High-Voltage Solution in Radial Power Networks: Existence, Properties, and Equivalent Algorithms

Krishnamurthy Dvijotham, *Member, IEEE*, Enrique Mallada, *Member, IEEE*, and John W. Simpson-Porco, *Member, IEEE*

***Abstract*—The ac power flow equations describe the steady-state behavior of the power grid. While many algorithms have been developed to compute solutions to the power flow equations, few theoretical results are available characterizing when such solutions exist, or when these algorithms can be guaranteed to converge. In this letter, we derive necessary and sufficient conditions for the existence and uniqueness of a power flow solution in balanced radial distribution networks with homogeneous (uniform *R/X* ratio) transmission lines. We study three distinct solution methods: 1) fixed point iterations; 2) convex relaxations; and 3) energy functions—we show that the three algorithms successfully find a solution if and only if a solution exists. Moreover, all three algorithms always find the unique high-voltage solution to the power flow equations, the existence of which we formally establish. At this solution, we prove that: 1) voltage magnitudes are increasing functions of the reactive power injections; 2) the solution is a continuous function of the injections; and 3) the solution is the last one to vanish as the system is loaded past the feasibility boundary.**

***Index Terms*—Power systems, smart grid, stability of nonlinear systems.**

I. Introduction

THE AC power flow equations are one of the most widely used modeling tools in power systems. They characterize the steady-state relationship between power injections at each bus and the voltage magnitudes and phase angle that are necessary to transmit power from generators to loads. They are embedded in every power system operations activity, including optimal power flow, state estimation, security/stability assessment, and controller design [1].

Since the power flow equations are nonlinear, solutions may not exist, and even when a solution exists, there may be multiple solutions. The insolvability of these equations may indicate a system-wide instability, such as voltage collapse [2]. Nonetheless, decades of experience show that there is typically a unique "high-voltage" solution [3]. The high-voltage (small-current) solution is typically assumed to be the desired operating point for the network [4], although there are exceptions to this when the system is operated close to voltage collapse [5]. In general then, establishing the existence, uniqueness, and properties of this high-voltage solution is a prerequisite for static and dynamic network analysis.

The second implication of power flow nonlinearity is that — even when solutions exist — finding them can be challenging. While many iterative algorithms (e.g., Newton-Raphson and Gauss-Seidel variants [6]) are effective at solving power flow equations, they lack useful theoretical guarantees. In particular, when these algorithms fail to find a solution, it could be (i) due to an initialization issue, (ii) due to numerical instability, or (iii) that no solution exists to be found. This uncertainty introduces conservatism into system operation: when a perfectly good solution exists, but the solver fails to find it, the operator would mistakenly declare the case to be unsafe.

These limitations of conventional nonlinear equation solvers have spurred the development of new conceptual frameworks for studying power flow equations [6]. Each framework has a body of supporting theory and offers an algorithmic approach for solving the power flow equations. We consider three frameworks:

(i) fixed point approaches, which exploit contraction/monotonicity properties of an operator to iteratively find a power flow solution [7]–[12]; here we propose and study a novel fixed point approach.

(ii) convex relaxations, which cast the problem of finding a power flow solution as a convex optimization problem; a convex super-set containing all power flow solutions is defined, and minimization of a carefully chosen function over this set yields the power flow solution [13]–[16].

Manuscript received March 5, 2017; revised May 20, 2017; accepted June 6, 2017. Date of publication June 20, 2017; date of current version June 30, 2017. The work of K. Dvijotham was supported by the Laboratory Directed Research and Development Project through the Control of Complex Systems Initiative at Pacific Northwest National Laboratory. The work of E. Mallada was supported in part by NSF CPS under Grant CNS 1544771 and in part by Johns Hopkins E2SHI Seed Grant. The work of J. W. Simpson-Porco was supported by the University of Waterloo Startup Grant. Recommended by Senior Editor R. S. Smith. *(Corresponding author: Krishnamurthy Dvijotham.)*

K. Dvijotham is with the Optimization and Control Group, Pacific Northwest National Laboratory, Richland, WA 99354 USA (e-mail: dvij@cs.washington.edu).

E. Mallada is with the Department of Electrical and Computer Engineering, Johns Hopkins University, Baltimore, MD 21218 USA.

J. W. Simpson-Porco is with the Department of Electrical and Computer Engineering, University of Waterloo, Waterloo, ON N2L 3G1, Canada.

Digital Object Identifier 10.1109/LCSYS.2017.2717578

2475-1456 © 2017 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission.
See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.

(iii) energy functions, where the stable power flow solution is characterized as a local minimizer of an appropriately defined scalar function; the function can then be minimized by gradient descent to find the solution [17].[1]

For a given set of power injections, we show that there are only two possibilities. Either

1) there are no power flow solutions, or
2) there is a unique "high voltage" power flow solution,[2] and all three approaches described above find this solution.

The properties we establish (existence of a unique high-voltage/voltage-regular/stable solution) have been conjectured to be true for distribution networks. To the best of our knowledge though, formal proofs of these properties have never been presented. Rigorous proofs are difficult and require special assumptions on the distribution system. In the next subsection, we summarize these assumptions and precisely define the contributions of this letter.

A. Assumptions Made and Contributions of This Letter

We make several assumptions on the distribution network:

(i) *Simplified line model:* the shunt elements typically present in the Π-model are neglected, along with voltage control mechanisms like capacitor banks and tap-changing transformers; all distribution lines are modeled as series impedances.

(ii) *Balanced operation:* the three phases of the system are balanced, allowing for a single-phase representation.

