Synchronization of Phase-coupled Oscillators with Arbitrary Topology

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***Abstract*— This paper studies networks of identical phase-coupled oscillators with arbitrary underlying connected graph. By using results from algebraic graph theory, a sufficient condition is obtained which can be used to check equilibrium stability. This condition generalizes existing results and can solve some previously unsolved cases. It also leads to the first sufficient condition on the coupling function with which the system is guaranteed to reach synchronization. Throughout the paper, several examples are used to verify and illustrate the theory. We also correct some mistakes in the existing literature.**

I. Introduction

The model of phase-coupled oscillators has been of considerable interest to different research communities. This is mainly due to its wide application in modeling different systems. Recent examples include the interaction of cells [8], Josephson junction circuits [11], and the coupling of oscillatory neurons [17].

The possible behavior of such systems is complex and can depend on different factors. For example, the intrinsic symmetry of the network can produce multiple limit cycles or equilibria with relatively fixed phases (phase-locked trajectories) [1], which in many cases can be stable [4]. Also, the heterogeneity in the natural oscillation frequency can lead to incoherence [9] or even chaos [14].

One particular interesting question is whether the coupled oscillators will synchronize in phase in the long run. Besides its clear theoretical value, it also has rich applications in practice such as clock synchronization in distributed systems. There has been active research work regarding this question, see e.g., [3], [6], [10], [12]. However, most of them either study special coupling functions or focus on graphs with special symmetries.

This paper studies synchronization of identical phase-coupled oscillators with arbitrary underlying connected graph for a large class of coupling functions. Our main contribution is to develop the negative cut instability condition (Theorem 2). It is a sufficient condition for an equilibrium to be unstable. It holds for arbitrary connected graph and any odd and symmetric coupling function. Analytically, the condition leads to the characterization of a class of coupling functions with which the system is guaranteed to reach synchronization in phase.

The paper is organized as follows. After briefly introducing the model and notation in Section II, we study two motivating examples in Section III. These examples provide basic intuition about the possible complex behavior of networks with phase-coupled oscillators. They also demonstrate some existing results are incorrect and motivate further study. With some facts from algebraic graph theory in section IV-A, we present the negative cut instability theorem in section V-A to check whether an equilibrium is unstable. This then leads to Theorem 6 in section V-B which identifies a large class of coupling functions with which the system always synchronizes in phase. We conclude the paper in Section VI.

II. Model

Consider the set of N oscillators, $\mathcal{N}$, whose state can be represented by phase variables $\varphi_i \in \mathbb{S}^1$ for each oscillator $i \in \mathcal{N}$. In the absence of coupling, we have

$$\dot{\varphi}_i = \omega_i, \ \forall i \in \mathcal{N}.$$

Here, $\mathbb{S}^1$ represents the unit circle, or equivalently the interval $[0, 2\pi]$ with 0 and 2π glued together ($0 \equiv 2\pi$), and $\omega_i = \frac{2\pi}{T_i}$ denotes the natural frequency of oscillation.

We assume that the coupling is weak enough such that the dynamics can still be represented by their phases φ_i in the following way,

$$\dot{\varphi}_i = \omega_i + \sum_{j \in \mathcal{N}_i} f_{ij}(\varphi_j - \varphi_i), \ \forall i \in \mathcal{N}, \tag{1}$$

where f_{ij} is a 2π-periodic function and $\mathcal{N}_i \subset \mathcal{N}$ is the set of “neighboring” oscillators j that are coupled with i. When $f_{ij} = \sin$, (1) yields the classical Kuramoto model [9].

The state space of (1) is the N dimensional torus $\mathcal{T}^N$ which corresponds to the direct sum of N unit circles $\mathbb{S}^1$, i.e. $\varphi = (\varphi_1, \dots, \varphi_N)^T \in \mathcal{T}^N = \mathbb{S}^1_1 \oplus \dots \oplus \mathbb{S}^1_N$.

In this paper, we assume that all oscillators are identical ($\omega_i = \omega \ \forall i$) so that if we subtract ωt from $\varphi_i \ \forall i$, (1) yields

$$\dot{\varphi}_i = \sum_{j \in \mathcal{N}_i} f_{ij}(\varphi_j - \varphi_i). \tag{2}$$

We will concentrate on (2). Clearly, any solution to it can be immediately translated to (1) by adding ωt. For example, if φ^* is an equilibrium of (2), by adding ωt, we obtain a limit cycle in (1).

