, \mathbf{q}^r_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1) = 0, \tag{33e}$$

where:

$$\boldsymbol{\pi}^d = (1/K)\left(\boldsymbol{\pi}(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1) + \boldsymbol{\delta}\right); \tag{33f}$$
$$\boldsymbol{\pi}^r_1 = (1/K)\boldsymbol{\pi}(\lambda_1, \underline{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_1). \tag{33g}$$

Similarly, from the KKT conditions, $\mathbf{q}^r_s$ is an optimal solution to (32) if and only if $(\mathbf{q}^d, \mathbf{q}^r_1, \mathbf{q}^r_s - \mathbf{q}^r_1) \in \Omega(\mathbf{d}_s)$ and there exists $\underline{\boldsymbol{\nu}}_s, \bar{\boldsymbol{\nu}}_s \in \mathbb{R}^N_+, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s \in \mathbb{R}^L_+, \lambda_s \in \mathbb{R}$ such that:

$$\left(c^{r\prime}_n((\gamma^r_n/\alpha^r_n)q^r_{s,n}), n \in N\right) + \bar{\boldsymbol{\nu}}_s - \underline{\boldsymbol{\nu}}_s - \boldsymbol{\pi}^r_s = 0; \tag{34a}$$
$$L^r(\mathbf{q}^r_s, \underline{\boldsymbol{\nu}}_s, \bar{\boldsymbol{\nu}}_s) = 0; \tag{34b}$$
$$L^f(\mathbf{q}^d, \mathbf{q}^r_s, \underline{\boldsymbol{\mu}}_s, \bar{\boldsymbol{\mu}}_s) = 0, \tag{34c}$$

where:

$$\boldsymbol{\pi}^r_s = \boldsymbol{\pi}(\lambda_s, \underline{\boldsymbol{\mu}}_s, \boldsymbol{\mu}_s). \tag{34d}$$

Step 4: Showing that, at an equilibrium, the production schedule is the unique optimal solution to $\hat{ED}$-$\hat{FR}$. Let $(\mathbf{q}^d, \mathbf{q}^r)$ be an optimal solution to $\hat{ED}(\mathbf{d}_1)$ and $\mathbf{r}^r_s$ be an optimal solution to $\hat{FR}(\mathbf{q}^d, \mathbf{q}^r, \mathbf{d}_s)$. We will show that:

$$\mathbf{q}^d = \left([\alpha^d_n s^d_n(\pi^d_n)]^{\bar{q}^d_n}_{\underline{q}^d_n}, n \in N\right);$$
$$\mathbf{q}^r = \left([\alpha^r_n s^r_n(\pi^r_{1,n})]^{\bar{q}^r_n}_{\underline{q}^r_n}, n \in N\right);$$
$$\mathbf{r}^r_s = \left([\alpha^r_n s^r_n(\pi^r_{s,n})]^{\bar{q}^r_n}_{\underline{q}^r_n} - [\alpha^r_n s^r_n(\pi^r_{1,n})]^{\bar{q}^r_n}_{\underline{q}^r_n}, n \in N\right).$$

It suffices to show that, if $(\mathbf{q}^d, \mathbf{q}^r_1)$ is an optimal solution to (31) and $\mathbf{q}^r_s$ is an optimal solution to (32), then:

$$\mathbf{q}^d = \left([\alpha^d_n s^d_n(\pi^d_n)]^{\bar{q}^d_n}_{\underline{q}^d_n}, n \in N\right); \tag{35}$$
$$\mathbf{q}^r_s = \left([\alpha^r_n s^r_n(\pi^r_{s,n})]^{\bar{q}^r_n}_{\underline{q}^r_n}, n \in N\right). \tag{36}$$

By rewriting (33a) for dispatch generator n, we infer that:

$$q^d_n = \alpha^d_n s^d_n\left(\pi^d_n + \underline{\xi}_n/K - \bar{\xi}_n/K\right).$$

If $\underline{q}^d_n < q^d_n < \bar{q}^d_n$, then from (33b), we infer that $\bar{\xi}_n = \underline{\xi}_n = 0$, which implies that $q^d_n = \alpha^d_n s^d_n(\pi^d_n)$. If $q^d_n = \underline{q}^d_n$, then from (33b), we infer that $\bar{\xi}_n = 0$ and $\underline{\xi}_n \ge 0$, which implies that $\underline{q}^d_n = q^d_n = \alpha^d_n s^d_n(\pi^d_n + \underline{\xi}_n/K) \ge \alpha^d_n s^d_n(\pi^d_n)$, where the last inequality follows from the fact that c^d_n is strictly convex. If $q^d_n = \bar{q}^d_n$, then from (33b), we infer that $\underline{\xi}_n = 0$ and $\bar{\xi}_n \ge 0$, which implies that $\bar{q}^d_n = q^d_n = \alpha^d_n s^d_n(\pi^d_n - \bar{\xi}_n/K) \le \alpha^d_n s^d_n(\pi^d_n)$, where the last inequality follows from the fact that c^d_n is strictly convex. Hence, we conclude that $\mathbf{q}^d$ is given by (35). By making similar arguments, we conclude that $\mathbf{q}^r_s$ is given by (36).