(iii) *Constant r/x ratios of lines longitudinal impedance parameters:* all distribution lines have an equal ratio of resistance to reactance.

(iv) *Strictly positive voltage magnitudes:* we consider power flow solutions with strictly positive voltage magnitudes at every bus.

While these assumptions limit the applicability of our results to practical distribution systems, we leverage them to derive equally strong results:

(i) *Existence conditions:* We establish necessary and sufficient conditions for the existence of solutions. This is in contrast to previous works that only establish sufficient [8], [10], [11] or necessary [18] conditions.

(ii) *Connections between approaches:* We establish connections between the three approaches (fixed point, relaxation, energy function) and show that all three approaches either find the *same high-voltage solution* (if one exists) or fail (if no power flow solution exists). These guarantees are stronger than the guarantees established in previous work on computing power flow solutions [7], [8], [17], [19].

(iii) *Properties of the power flow solution:* We establish several properties of the power flow solution (voltage regularity, stability and continuity). While these properties have been conjectured in previous work [20], [21], we provide rigorous proofs of these from first principles. We envision that the strong results established under the special assumptions in this letter will form the basis of further studies that will extend the results to be applicable to practical distribution networks.

The rest of this letter is organized as follows. Section II introduces the power flow equations. Section III describes three solution algorithms based on fixed-point iterations, convex relaxations, and energy functions. Our main contributions are developed in Section IV where we show the equivalence of the three representations and describe additional properties of the high-voltage solution. Section V concludes this letter.

II. AC Power Flow Equations

We start with notation that is used in this letter:

$\mathbb{C}$:	Set of complex numbers,
$\mathbf{j}$:	$\sqrt{-1}$
$\arg(x)$:	Phase of the complex number x
$\lvert x\rvert$:	Magnitude of the complex number x
$\log(x)$:	The vector with entries $\log(x_i)$ for any $x \in \mathbb{R}^n$
$\exp(x)$:	The vector with entries $\exp(x_i)$ for any $x \in \mathbb{R}^n$
$[a, b]$:	For $a, b \in \mathbb{R}^n$ with $a \leq b$ (componentwise), this denotes $\{x \in \mathbb{R}^n : a \leq x \leq b\}$
Z-matrix:	Square matrix with non-positive off-diagonal entries
e_i:	Vector with all entries 0 except the i-th entry (equal to 1)
$[n]$:	$\{1, 2, \ldots, n\}$ for any natural number n
$\mathbb{1}_n$:	Vector in $\mathbb{R}^n$ with all entries equal to 1

We will focus exclusively on radial (tree) AC power networks in steady-state. The grid topology is that of a rooted oriented tree $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{0, \ldots, n\}$ is the set of buses and (by convention) the lines $(i, k) \in \mathcal{E}$ are oriented away from the substation bus 0. Each line $(i, k) \in \mathcal{E}$ in the network has an associated complex admittance $Y_{ik} = Y_{ki} = G_{ik} - \mathbf{j}B_{ik}$, where $G_{ik} \geq 0$ and $B_{ik} > 0$. We let $\mathcal{N}(i) = \{k : (i, k) \in \mathcal{E} \text{ or } (k, i) \in \mathcal{E}\}$ denote the set of neighbors of bus i.

At every bus $i \in \mathcal{V}$, we denote the voltage phasor by $V_i \in \mathbb{C}$, the squared voltage magnitude by $v_i = |V_i|^2$, the voltage phase by $\theta_i = \arg(V_i)$, (net) active power injection by p_i, and (net) reactive power injection by q_i. Bus 0 is interpreted as a substation, and taken to be the reference (slack) bus with voltage phasor $1\exp(\mathbf{j}0)$ per unit. Every other bus $i \in \{1, \ldots, n\}$ is a PQ bus with p_i, q_i specified and v_i, θ_i to be determined. Under assumptions (i) and (ii) (Section I-A), the power balance equation at bus $i \in \mathcal{V}$ is

$$p_i + \mathbf{j}q_i = V_i \sum_{k \in \mathcal{N}(i)} \overline{Y_{ik}}(\overline{V_i} - \overline{V_k}), \tag{1}$$

where $\overline{x}$ denotes the complex conjugate of x. The voltage phase difference across line $(i, k) \in \mathcal{E}$ is denoted by $\phi_{ik} = \theta_i - \theta_k$. Substituting $V_i = \sqrt{v_i}\exp(\mathbf{j}\theta_i)$ and taking real and imaginary parts of (1), a simple calculation shows that

$$p_i = \mathbf{G}_i v_i + \sum\nolimits_{k \in \mathcal{N}(i)} (B_{ik}s_{ik} - G_{ik}c_{ik}) \tag{2a}$$

$$q_i = \mathbf{B}_i v_i + \sum\nolimits_{k \in \mathcal{N}(i)} (-G_{ik}s_{ik} - B_{ik}c_{ik}) \tag{2b}$$

for each $i \in \mathcal{V} \setminus \{0\}$, where

$$\mathbf{G}_i := \sum\nolimits_{k \in \mathcal{N}(i)} G_{ik} \qquad \mathbf{B}_i := \sum\nolimits_{k \in \mathcal{N}(i)} B_{ik}$$

[1]Note that in this letter we use energy function in a generalized sense so that it may not actually correspond to a Lyapunov function of the dynamics. However, we still abuse terminology to refer to minimizers of the energy function as "stable" solutions.