We are interested in the attracting properties of phase-locked invariant orbits within $\mathcal{T}^N$, which can be represented by

$$\varphi(t) = \omega^* t \mathbf{1}_N + \varphi^*,$$

where $\mathbf{1}_N = (1, \dots, 1)^T \in \mathcal{T}^N$, and φ^* and ω^* are solutions to

$$\omega^* = \sum_{j \in \mathcal{N}_i} f_{ij}(\varphi^*_j - \varphi^*_i), \ \forall i. \tag{3}$$

Whenever the system reaches one of these orbits, we say that it is synchronized or phase-locked, and if all the elements of φ^* are equal, we say the system is synchronized **in-phase** or that it is **in-phase** locked.

It is easy to check that for a given equilibrium φ^* of (2), any solution of the form $\varphi^* + \lambda \mathbf{1}_N$, with $\lambda \in \mathbb{R}$, is also an equilibrium that identifies the same limit cycle. Therefore, two equilibria $\varphi^{1,*}$ and $\varphi^{2,*}$ will be considered to be equivalent, if both identifies the same orbit, or equivalently, if both belongs to the same set of equilibria

$$E_{\varphi^*} := \{\varphi \in \mathcal{T}^N | \varphi = \varphi^* + \lambda \mathbf{1}_N, \lambda \in \mathbb{R}\}.$$

Throughout this paper we concentrate on the class of coupling function f_{ij} with the following characteristics:

Assumption 1 *Properties of* f_{ij}*:*

(a) *Symmetric coupling:* $f_{ij} = f_{ji}\ \forall ij$.
(b) *Odd:* $f_{ij}(-\theta) = -f_{ij}(\theta)$.
(c) C^1*:* f_{ij} *is continuously differentiable.*

To the best of our knowledge, the first attempt to study (2) with fairly general f_{ij} and arbitrary coupling topology was in [10], using previous results in multi-agent control and consensus algorithms [13]. There, they tried to show that whenever f_{ij} is odd and $\theta f_{ij}(\theta) > 0\ \forall \theta \neq 0$ the system globally synchronize in-phase. However, the analysis there was incorrect since the analogy between these two systems is incomplete; the main difference is the periodicity of f_{ij}, which makes $\theta f_{ij}(\theta) > 0\ \forall \theta \neq 0$ impossible and gives (2) other non in-phase equilibria. We will discuss this further with concrete examples in the next section.

III. Motivating Examples

In order to gain insight on the complexity of this problem, and its dependence on the shape of f_{ij} as well as the network topology, we study two examples that illustrate the behaviors not captured in [10].

Example 1 (Two Oscillators) *We first consider two connected oscillators, i.e.,* $\mathcal{N} = \{1, 2\}$. *Let* f *has a shape as in Figure 1 with* $f(0) = f(\pi) = 0$. *In this case* (2) *reduces to*

$$\dot{\varphi}_i = f(\varphi_j - \varphi_i).\ i, j \in \{1, 2\},\ j \neq i. \tag{4}$$

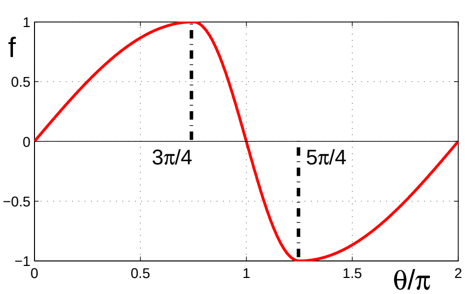


Fig. 1. Plot of function f

Equation (4) *has only two equilibria sets* $E_{\varphi_0^*}$ *and* $E_{\varphi_\pi^*}$*; one in-phase identified by* $\varphi_0^* = \mathbf{1}_N = (1, 1)^T$ *and another in anti-phase identified by* $\varphi_\pi^* = (0, \pi)^T$. *Also, because* $f(\theta) \geq 0$ *for* $\theta \in [0, \pi]$ *and* $f(\theta) \leq 0$ *for* $\theta \in [\pi, 2\pi]$, *for any* $|\varphi_j - \varphi_i| \notin \{0, \pi\}$, *the drift pushes the phases together; see Figure 2. Therefore, unless the initial conditions belong to* $E_{\varphi_\pi^*}$ *both oscillators will always converge to* $E_{\varphi_0^*}$ *and thus synchronize in-phase.*

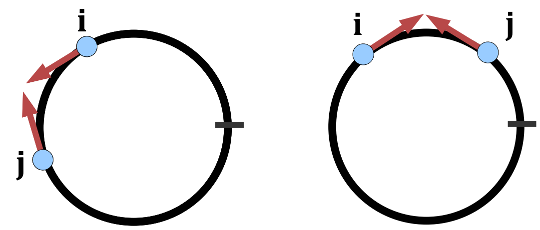
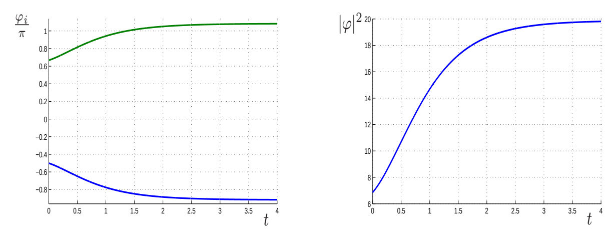