Step 5: Showing that any production schedule $(\mathbf{q}^d, \mathbf{q}^r, \mathbf{r}^r)$ that solves $SYSTEM$ can be obtained using bids $(\boldsymbol{\gamma}^d, \boldsymbol{\gamma}^r)$ and the latter satisfy the characterizations in steps 1 to 3. By Theorem 1, $(\mathbf{q}^d, \mathbf{q}^r)$ is the unique solution to $ED(\mathbf{d}_1)$ and $\mathbf{r}^r_s$ is the unique solution to $FR(\mathbf{q}^d, \mathbf{q}^r, \mathbf{d}_s)$. Under bids $(\boldsymbol{\gamma}^d, \boldsymbol{\gamma}^r)$, the problems $ED(\mathbf{d}_1)$ and $\hat{ED}(\mathbf{d}_1)$ are equivalent. Hence, $(\mathbf{q}^d, \mathbf{q}^r)$ is the unique solution to $\hat{ED}$, and by step 4, the production in the first time period is $(\mathbf{q}^d, \mathbf{q}^r)$. Under bids $(\boldsymbol{\gamma}^d, \boldsymbol{\gamma}^r)$, the problems $FR(\mathbf{q}^d, \mathbf{q}^r, \mathbf{d}_s)$ and $\hat{FR}(\mathbf{q}^d, \mathbf{q}^r, \mathbf{d}_s)$ are equivalent. Hence, $\mathbf{r}^r_s$ is the unique solution to $\hat{FR}(\mathbf{q}^d, \mathbf{q}^r, \mathbf{d}_s)$, and by step 4, the recourse production is $\mathbf{r}^r_s$. Hence, the production schedule is $(\mathbf{q}^d, \mathbf{q}^r, \mathbf{r}^r)$.

It suffices to show that bids $(\boldsymbol{\gamma}^d, \boldsymbol{\gamma}^r)$ constitute an equilibrium. It is easy to check that $\boldsymbol{\alpha}^r = \boldsymbol{\gamma}^r$ and $\boldsymbol{\alpha}^d = \boldsymbol{\gamma}^d$ satisfy conditions (27) and (30) respectively for any prices $(\boldsymbol{\pi}^d, \boldsymbol{\pi}^r)$. Hence, simply choose $(\boldsymbol{\pi}^d, \boldsymbol{\pi}^r)$ based on equations (33) and (34). This proves part (a) of the proposition.

Step 6: Showing that any bids $(\boldsymbol{\alpha}^d, \boldsymbol{\alpha}^r)$ that satisfy the characterizations in steps 1 to 3 give the same dispatch as that under bids $(\boldsymbol{\gamma}^d, \boldsymbol{\gamma}^r)$. Suppose that $(\boldsymbol{\alpha}^d, \boldsymbol{\alpha}^r)$ satisfy the characterizations in step 4 with productions $(\mathbf{q}^d, \mathbf{q}^r_1, \dots, \mathbf{q}^r_S)$, Lagrange multipliers $(\underline{\boldsymbol{\xi}}, \bar{\boldsymbol{\xi}}, \underline{\boldsymbol{\nu}}, \bar{\boldsymbol{\nu}}, \underline{\boldsymbol{\mu}}, \bar{\boldsymbol{\mu}}, \boldsymbol{\lambda})$, and prices $(\boldsymbol{\pi}^d, \boldsymbol{\pi}^r)$. We will construct $\underline{\boldsymbol{\xi}}', \bar{\boldsymbol{\xi}}' \in \mathbb{R}^N_+$ and $\underline{\boldsymbol{\nu}}'_1, \bar{\boldsymbol{\nu}}'_1 \in \mathbb{R}^N_+$ such that:

$$\big(K c_n^{d\prime}(q_n^d), n \in N\big) + \bar{\boldsymbol{\xi}}' - \underline{\boldsymbol{\xi}}' - K\boldsymbol{\pi}^d = 0; \tag{37a}$$

$$L^d(\mathbf{q}^d, \underline{\boldsymbol{\xi}}', \bar{\boldsymbol{\xi}}') = 0; \tag{37b}$$

$$\big(K c_n^{r\prime}(q_{1,n}^r), n \in N\big) + \bar{\boldsymbol{\nu}}'_1 - \underline{\boldsymbol{\nu}}'_1 - K\boldsymbol{\pi}^r_1 = 0; \tag{37c}$$

$$L^r(\mathbf{q}^r_1, \underline{\boldsymbol{\nu}}'_1, \bar{\boldsymbol{\nu}}'_1) = 0, \tag{37d}$$

and $\underline{\boldsymbol{\nu}}'_s, \bar{\boldsymbol{\nu}}'_s \in \mathbb{R}^N_+$ for all $s \in S \setminus \{1\}$ such that:

$$\big(c_n^{r\prime}(q_{s,n}^r), n \in N\big) + \bar{\boldsymbol{\nu}}'_s - \underline{\boldsymbol{\nu}}'_s - \boldsymbol{\pi}^r_s = 0; \tag{38a}$$

$$L^r(\mathbf{q}^r_s, \underline{\boldsymbol{\nu}}'_s, \bar{\boldsymbol{\nu}}'_s) = 0, \tag{38b}$$

which are the KKT conditions for (31) and (32) under bids $(\boldsymbol{\gamma}^d, \boldsymbol{\gamma}^r)$. Then, step 5 allows us to infer that the production schedule is an optimal solution to $SYSTEM$. Our construction is given by:

$$\underline{\xi}'_n = \begin{cases} K\left(c_n^{d\prime}(\underline{q}_n^d) - \pi_n^d\right), & \text{if } q_n^d = \underline{q}_n^d; \\ 0, & \text{else,} \end{cases}$$

$$\bar{\xi}'_n = \begin{cases} K\left(\pi_n^d - c_n^{d\prime}(\bar{q}_n^d)\right), & \text{if } q_n^d = \bar{q}_n^d; \\ 0, & \text{else,} \end{cases}$$

$$\underline{\nu}'_{1,n} = \begin{cases} K\left(c_n^{r\prime}(\underline{q}_n^r) - \pi_{1,n}^r\right), & \text{if } q_{1,n}^r = \underline{q}_n^r; \\ 0, & \text{else,} \end{cases}$$

$$\bar{\nu}'_{1,n} = \begin{cases} K\left(\pi_{1,n}^r - c_n^{r\prime}(\bar{q}_n^r)\right), & \text{if } q_{1,n}^r = \bar{q}_n^r; \\ 0, & \text{else,} \end{cases}$$

and:

$$\underline{\nu}'_{s,n} = \begin{cases} c_n^{r\prime}(\underline{q}_n^r) - \pi_{s,n}^r, & \text{if } q_{s,n}^r = \underline{q}_n^r; \\ 0, & \text{else,} \end{cases}$$

$$\bar{\nu}'_{s,n} = \begin{cases} \pi_{s,n}^r - c_n^{r\prime}(\bar{q}_n^r), & \text{if } q_{s,n}^r = \bar{q}_n^r; \\ 0, & \text{else,} \end{cases}$$

for all $s \in S \setminus \{1\}$.

First, we show that $\underline{\boldsymbol{\xi}}', \bar{\boldsymbol{\xi}}', \underline{\boldsymbol{\nu}}'_s, \bar{\boldsymbol{\nu}}'_s \geq 0$. Suppose $q_n^d = \underline{q}_n^d$. Then, from (30a), we infer that $c_n^{d\prime -1}(\pi_n^d) \leq \underline{q}_n^d$, and since c_n^d is strictly convex, we infer that $\pi_n^d \leq c_n^{d\prime}(\underline{q}_n^d)$, and hence $\underline{\xi}'_n \geq 0$. Suppose $q_n^d = \bar{q}_n^d$. Then, from (30c), we infer that $c_n^{d\prime -1}(\pi_n^d) \geq \bar{q}_n^d$, and since c_n^d is strictly convex, we infer that $\pi_n^d \geq c_n^{d\prime}(\bar{q}_n^d)$, and hence $\bar{\xi}'_n \geq 0$. By similar arguments, we infer that $\underline{\nu}'_{s,n} \geq 0$ and $\bar{\nu}'_{s,n} \geq 0$.

Second, we show that this construction satisfies (37) and (38). It is easy to check that the complementary slackness conditions (37b), (37d), (38b) are satisfied. Suppose $\underline{q}_n^d < c_n^{d\prime -1}(\pi_n^b) < \bar{q}_n^d$. From (30b), we infer that $\alpha_n^d = \gamma_n^d$. From the fact that $q_n^d = \left[\alpha_n^d s_n^d(\pi_n^d)\right]_{\underline{q}_n^d}^{\bar{q}_n^d} = c_n^{d\prime -1}(\pi_n^d)$, we infer that $\underline{q}_n^d < q_n^d < \bar{q}_n^d$. From (33b), we infer that $\underline{\xi}_n = \bar{\xi}_n = 0$. Substituting into (33a), we infer that our construction satisfies (37a). Suppose $c_n^{d\prime -1}(\pi_n^d) \leq \underline{q}_n^d$. From (30a), we infer that $q_n^d = \underline{q}_n^d$. Hence, our construction satisfies (37a). Suppose $\bar{q}_n^d \leq c_n^{d\prime -1}(\pi_n^d)$. From (30c), we infer that $q_n^d = \bar{q}_n^d$. Hence, our construction satisfies (37a). Using similar arguments, we can infer that our construction satisfies (37c) and (38a). This proves part (b) of the proposition. □