[2]High-voltage meaning that the voltage magnitude at each bus is higher than the corresponding voltage magnitude of any other solution.

denote the total conductance/susceptance incident to bus i, and we have used the simplifying variables

$$s_{ik} := \sqrt{v_i v_k}\sin(\phi_{ik}), \quad c_{ik} := \sqrt{v_i v_k}\cos(\phi_{ik}). \tag{3}$$

The power flow equations can be written using only the variables c_{ik}, s_{ik} and v_i, with the additional constraints that

$$c_{ik}^2 + s_{ik}^2 = v_i v_k, \quad (i,k) \in \mathcal{E}. \tag{4}$$

The phase differences $\phi = \{\phi_{ik}, (i,k) \in \mathcal{E}\}$ can be recovered uniquely (modulo 2π) via $\sin(\phi_{ik}) = \frac{s_{ik}}{\sqrt{v_i v_k}}$.[3] Since the network is a tree, $|\mathcal{E}| = n$, and once $\theta_0 = 0$ is fixed there is a one-to-one mapping between ϕ and $\theta = \{\theta_i\}_{i=1}^n$, which allows the phase angles θ to also be uniquely recovered. Since all lines are homogeneous (assumption (iv) from Section I-A), $G_{ik}/B_{ik} = \kappa$ some $\kappa > 0$ and for all $(i,k) \in \mathcal{E}$. We create a new system of equations by subtracting κ times (2b) from (2a), and adding κ times (2a) to (2b), to obtain

$$\tilde{p}_i := p_i - \kappa q_i = \sum\nolimits_{k\in\mathcal{N}(i)} \tilde{B}_{ik} s_{ik}, \tag{5a}$$

$$\tilde{q}_i := q_i + \kappa p_i = \tilde{\mathbf{B}}_i v_i - \sum\nolimits_{k\in\mathcal{N}(i)} \tilde{B}_{ik} c_{ik}, \tag{5b}$$

where $\tilde{B}_{ik} := (1+\kappa)B_{ik}$ and $\tilde{\mathbf{B}}_i := \mathbf{B}_i(1+\kappa)$. For notational simplicity, going forward we will drop the $\tilde{}$ and use $B_i, \mathbf{B}_i, p_i, q_i$ instead of $\tilde{B}_i, \tilde{\mathbf{B}}_i, \tilde{p}_i, \tilde{q}_i$. The equation (5a) is a square, full-rank linear system in the variables $s = \{s_{ik}\}$. Thus (5a) can be uniquely solved for the flows s as a function of p, and we denote this solution by $s = s(p)$. The power flow equations (5) with the constraints (4) then simplify to

$$\mathbf{B}_i v_i - \sum\nolimits_{k\in\mathcal{N}(i)} B_{ik} c_{ik} = q_i, \quad \forall i \in [n] \tag{6a}$$

$$s_{ik}(p)^2 + c_{ik}^2 = v_i v_k, \quad \forall (i,k) \in \mathcal{E}. \tag{6b}$$

We will sometimes drop the dependence $s_{ik}(p)$ for brevity and simply write s_{ik}. Let $\mathsf{L} = \mathsf{L}^\mathsf{T} \in \mathbb{R}^{(n+1)\times(n+1)}$ be the *Laplacian matrix* of the (undirected) graph, with entries $\mathsf{L}_{ik} = -B_{ik}$ and $\mathsf{L}_{ii} = \mathbf{B}_i$. We may write L as the block matrix

$$\mathsf{L} = \begin{pmatrix} \mathsf{L}_{00} & \mathsf{L}_{0\mathrm{r}} \\ \mathsf{L}_{\mathrm{r}0} & \mathsf{L}_{\mathrm{red}} \end{pmatrix}$$

where the first row/column corresponds to the substation and $\mathsf{L}_{\mathrm{red}} \in \mathbb{R}^{n\times n}$ is the *reduced Laplacian matrix*. We let $\mathbf{B} = \mathrm{diag}(\mathbf{B}_i)_{i=0}^n$ be the diagonal matrix with entries $\mathbf{B}_i$, and $\mathbf{B}_{\mathrm{red}} = \mathrm{diag}(\mathbf{B}_i)_{i=1}^n$ be the corresponding reduced matrix.

III. Solving the Power Flow Equations

We now describe three approaches for computing solutions to the power flow equations (6).

A. Convex Relaxation Approach

We start with the power flow equations (6) and relax both equations to inequalities:

$$\mathbf{B}_i v_i - \sum_{k\in\mathcal{N}(i)} B_{ik} c_{ik} \le q_i, \quad (s_{ik}(p))^2 + c_{ik}^2 \le v_i v_k, \tag{7}$$

for all $i \in [n]$, and $(i,k) \in \mathcal{E}$, respectively. The inequalities (7) are equivalent to the constraints of the second-order conic relaxation described in [19]. While [19] was the first to formulate this conic relaxation, no guarantees were established on when this approach provably finds a power flow solution. In the context of the optimal power flow problem, [16] describes several results proving tightness of the convex relaxation, that is, that the global optimum of the non-convex OPF problem can be computed by solving a convex relaxation. Here, we are concerned with simply *solving the power flow equations*, and indeed, the results of [16] imply that the power flow equations for radial networks can be solved by solving a convex optimization problem. In this letter, we use the convex relaxation formulation as a tool to establish several properties of the power flow solution for radial networks.