Fig. 2. Attracting drift

However, this implies that Theorem 2 of [10] cannot hold, i.e. the in-phase equilibria set $E_{\varphi_0^*}$ *is* **not** *globally asymptotically stable, since there is a set of initial conditions that do not converge to it. Furthermore, the Lyapunov function used in [10],* $|\varphi|^2$, *does not work in the whole the state space. This can be readily verified in Figure 3(b), where* $|\varphi(t)|^2$ *can increase along trajectories even when the system synchronizes in-phase.*



(a) Phases　　(b) Lyapunov candidate increasing

Fig. 3. A counter example for the Lyapunov function in [10]

The preceding example is still simple in the sense that the anti-phase equilibria set $E_{\varphi_\pi^*}$ is unstable (as it is shown in Section V-A). However, when $N > 2$, new **stable** orbits E_{φ^*} can appear as we show in the next example.

Example 2 (Three oscillators) *Consider now three oscillators coupled all to all with the same function* f *as before. Again, since* f *is odd, each phase locked solution of* (3) *must be an equilibrium (see Corollary 2.2 of [3]). However, due to network symmetry, a new* **stable** E_{φ^*} *appears, with* $\varphi^* = (-\frac{2\pi}{3}, 0, \frac{2\pi}{3})^T$. *Figure 4 illustrates it by showing the trajectories starting close to the set eventually converge to it, which suggests that the equilibrium set* E_{φ^*} *is stable. This example hints that the problem can be very complex. Since as* N *grows, the number of sets* E_{φ^*} *can explode and become difficult to locate for an arbitrary graph.*

In the rest of this paper we progressively show how Assumption 1 with some extra conditions on f_{ij} guarantees in-phase synchronization for arbitrary graph. Since we know that the network can have many other phase-locked trajectories besides the in-phase one, our target is an **almost global stability** result [15], meaning that the set of initial conditions that does not eventually lock in-phase has zero measure.

IV. Preliminaries

In this section we briefly introduce some prerequisites used in our analysis together with some new results that are easy

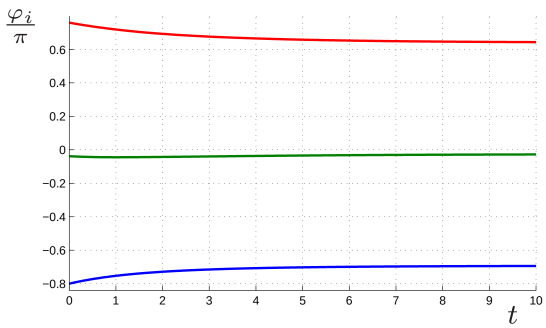


Fig. 4. A stable non-in-phase equilibrium in Example 2

generalization of previous works.

A. Algebraic Graph Theory

We start by reviewing basic definitions and properties from graph theory [2], [5] that are used in next sections. Let G be the connectivity graph that describes the coupling configuration. Each graph is composed by two sets, $V(G)$ and $E(G) \subset V(G) \times V(G)$, which are called vertex set and edge set, respectively. Each individual vertex is represented by i or j, and each edge by either e or the pair ij. If the graph is undirected, then ij and ji represent the same edge. Using this new framework, $\mathcal{N} = V(G)$ and $\mathcal{N}_i$ can be compactly defined as $\mathcal{N}_i = \{j \in V(G)|ij \in E(G)\}$.

An undirected graph G can be directed by giving a specific orientation σ to the elements in the set $E(G)$. That is, for any given edge $e \in E(G)$, we designate one of the vertices to be the *head* and the other to be the *tail*. The outcome of this operation is a directed graph G^σ, induced by G, where $V(G) = V(G^\sigma)$ and for any $ij \in E(G)$, either $ij \in E(G^\sigma)$ or $ji \in E(G^\sigma)$, but not both. We also attach to the convention that if $e = ij \in E(G^\sigma)$ then i is the tail of e and j is its head.