Lemma 1 shows that the relaxation (7) is feasible if and only if there exists a voltage vector v such that

$$\mathbf{B}_i v_i - \sum_{k\in\mathcal{N}(i)} B_{ik}\sqrt{v_i v_k - s_{ik}^2} \le q_i, \quad \forall i \in [n]. \tag{8}$$

We will also refer to the equality form of the above constraint

$$\mathbf{B}_i v_i - \sum_{k\in\mathcal{N}(i)} B_{ik}\sqrt{v_i v_k - s_{ik}^2} = q_i, \quad \forall i \in [n]. \tag{9}$$

We propose computing a power flow solution by maximizing a weighted linear combination of log-voltage magnitudes:

$$\underset{v\in\mathbb{R}^n_{>0}}{\text{maximize}} \sum\nolimits_{i=1}^n w_i \log(v_i) \quad \text{subject to} \quad (8) \tag{10}$$

were $w_i > 0$ are arbitrary positive weights.

Definition 1: The relaxation approach *succeeds* if (10) is feasible and the optimal solution satisfies (9). In this case, a solution to (6) can be computed by taking the optimal solution $v^\star$ and defining $s_{ik}^\star := s_{ik}(p)$, $c_{ik}^\star := \sqrt{v_i v_k - (s_{ik}^\star)^2}$. Otherwise, we say the approach *fails*.

B. Energy Function Approach

Power flow solutions can also be interpreted as stationary points of the *energy function* introduced in [17] and [22]. In the context of transient stability in lossless transmission networks, it is known that the energy function is a Lyapunov function of the power system swing dynamics, and that any power flow solution at which the energy function is locally convex is asymptotically stable. We do not pursue such a stability analysis here, and instead use the energy function in a broader sense — it is simply a function whose stationary points correspond to power flow solutions. We will nonetheless refer to power flow solutions at which the energy function is locally convex (i.e., its Hessian is positive definite) as "stable" solutions. Our main results will establish that such a solution indeed exists, and is unique. Our approach here builds on [17]. The energy function $E : \mathbb{R}^n_{>0} \times \mathbb{R}^n \to \mathbb{R}$ is defined as

$$E(v,\theta) := \sum_{(i,k)\in\mathcal{E}} B_{ik}(v_i + v_k - 2\sqrt{v_i v_k}\cos(\theta_i - \theta_k)) + \sum\nolimits_{i=1}^n p_i\theta_i + \sum\nolimits_{i=1}^n q_i \log(v_i)/2. \tag{11}$$

[3] As mentioned before, we assume $v_i > 0$ for all $i \in \mathcal{V}$. Having $v_i = 0$ for some bus i necessarily implies that no power is being transferred through the lines adjacent to it which implies that no load can exist at that bus.

One may check directly that

$$\frac{\partial E}{\partial \log(v)} = 0 \equiv \text{ (5b)}, \quad \frac{\partial E}{\partial \theta} = 0 \equiv \text{ (5a)}, \tag{12}$$

which shows that power flow solutions are simply stationary points of the energy function.

Theorem 1 [17]: The energy function E is jointly convex in $(\log(v), \theta)$ provided that

$$\sum_{i=1}^{n} \mathbf{B}_i [2v_i]_i - \sum_{(i,k)\in\mathcal{E}} \frac{B_{ik}\sqrt{v_i v_k}}{c_{ik}} \left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right]_{i,k} \succeq 0 \tag{13a}$$

$$c_{ik} = \sqrt{v_i v_k}\cos(\theta_i - \theta_k) > 0 \tag{13b}$$

where

$$[a]_i = a\mathrm{e}_i\mathrm{e}_i^\mathsf{T}, \left[\begin{pmatrix} a & b \\ b & c \end{pmatrix}\right]_{i,k} = a\mathrm{e}_i\mathrm{e}_i^\mathsf{T} + b(\mathrm{e}_i\mathrm{e}_k^\mathsf{T} + \mathrm{e}_k\mathrm{e}_i^\mathsf{T}) + c\mathrm{e}_k\mathrm{e}_k^\mathsf{T}$$

The constraints (13) define a convex set in $(\log(v), \theta)$.[4]

The power flow equations (5) can therefore be solved by solving the following optimization problem:

$$\underset{v,\theta}{\text{minimize}}\ E(v,\theta) \quad \text{subject to} \quad (13). \tag{14}$$

Convexity allows us to draw strong conclusions: if there is a power flow solution in the interior of the domain of the set (13), it is the unique power flow solution in the set (13). Conversely, if the optimizer of (14) is not a solution, there are no solutions to the PF equations in the interior of the set (13).

Definition 2: The energy function approach *succeeds* if there exists an optimizer of (14) that satisfies (12). In this case, a solution to (6) can be computed by taking the optimizer $(v^\star, \theta^\star)$ and defining $s_{ik}^\star = \sqrt{v_i^\star v_k^\star}\sin(\theta_i^\star - \theta_k^\star)$, $c_{ik}^\star = \sqrt{v_i^\star v_k^\star}\cos(\theta_i^\star - \theta_k^\star)$. Otherwise, we say the approach *fails*.

C. Fixed-Point Approach

Simply by rearranging, the power flow equations (9) can be rewritten as fixed-point equations:

$$v_i = g_i(v) := \frac{q_i}{\mathbf{B}_i} + \sum_{k\in\mathcal{N}(i)} \frac{B_{ik}}{\mathbf{B}_i}\sqrt{v_i v_k - s_{ik}^2}, \quad \forall i \in [n].$$

This system can be written in vector form as $v = g(v)$. We can therefore define a fixed-point iteration

$$v^{(i+1)} \leftarrow g(v^{(i)}), \quad v^{(0)} = v^{\max} := \mathbb{1}_n + 2(\mathsf{L}_{\mathrm{red}})^{-1}q. \tag{15}$$

Lemma 2 shows that $v^{\max}$ is an upper bound on the voltage magnitudes of any power flow solution.