Remark 1 *Although all definitions described from now on implicitly require the graph to be oriented, the properties used in this paper are independent of a particular orientation σ. We therefore drop the superscript σ from G^σ with the understanding that G is now an induced directed graph with some fixed, but arbitrarily chosen, orientation.*

Each set $V(G)$ and $E(G)$ can be associated with a vector space over the real field. The **vertex space** $\mathcal{V}(G)$ is the space of functions that maps $V(G)$ into $\mathbb{R}^{|V(G)|}$, and the **edge space** $\mathcal{E}(G)$ is the analogous for $E(G)$ and $\mathbb{R}^{|E(G)|}$.

An oriented cycle L of the *oriented* graph G consists of an ordered sequence of vertices $i_1 i_2 i_3 \dots i_l$ where $i_1 = i_l$, $i_k \neq i_1$ appears only once for $k \in \{2, 3, ..., l-1\}$ and either $i_k i_{k+1} \in E(G)$ or $i_{k+1} i_k \in E(G)$. Let $E(L)$ denote the set of edges that are in L, if $e = i_k i_{k+1} \in E(L)$, we say that e is oriented as L. Each oriented cycle L determines an element $z_L \in \mathcal{E}(G)$ as follows:

$$z_L(e_i) = \begin{cases} 1 & \text{if } e \in E(L) \text{ is oriented as } L, \\ -1 & \text{if } e \in E(L) \text{ is not oriented as } L, \\ 0 & \text{if } e \notin E(L). \end{cases}$$

If we let $\mathcal{L}(G)$ be the set of all cycles within G, then the subspace $\mathcal{Z}(G)$ of $\mathcal{E}(G)$ spanned by all the vectors $\{z_L\}_{L \in \mathcal{L}(G)}$,

$$\mathcal{Z}(G) := \text{span}\{z_L\}_{L \in \mathcal{L}(G)}$$

is called the **cycle space**.

Now let $P = (V^-, V^+)$ be a partition of the vertex set $V(G)$ such that $V(G) = V^- \cup V^+$ and $V^- \cap V^+ = \emptyset$. The cut $C(P)$ associated with P, or equivalently $C(V^-, V^+)$, is defined as $C(P) := \{ij \in E(G)|i \in V^-, j \in V^+, \text{ or viceversa.}\}$. Again, each partition also determines a vector $c_P \in \mathcal{E}(G)$:

$$c_P(e) = \begin{cases} 1 & \text{if } e \text{ goes from } V^- \text{ to } V^+, \\ -1 & \text{if } e \text{ goes from } V^+ \text{ to } V^-, \\ 0 & \text{if } e \notin E(V^-, V^+). \end{cases}$$

Analogously, the space spanned by all vectors c_P is called the **cut space** and denoted by $\mathcal{C}(G)$. A basic property of $\mathcal{C}(G)$ and $\mathcal{Z}(G)$ is that they are orthogonal complement; i.e. $\mathcal{C}(G) \bigoplus \mathcal{Z}(G) = \mathcal{E}(G)$ and $\mathcal{C}(G) \perp \mathcal{Z}(G)$.

There are several matrices associated with the oriented graph G that embed information about its topology. However, the one with most significance to this work is the *oriented incidence matrix* $B \in \mathbb{R}^{|V(G)| \times |E(G)|}$ where

$$B(i, e) = \begin{cases} 1 & \text{if } i \text{ is the head of } e, \\ -1 & \text{if } i \text{ is the tail } e, \\ 0 & \text{otherwise.} \end{cases}$$

We now list some properties of B that are used in subsequent sections.

(a) The null-space of B is the cycle space $\mathcal{Z}(G)$. i.e. $Bz = 0 \iff z \in \mathcal{Z}(G)$.
(b) The range of B^T is the cut space $\mathcal{C}(G)$, i.e., if $z \in \mathcal{E}(G)$ is equal to $B^T x$ for some $x \in \mathcal{V}(G)$, then $z \in \mathcal{C}(G)$. Or in other words, the column vectors of B^T span $\mathcal{C}(G)$.
(c) If G is connected, then $\ker(B^T) = \text{span}(\mathbf{1}_N)$.

B. Potential Dynamics

In Assumption 1, although f_{ij} being C^1 is standard in order to study local stability and sufficient to apply LaSalle's invariance principle [7], the symmetry and odd assumptions have a stronger effect on the dynamics. For example, under these assumptions the system (2) can be compactly rewritten in a vector form as

$$\dot{\varphi} = -BF(B^T \varphi) \tag{5}$$

where B is the adjacency matrix defined in Section IV-A and the map $F : \mathcal{E}(G) \to \mathcal{E}(G)$ is

$$F(y) = (f_{ij}(y_{ij}))_{ij \in E(G)}.$$

This new representation has several properties. First, from the properties of B one can easily show that (3) can only hold with $\omega^* = 0$ for arbitrary graphs [3] (since $N\omega^* = \omega^* \mathbf{1}_N^T \mathbf{1}_N = -\mathbf{1}_N^T BF(B^T \varphi) = 0$), which implies that every phase-locked solution is an equilibrium of (2) and that every limit cycle of the original system (1) can be represented by some E_φ^* on (2).