Definition 3: The fixed-point approach *succeeds* if the iteration (15) converges to a point $v^\star \in \mathbb{R}^n_{>0}$ satisfying $g(v) = v$. In this case, a solution to (6) can be computed by taking the fixed point $v^\star$ and defining $s_{ik}^\star = s_{ik}(p)$, $c_{ik}^\star = (v_i v_k - (s_{ik}^\star)^2)^{\frac{1}{2}}$. Otherwise, we say the approach *fails*.

[4]The condition $c_{ik} > 0$ is equivalent to requiring that the phase differences between neighboring buses are smaller than $\frac{\pi}{2}$. In Theorem 2, it is shown that this is always the case for the high-voltage power flow solution.

IV. Theoretical Results on PF Approaches and Properties of the PF Solution

For each approach presented in Section III, it is of interest to understand when the approach succeeds and when it fails. The following theorem addresses these questions.

Theorem 2 (Equivalent Power Flow Approaches): The following two statements are equivalent:

(i) the power flow equations (6) have a solution;

(ii) the approaches (10), (14), and (15) succeed.

If either of these equivalent statements are true, all three approaches compute the same power flow solution.

Proof: We begin by proving (i) $\Longrightarrow$ (ii). Suppose that a power flow solution exists. The reduced power flow equations (9) have a solution, and hence $\exists v^a$ such that $g(v^a) = v^a$ and hence $\exists v^a$ such that $g(v^a) \geq v^a$. Lemma 2 shows that $v^a \leq v^{\max}$ and $g(v^{\max}) \leq v^{\max}$. Note that the map g is monotone, since each component of g is a non-decreasing function of each v_i. Further, we have for any $v \in [v^a, v^{\max}]$ that $g(v) \geq g(v^a) \geq v^a$, and that $g(v) \leq g(v^{\max}) \leq v^{\max}$. We invoke Theorem 4 (Appendix) with $a = v^a, b = v^{\max}$, and $F = g$, establishing that the fixed-point iteration (15) converges to a fixed point and in fact converges to a maximal fixed point $v^\star$, such that $v^\star \geq v$ for every

$$v \in \{v \text{ s.t } v^a \leq v \leq v^{\max}, g(v) \geq v\}.$$

a) Proof that the fixed-point approach succeeds: The above argument proves that the fixed point approach succeeds since (15) converges to the PF solution $v^\star$.

b) Proof that the convex relaxation approach succeeds: By scaling each constraint in (10) by $\mathbf{B}_i$ and rearranging, the feasible set of (10) can be written as $g(v) \geq v$. Since there is a power flow solution, (10) is feasible. Given any feasible solution of v' of (10), the argument from the beginning of this proof with $v^a = v'$ establishes that $v^\star \geq v'$. Since the objective of (10) is strictly increasing in each component of v, $v^\star$ must be the optimal solution since $v^\star \geq v'$ for every feasible solution v'. Thus, if there is a power flow solution, (10) has a unique optimizer that satisfies (9) and hence the relaxation approach succeeds.

c) Proof that the energy function approach succeeds: Define a new mapping $\tilde{g} : \mathbb{R}^n \to \mathbb{R}^n$ as $\tilde{g}(\gamma) = \mathbf{B}_{\mathrm{red}}(\exp(\gamma) - g(\exp(\gamma)))$, where $\gamma \in \mathbb{R}^n$. Lemma 3 shows that $\frac{\partial \tilde{g}}{\partial \gamma}$ is symmetric and that the convexity conditions (13) are equivalent to the condition $\frac{\partial \tilde{g}}{\partial \gamma} \succeq 0$ on the Jacobian of $\tilde{g}$. Further, since $\tilde{g}_i(\gamma) = \mathbf{B}_i(\exp(\gamma_i) - g_i(\exp(\gamma_i)))$ and g_i is increasing in each component of γ, $\frac{\partial \tilde{g}_i}{\partial \gamma_k} \leq 0$ for each $k \neq i$. Thus, the matrix $\frac{\partial \tilde{g}}{\partial \gamma}$ is a Z-matrix. We rewrite (10) as an optimization problem where the decision variables are $\gamma = \log(v)$ as follows:

$$\underset{\gamma}{\text{maximize}} \sum_i w_i\gamma_i \quad \text{subject to} \quad \tilde{g}(\gamma) \leq q,$$

and the unique optimal solution (from part (b)) is $\log(v^\star)$. Writing the KKT conditions for this problem establishes existence of a vector $\lambda \geq 0$ such that $\frac{\partial \tilde{g}}{\partial \gamma}\lambda = w$. By Lemma 4, this implies that $\frac{\partial \tilde{g}}{\partial \gamma}$ must be positive definite at the optimal solution. Define $\gamma^\star = \log(v^\star)$ and $\theta^\star$ such that $\theta_i^\star - \theta_k^\star = \arcsin(s_{ik}/\sqrt{v_i^\star v_k^\star})$, with $\theta_0^\star = 0$. The positive definiteness of

$\frac{\partial \tilde{g}}{\partial \gamma}$ at $\gamma = \gamma^\star$ shows that the energy function optimization problem (14) has a stationary point $(\gamma^\star, \theta^\star)$ in the interior of the feasible set. Since the energy function is strongly convex in the interior, this stationary point is an isolated local minimum. Since (14) is a convex problem, this stationary point is the unique global optimum.

Proof of converse (ii) $\Longrightarrow$ (i): Conversely, if there are no power flow solutions, then (10) must be infeasible. If this was not true, then $\exists v^a$ such that $g(v^a) \geq v^a$. Using Lemma 2, $\exists v^{\max}$ such that $g(v^{\max}) \leq v^{\max}$, $v^a \leq v^{\max}$ so that $v^a \leq g(v^a) \leq g(v^{\max}) \leq v^{\max}$. Hence by Theorem 4, there is a fixed point $g(v) = v$ and hence a power flow solution exists (which is a contradiction). Thus, the convex relaxation must be infeasible and the convex relaxation approach cannot succeed. Since g has no fixed points (as there are no power flow solutions) the iterative procedure cannot converge to a fixed point and hence the fixed-point approach fails. Finally, the energy function approach succeeds only if the optimum of (14) is a stationary point of the energy function. Since stationary points are power flow solutions, there are no stationary points and hence the energy function approach cannot succeed. ■

Theorem 2 says that the three approaches from Section III all find a power flow solution if and only if a power flow solution exists, and in fact, they all find the same solution. If any of them fail, then no power flow solution exists. To state our next result, which establishes several desirable properties of the power flow solution, we define some notation. Let

$$\mathcal{S} := \{(p, q) \in \mathbb{R}^n \times \mathbb{R}^n \text{ s.t. } (10) \text{ is feasible}\}$$

be the feasible set of injections. For each $(p, q) \in \mathcal{S}$, let $v(p, q)$ be the unique optimal solution of (10), and define $\theta(p, q) \in \mathbb{R}^{n+1}$ as the unique solution to

$$\theta_i - \theta_k = \arcsin\left(\frac{s_{ik}(p)}{\sqrt{v_i(p, q), v_k(p, q)}}\right) \quad \forall (i, k) \in \mathcal{E}, \theta_0 = 0.$$

Theorem 3 (Properties of High-Voltage Solution): The following statements hold:

(i) $\mathcal{S}$ is a convex set and the map $(p, q) \mapsto (v(p, q), \theta(p, q))$ is continuous on $\mathcal{S}$;

(ii) for any $(p, q) \notin \mathcal{S}$, there are no solutions to (6);

(iii) for any $(p, q) \in \mathcal{S}$, $(v(p, q), \theta(p, q))$ is the *unique solution* to the power flow equations that satisfies the following properties:

High-voltage: $v(p, q) \geq v'$ for every solution v' of (9).

Stability: $(v(p, q), \theta(p, q))$ lies in the domain of convexity of the energy function (i.e., satisfies (13));

Voltage-regularity: the matrix $\frac{\partial v}{\partial q}$ is element-wise positive at the solution.

Proof: **(i):** Since $\sqrt{xy - z^2}$ is a concave function in (x, y, z), the feasible set of (10) is jointly convex in (v, p, q). The injection set $\mathcal{S}$ is simply the projection of this set onto the last two arguments, and is therefore convex. Further, (10) has a strongly concave objective and hence a unique optimum. It follows from standard results in parametric convex optimization [23, Sec. 6.1] that $v(p, q)$ is a continuous map, and hence so is $\theta(p, q)$.

(ii): If $(p, q) \notin \mathcal{S}$, (10) is infeasible and hence there are no power flow solutions.

(iii): The first two properties of the specified solution follow immediately from the proof of Theorem 2. For the third, that same proof showed that

$$\frac{\partial \tilde{g}}{\partial \gamma} = \frac{\partial q}{\partial \log(v)} = \frac{\partial q}{\partial v}\left(\frac{\partial \log(v)}{\partial v}\right)^{-1} = \frac{\partial q}{\partial v}\operatorname{diag}(v) \tag{16}$$

evaluated at the solution is a positive-definite Z-matrix. Since the network is connected, this matrix is irreducible. The result follows by applying Lemma 4. ■

V. Conclusion

We have developed and analyzed several approaches to solving the power flow equations for balanced radial networks with transmission lines characterized by homogeneous ratios of longitudinal impedance parameters. We showed these approaches are equivalent: they all either succeed and find the high-voltage power flow solution, or they all fail and no power flow solution exists to be found. In the former case, we established several desirable properties of the power flow solution found by each method. While some of these approaches were known, this is the first paper to systematically study the connections between these approaches.

These results form a solid foundation for further investigation. Immediate future work will study relaxing the assumptions from Section I-A. Establishing analogous results for meshed power networks remains an open problem.

Appendix

Supporting Lemmas

Lemma 1: There exist $v \in \mathbb{R}^n_{>0}$ and $c \in \mathbb{R}^{|\mathcal{E}|}$ satisfying (7) if and only if there exists $v \in \mathbb{R}^n_{>0}$ satisfying

$$\mathbf{B}_i v_i - \sum_{k \in \mathcal{N}(i)} B_{ik}\sqrt{v_i v_k - s_{ik}^2} \leq q_i, \quad \forall i \in [n]. \tag{17}$$

Proof: The second constraint of (7) implies that

$$-\sqrt{v_i v_k - s_{ik}^2} \leq c_{ik} \leq \sqrt{v_i v_k - s_{ik}^2}.$$

If (7) does not hold for $c_{ik} = \sqrt{v_i v_k - s_{ik}^2}$, it cannot hold for any other value of c_{ik}. We can therefore rewrite the relaxation as $\mathbf{B}_i v_i - \sum_{k \in \mathcal{N}(i)} B_{ik}\sqrt{v_i v_k - s_{ik}^2} \leq q_i$. ■

Lemma 2: Any solution to $g(v) \geq v$ satisfies $v \leq v^{\max}$. Further, we have $g(v^{\max}) \leq v^{\max}$.