Additionally, (5) makes evident the difference between two classes of E_{φ^*}. In the first, φ^* is an equilibrium because $F(B^T\varphi^*) = 0$ and therefore $\dot{\varphi} = -BF(B^T\varphi^*) = -B0 = 0$. However, in the second class $F(B^T\varphi^*) \neq 0$ but $F(B^T\varphi^*) = z \in \mathcal{Z}(G)$, and therefore when $F(B^T\varphi^*)$ is multiplied by B we get $\dot{\varphi} = -BF(B^T\varphi^*) = -Bz = 0$.

However, the most interesting consequence of (5) comes from interpreting $F(y)$ as the gradient of a potential function

$$W(y) = \sum_{ij\in E(G)} \int_0^{y_{ij}} f_{ij}(s)ds.$$

Then, by defining $V(\varphi) = (W \circ B^T)(\varphi) = W(B^T\varphi)$, (5) becomes a gradient descent law for $V(\varphi)$, i.e.,

$$\dot{\varphi} = -BF(B^T\varphi) = -B\nabla W(B^T\varphi) = -\nabla V(\varphi),$$

where in the last step above we used the property $\nabla(W \circ B^T)(\varphi) = B\nabla W(B^T\varphi)$. This makes $V(\varphi)$ a natural Lyapunov function candidate since

$$\dot{V}(\varphi) = \langle \nabla V(\varphi), \dot{\varphi}\rangle = -|\nabla V(\varphi)|^2 = -|\dot{\varphi}|^2 \leq 0.$$

Furthermore, since the trajectories of (5) are constrained into the N-dimensional torus $\mathcal{T}^N$, which is compact, we are ready to apply LaSalle's invariance principle. Therefore, for every initial condition, the trajectory converges to the largest invariant set $M \subset \{\dot{V} \equiv 0\}$.

Finally, since $\dot{V}(\varphi) \equiv 0$ implies $|\dot{\varphi}| \equiv 0$ we conclude that M equals the set of all equilibria $E = \{\varphi \in \mathcal{T}^N | \dot{\varphi} \equiv 0\} = \bigcup_{\varphi^*} E_{\varphi^*}$. So we have proved the following proposition.

Proposition 1 (Global convergence) *The dynamics* (2) *under Assumption 1 converges for* **every initial condition** *to the set of equilibrium points E.* ■

Remark 2 *Proposition 1 is a generalization of the results of [6] where only the Kuramoto model was considered. Clearly, this is not enough to show almost global stability, since it is possible to have other stable phase-locked equilibria sets besides the in-phase one. However, if we are able show that all the non-in-phase equilibria are unstable, then almost global stability follows. That is the focus of the next section.*

V. Main Results

We now present the main results of the paper. Our technique can be viewed as a generalization of [12]. By means of algebraic graph theory, we provide a better stability analysis of the equilibria under a more general framework. We further use the new stability results to characterize f_{ij} that guarantees almost global stability.

A. Local Stability Analysis

Given an equilibrium point φ^*, the first order approximation of (5) around φ^* is

$$\delta\dot{\varphi} = -B\left[\frac{\partial}{\partial y}F(B^T\varphi^*)\right]B^T\delta\varphi,$$

were $\delta\varphi = \varphi - \varphi^*$ is the incremental phase variable, and $\frac{\partial}{\partial y}F(B^T\varphi^*) \in \mathbb{R}^{|E(G)|\times|E(G)|}$ is the Jacobian of $F(y)$ evaluated at $B^T\varphi^*$, i.e.,

$$\frac{\partial}{\partial y}F(B^T\varphi^*) = \mathrm{diag}\left(\{f'_{ij}(\varphi_j^* - \varphi_i^*)\}_{ij\in E(G)}\right).$$

Now let $A = -B\left[\frac{\partial}{\partial y}F(B^T\varphi^*)\right]B^T$ and consider the linear system

$$\delta\dot{\varphi} = A\delta\varphi.$$

Lyapunov's indirect method [7] asserts:

- If there exists an eigenvalue λ of A with $\mathrm{Re}\lambda > 0$ then the equilibrium is unstable.

Although it is possible to numerically calculate the eigenvalues of A given φ^*, here we use the special structure of A to provide a sufficient condition for instability that has nice graph theoretical interpretations.