Proof: The proof relies on the inequality $\sqrt{v_i v_k - s_{ik}^2} \leq \sqrt{v_i v_k} \leq \frac{v_i + v_k}{2}$. We apply this to g to obtain:

$$\begin{aligned} g_i(v) - v_i &\leq \frac{q_i}{\mathbf{B}_i} + \sum_{k \in \mathcal{N}(i)} \frac{B_{ik}}{\mathbf{B}_i}\frac{v_i + v_k}{2} - v_i \\ &= \frac{q_i + \sum_{k \in \mathcal{N}(i)} \frac{B_{ik}}{2}(v_k - v_i)}{\mathbf{B}_i} \end{aligned}$$

In vector notation, this inequality can be written as

$$g(v) - v \leq (\mathbf{B}_{\text{red}})^{-1}\left(q - \frac{1}{2}(\mathsf{L}_{\text{red}}v + \mathsf{L}_{\text{r0}})\right) = \frac{1}{2}(\mathbf{B}_{\text{red}})^{-1}\mathsf{L}_{\text{red}}(v^{\max} - v) \tag{18}$$

where $A = \mathbf{B}_{\text{red}}^{-1}\mathsf{L}_{\text{red}}/2$ and we used $-\mathsf{L}_{\text{red}}^{-1}\mathsf{L}_{\text{r0}} = \mathbb{1}_n$ and the definition of $v^{\max}$. A is an M-matrix (since L_{red} is a principal submatrix of a Laplacian of a connected graph and hence an M-matrix, $\mathbf{B}_{\text{red}}$ is diagonal matrix with positive entries). By Lemma 4, we find that $v \leq v^{\max}$. Further, plugging in $v = v^{\max}$ in (18), we obtain $g(v^{\max}) \leq v^{\max}$. ■

Lemma 3: Define $\tilde{g}(\gamma) = \mathbf{B}_{\text{red}}(\exp(\gamma) - g(\exp(\gamma)))$. Then $2\frac{\partial \tilde{g}}{\partial \gamma}$ is equal to the matrix in the LHS of (13a) after substituting $v = \exp(\gamma)$ and $c_{ik} = \sqrt{\exp(\gamma_i + \gamma_k) - s_{ik}^2}$.

Proof: Using the definitions of $g, \tilde{g}$, we have

$$\tilde{g}_i(\gamma) = \mathbf{B}_i \exp(\gamma_i) - \sum_k B_{ik}\sqrt{\exp(\gamma_i + \gamma_k) - s_{ik}^2} - q_i.$$

For any $k \neq i$, we have

$$\frac{\partial \tilde{g}_i}{\partial \gamma_k} = -\frac{B_{ik}}{2}\frac{\exp(\gamma_i + \gamma_k)}{\sqrt{\exp(\gamma_i + \gamma_k) - s_{ik}^2}} = -\frac{B_{ik}}{2}\frac{\sqrt{v_i v_k}}{c_{ik}}.$$

For $k = i$, we have

$$\frac{\partial \tilde{g}_i}{\partial \gamma_i} = \mathbf{B}_i \exp(\gamma_i) - \sum_k \frac{B_{ik}}{2}\frac{\exp(\gamma_i + \gamma_k)}{\sqrt{\exp(\gamma_i + \gamma_k) - s_{ik}^2}} = \mathbf{B}_i v_i - \sum_k \frac{B_{ik}}{2}\frac{\sqrt{v_i v_k}}{c_{ik}}.$$

Comparing these values with the LHS of (13a), it is easy to see that the matrix on the LHS of (13a) is simply $2\frac{\partial \tilde{g}}{\partial \gamma}$. ■

Theorem 4 (Knaster-Tarski Theorem [24]/Kantorovich Lemma [25]): Let $F : [a, b] \mapsto [a, b]$ be a continuous map where $a, b \in \mathbb{R}^n$, $a \leq b$ such that F is monotone:

$$F(x) \geq F(x') \quad \forall x, x' \in [a, b],\ x \geq x'.$$

Define $A = \{x \in [a, b] : F(x) \geq x\}$, $B = \{x \in [a, b] : F(x) \leq x\}$. F has a maximal fixed point $x^\star$ ($x \leq x^\star\, \forall x \in A$) and a minimal fixed point $x_\star$ ($x_\star \leq x\, \forall x \in B$). The iteration $x^{(i+1)} \leftarrow F(x^{(i)})$ initialized at $x^{(0)} = b$ converges to $x^\star$.

Lemma 4 [26]: If $A \in \mathbb{R}^{n\times n}$ is an irreducible Z-matrix, the following statements are equivalent: (i) A is an M-matrix; (ii) there exists $x \geq 0$ such that $Ax > 0$; (iii) $-A$ is Hurwitz; (iv) $A^{-1} > 0$ entry-wise; (v) $Ax \geq 0$ entry-wise $\Rightarrow x \geq 0$ entry-wise for all $x \in \mathbb{R}^n$.

Acknowledgment

The authors thank Florian Dorfler for encouraging their work on this problem and Florian Dorfler and Saverio Bolognani for helpful comments and feedback on this letter.

References

[1] J. Machowski, J. W. Bialek, and J. R. Bumby, *Power System Dynamics*, 2nd ed. Hoboken, NJ, USA: Wiley, 2008.

[2] T. Van Cutsem, "Voltage instability: Phenomena, countermeasures, and analysis methods," *Proc. IEEE*, vol. 88, no. 2, pp. 208–227, Feb. 2000.