Theorem 2 (Negative cut instability condition) *Given an equilibrium φ^* of the system* (5)*, with connectivity graph G and f_{ij} satisfying Assumption 1. If there exists a cut $C(P)$ such that the sum*

$$\sum_{ij\in C(P)} f'_{ij}(\varphi_j^* - \varphi_i^*) < 0, \tag{6}$$

the equilibrium φ^* is **unstable.**

Proof: Let $D := \frac{\partial}{\partial y}F(B^T\varphi^*)$. In order to apply the instability criterion of Lyapunov's indirect method, we need to find at least one eigenvalue of BDB^T with $\mathrm{Re}\lambda < 0$. If such eigenvalue existed, it would imply the existence of an eigenvalue of A with $\mathrm{Re}\lambda_A > 0$, and therefore the instability of φ^*.

Since BDB^T is symmetric (recall D is diagonal), it is enough to find some direction $x \in \mathcal{V}(G)$ such that $x^TBDB^Tx < 0$, since that would imply the existence of such negative eigenvalue. Also, since we know that the range of B^T is the cut space $\mathcal{C}(G)$, for any $y \in \mathcal{C}(G)$ there exists an $x \in \mathcal{V}(G)$ such that $y = B^Tx$.

Now let $y = c_P$ for some partition $P = (V^-, V^+)$ as defined in (IV-A) and let $x_P \in \mathcal{V}(G)$ such that $c_P = B^Tx_P$. Note that

$$\sum_{ij\in C(P)} f'_{ij}(\varphi_j^* - \varphi_i^*) = c_P^TDc_P = x_P^TBDB^Tx_P.$$

Therefore, when condition (6) holds, there exists some $x_P \in \mathcal{V}(G)$ with $x_P^TBDB^Tx_P < 0$, which implies that $A = -BDB^T$ has at least one eigenvalue whose real part is **positive**. ■

Remark 3
- *Theorem 2 provides a* **sufficient** *condition for instability; it is not clear what happens when* (6) *does not hold. However, it gives a graph-theoretical interpretation that can be used to provide stability results for general topologies. That is, if the* **minimum cut cost** *is negative, the equilibrium is unstable.*
- *There are several fast algorithms like [16] to find the minimum cut cost of an arbitrary graph that can be used. Thus this Theorem can also provide a computational instability check, alternative to calculating all the eigenvalues of A, which can be computationally demanding for large networks.*

When (6) is specialized to $P = (\{i\}, V(G)\backslash\{i\})$ and $f_{ij}(\theta) = \sin(\theta)$, it reduces to the instability condition in

Lemma 2.3 of [12]; i.e.,

$$\sum_{j\in\mathcal{N}_i} cos(\varphi_j^*-\varphi_i^*)<0. \tag{7}$$

However, (6) has a broader applicability spectrum as the following example shows.

Example 3 *Consider a six oscillators network as in Figure 5, where each node is linked with its four closest neighbors and $f_{ij}(\theta)=\sin(\theta)$. Then, by symmetry, it is easy to verify that*

$$\varphi^*=\left[0,\frac{\pi}{3},\frac{2\pi}{3},\pi,\frac{4\pi}{3},\frac{5\pi}{3}\right]^T \tag{8}$$

is an equilibrium of (2)*.*

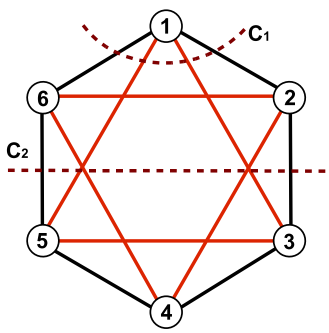


Fig. 5. Network of six oscillators (Example 3)

We first study the stability of φ^* using (7) *as in [12]. By substituting* (8) *in* $\cos(\varphi_j^*-\varphi_i^*)$ $\forall ij\in E(G)$ *we find that the edge weights can only take two values:*

$$\cos(\varphi_j^*-\varphi_i^*)=\begin{cases}\cos(\frac{\pi}{3})=\frac{1}{2}, & \text{if } j=i\pm1 \mod 6\\ \cos(\frac{2\pi}{3})=-\frac{1}{2}, & \text{if } j=i\pm2 \mod 6\end{cases}$$

Then, since any cut that isolates one node from the rest (like $C_1=C(\{1\},V(G)\backslash\{1\})$ in Figure 5) will always have two edges of each type, their sum is **zero***. Therefore,* (7) *cannot be used to determine stability.*

If we now use Theorem 2 instead, we are allowed to explore a wider variety of cuts that can potentially have smaller costs. In fact, if instead of C_1 we sum over $C_2=C(\{1,2,6\},\{3,4,5\})$, we obtain,

$$\sum_{ij\in C_2}\cos(\varphi_j^*-\varphi_i^*)=-1<0,$$

which implies that φ^* is unstable.

Figure 6 verifies the equilibrium instability. By starting with an initial condition $\varphi_0=\varphi^*+\delta\varphi$ close to the equilibrium φ^*, we can see how the system slowly starts to move away from φ^* towards a **stable** *equilibrium set.*

B. Almost Global Stability

Condition (6) also provides insight on which class of coupling functions can potentially give us almost global convergence to the in-phase equilibrium set $E_{\mathbf{1}_N}$. If it is possible to find some f_{ij} with $f'_{ij}(0)>0$, and that for any non-in-phase equilibrium φ^*, there is a cut C with $\sum_{ij\in C}f'_{ij}(\varphi_j^*-\varphi_i^*)<0$, then the in-phase equilibrium set

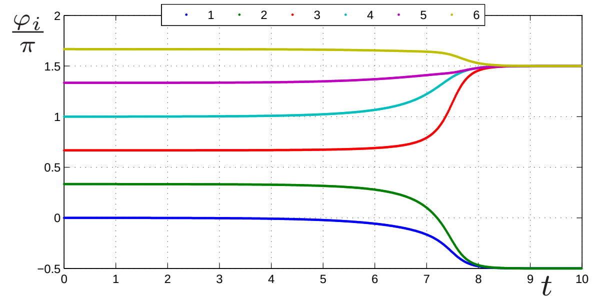


Fig. 6. Unstable equilibrium φ^*. Initial condition $\varphi_0=\varphi^*+\delta\varphi$

will be almost globally stable [4]. The main difficulty is that for general f_{ij} and arbitrary network G, it is not easy to locate every phase-locked equilibria and thus, it is not simple to know in what region of the domain of f_{ij} the slope should be negative.

We now concentrate on the one-parameter family of functions $\mathcal{F}_b$, with $b\in(0,\pi)$ defined by:

- Assumption 1
- $f'_{ij}(\theta;b)>0$, $\forall\theta\in(0,b)\cup(2\pi-b,2\pi)$, and
- $f'_{ij}(\theta;b)<0$, $\forall\theta\in(b,2\pi-b)$.

See Figure 1 for an illustration with $b=\frac{3\pi}{4}$. Also note that this definition implies that if $f_{ij}(\theta;b)\in\mathcal{F}_b$, $f_{ij}(\theta;b)>0$ $\forall\theta\in(0,\pi)$. This property will be used later.

In order to obtain almost global stability we need b to be small. However, since the equilibria position is not known a priori, it is not clear how small b should be or if there is any $b>0$ such that all nontrivial equilibria are unstable. We therefore first need to estimate the region of the state space that contains every non-trivial phase-locked solution.

Let I be a compact connected subset of $\mathbb{S}^1$ and let $l(I)$ be its length, e.g., if $I=\mathbb{S}^1$ then $l(I)=2\pi$. For any $S\subset V(G)$ and $\varphi\in\mathcal{T}^N$, define $I^*(\varphi,S)$ as the smallest interval I such that $\varphi_i\in I$ $\forall i\in S$, i.e.,

$$I^*(\varphi,S)=\underset{I:\varphi_i\in I,\ \forall i\in S}{\operatorname{argmin}}\ l(I).$$

Since S is finite and I is compact, I^* is always well defined. The extremes of I^* represent the two phases of S that are farthest away. Therefore, we can use I^* to define the separation width of the elements of S within $\mathbb{S}^1$ as

$$d(\varphi,S)=l(I^*(\varphi,S)).$$

Direct application Proposition 2.6 of [3] gives the following Lemma:

Lemma 3 *Let $f_{ij}(\cdot;b)\in\mathcal{F}_b$ $\forall ij\in E(G)$ and G be connected. If φ^* is an equilibrium point of* (5) *and $d(\varphi^*,V(G))<\pi$, then it must be the case that φ^* is an in-phase equilibrium, i.e. $\varphi^*=\lambda\mathbf{1}_N$ for $\lambda\in\mathbb{R}$.*

Proof: Suppose φ^* is a **non-in-phase** equilibrium with $d(\varphi^*,V(G))<\pi$. Then, all the phases are strictly contained in a half circle and for the oscillator with smallest phase i_0, all the phase differences $(\varphi_j^*-\varphi_{i_0}^*)\in(0,\pi)$. However, since $f_{ij}(\cdot;b)\in\mathcal{F}_b$ implies $f_{ij}(\theta;b)>0$ $\forall\theta\in(0,\pi)$,

$\dot{\varphi}^*_{i_0} = \sum_{j\in\mathcal{N}_{i_0}} f_{ij}(\varphi_j - \varphi_{i_0}) \neq 0$. Therefore, φ^* cannot be an equilibrium which is a contradiction. ∎

In other words, there is no equilibria, other than the in-phase one, with all the phases within any given half unit circle. We are now ready to establish a bound on the value of b that guarantees the instability of the non-in-phase equilibria.