[3] D. K. Molzahn, D. Mehta, and M. Niemerg, "Toward topologically based upper bounds on the number of power flow solutions," in *Proc. Amer. Control Conf.*, Boston, MA, USA, Jul. 2016, pp. 5927–5932.

[4] F. Dörfler, M. Chertkov, and F. Bullo, "Synchronization in complex oscillator networks and smart grids," *Proc. Nat. Acad. Sci. USA*, vol. 110, no. 6, pp. 2005–2010, 2013.

[5] R. B. Prada and L. J. Souza, "Voltage stability and thermal limit: Constraints on the maximum loading of electrical energy distribution feeders," *IEE Proc. Gener. Transm. Distrib.*, vol. 145, no. 5, pp. 573–577, Sep. 1998.

[6] D. Mehta, D. K. Molzahn, and K. Turitsyn, "Recent advances in computational methods for the power flow equations," in *Proc. Amer. Control Conf.*, Boston, MA, USA, Jul. 2016, pp. 1753–1765.

[7] S. Bolognani and S. Zampieri, "On the existence and linear approximation of the power flow solution in power distribution networks," *IEEE Trans. Power Syst.*, vol. 31, no. 1, pp. 163–172, Jan. 2016.

[8] S. Yu, H. D. Nguyen, and K. S. Turitsyn, "Simple certificate of solvability of power flow equations for distribution systems," in *Proc. IEEE Power Energy Soc. Gener. Meeting*, Denver, CO, USA, Jul. 2015, pp. 1–5.

[9] C. Wang, A. Bernstein, J. Y. Le Boudec, and M. Paolone, "Explicit conditions on existence and uniqueness of load-flow solutions in distribution networks," *IEEE Trans. Smart Grid*, to be published, doi: 10.1109/TSG.2016.2572060.

[10] J. W. Simpson-Porco, "A theory of solvability for lossless power flow equations—Part I: Fixed-point power flow," *IEEE Trans. Control Netw. Syst.*, to be published, doi: 10.1109/TCNS.2017.2711433.

[11] J. W. Simpson-Porco, "A theory of solvability for lossless power flow equations—Part II: Conditions for radial networks," *IEEE Trans. Control Netw. Syst.*, to be published, doi: 10.1109/TCNS.2017.2711859.

[12] K. Dvijotham, S. Low, and M. Chertkov, "Solving the power flow equations: A monotone operator approach," *ArXiv e-prints*, vol. abs/1506.08472, Jun. 2015. [Online]. Available: http://adsabs.harvard.edu/abs/2015arXiv150608472D

[13] L. Gan, N. Li, U. Topcu, and S. Low, "On the exactness of convex relaxation for optimal power flow in tree networks," in *Proc. IEEE Conf. Decis. Control*, Maui, HI, USA, Dec. 2012, pp. 465–471.

[14] S. H. Low, "Convex relaxation of optimal power flow—Part I: Formulations and equivalence," *IEEE Trans. Control Netw. Syst.*, vol. 1, no. 1, pp. 15–27, Mar. 2014.

[15] J. Lavaei, D. Tse, and B. Zhang, "Geometry of power flows and optimization in distribution networks," *IEEE Trans. Power Syst.*, vol. 29, no. 2, pp. 572–583, Mar. 2014.

[16] S. H. Low, "Convex relaxation of optimal power flow—Part II: Exactness," *IEEE Trans. Control Netw. Syst.*, vol. 1, no. 2, pp. 1–13, Jun. 2014.

[17] K. Dvijotham, S. Low, and M. Chertkov, "Convexity of energy-like functions: Theoretical results and applications to power system operations," *ArXiv e-prints*, Jan. 2015. [Online]. Available: http://adsabs.harvard.edu/abs/2015arXiv150104052D

[18] D. K. Molzahn, B. C. Lesieutre, and C. L. DeMarco, "A sufficient condition for power flow insolvability with applications to voltage stability margins," *IEEE Trans. Power Syst.*, vol. 28, no. 3, pp. 2592–2601, Aug. 2013.

[19] R. A. Jabr, "Radial distribution load flow using conic programming," *IEEE Trans. Power Syst.*, vol. 21, no. 3, pp. 1458–1459, Aug. 2006.

[20] H.-D. Chiang and M. E. Baran, "On the existence and uniqueness of load flow solution for radial distribution power networks," *IEEE Trans. Circuits Syst.*, vol. 37, no. 3, pp. 410–416, Mar. 1990.

[21] K. N. Miu and H.-D. Chiang, "Existence, uniqueness, and monotonic properties of the feasible power flow solution for radial three-phase distribution networks," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 47, no. 10, pp. 1502–1514, Oct. 2000.

[22] P. Varaiya, F. F. Wu, and R.-L. Chen, "Direct methods for transient stability analysis of power systems: Recent results," *Proc. IEEE*, vol. 73, no. 12, pp. 1703–1715, Dec. 1985.

[23] *Parameteric Optimization, IEOR 265 Lecture Notes, UC Berkeley*. Accessed on Apr. 3, 2017. [Online]. Available: http://ieor.berkeley.edu/~ieor265/lecture_notes/ieor265_lec11.pdf

[24] A. Tarski, "A lattice-theoretical fixpoint theorem and its applications," *Pac. J. Math.*, vol. 5, no. 2, pp. 285–309, 1955.

[25] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*. Philadelphia, PA, USA: SIAM, 2000.

[26] R. Plemmons, "M-matrix characterizations. I—Nonsingular M-matrices," *Linear Algebra Appl.*, vol. 18, no. 2, pp. 175–188, 1977.