Lemma 4 *Consider $f_{ij}(\cdot;b) \in \mathcal{F}_b$ $\forall ij \in E(G)$ and arbitrary connected graph G. Then for any $b \leq \frac{\pi}{N-1}$ and non-in-phase equilibrium φ^*, there is a cut C with*

$$f'_{ij}(\varphi^*_j - \varphi^*_i; b) < 0, \forall ij \in C$$

Proof: Suppose there is a non-in-phase equilibrium φ^* for which no such cut C exists. Let $V_0^- = \{i_0\}$ and $V_0^+ = V(G)\backslash\{i_0\}$ be a partition of $V(G)$ for some arbitrary node i_0.

Since such C does not exists, there exists some edge $i_0 j_1 \in C(V_0^-, V_0^+)$, with $j_1 \in V_0^+$, such that $f'_{i_0 j_1}(\varphi^*_{j_1} - \varphi^*_{i_0}; b) > 0$. Move j_1 from V_0^+ to V_0^- and define $V_1^- := V_0^- \cup \{j_1\}$ and $V_1^+ := V_0^+\backslash\{j_1\}$. Now since $f'_{i_0 j_1}(\varphi^*_{j_1} - \varphi^*_{i_0}; b) > 0$, then

$$d(\varphi^*, V_1^-) < b.$$

In other words, both phases should be within a distance smaller than b.

At the k^{th} iteration, given V_{k-1}^-, V_{k-1}^+, again we can find some $i_{k-1} \in V_{k-1}^-$, $j_k \in V_{k-1}^+$ such that $i_{k-1}j_k \in C(V_{k-1}^-, V_{k-1}^+)$ and $f'_{i_{k-1}j_k}(\varphi^*_{j_k} - \varphi^*_{i_{k-1}}; b) > 0$. Also, since at each step $d(\varphi^*, \{i_{k-1}, j_k\}) < b$,

$$d(\varphi^*, V_k^-) < b + d(\varphi^*, V_{k-1}^-).$$

Solving the recursion we get:

$$d(\varphi^*, V_k^-) < kb.$$

Then, after $N-1$ iterations we have $V_{N-1}^- = V(G)$ and $d(\varphi^*, V(G)) < (N-1)b$. Therefore, since $b \leq \frac{\pi}{N-1}$, we obtain

$$d(\varphi^*, V(G)) < (N-1)\frac{\pi}{N-1} = \pi.$$

Then, by Lemma 3 φ^* must be an in-phase equilibrium, which is a contradiction, since we supposed φ^* not to be in-phase. Therefore, for any non-in-phase φ^* and $b \leq \frac{\pi}{N-1}$, we can always find a cut C with $f_{ij}(\varphi^*_j - \varphi^*_i; b) < 0$, $\forall ij \in C$. ∎

Corollary 5 *Consider $f_{ij}(\theta; b) \in \mathcal{F}_b$ and an arbitrary connected graph G. If $b \leq \frac{\pi}{N-1}$, then any non-in-phase equilibrium φ^* is* **unstable**.

We now have all the needed elements to prove the following theorem.

Theorem 6 (Almost global stability) *Consider $f_{ij}(\theta; b) \in \mathcal{F}_b$ and an arbitrary connected graph G. Then, if $b \leq \frac{\pi}{N-1}$, the in-phase equilibrium set $E_{\mathbf{1}_N}$ is* **almost globally asymptotically stable**.

Proof: By Proposition 1, from every initial condition the system (5) converges to the set of equilibria E. Additionally, since $b \leq \frac{\pi}{N-1}$, by Corollary 5 any non-in-phase equilibrium φ^* is **unstable**. So the only possible stable solutions are such $B^T\varphi^* = 0$. Since G is connected, this only holds when $\varphi^* = \lambda\mathbf{1}_N$ which is always some in-phase equilibrium. Therefore, for almost every initial condition, the system (2) converges to the set of in-phase equilibria $E_{\mathbf{1}_N}$ and thus this set is almost globally asymptotically stable. ∎

VI. Conclusion

We have analyzed dynamics of identical phase-coupled oscillators with arbitrary connections. A general condition is developed to check equilibrium instability. We further characterize a large class of coupling functions with which the system is provable to reach in-phase synchronization. We are currently extending this work with the focus on the effect of topology on synchronization. We are also interested in investigating to what degree the odd coupling function assumption can be relaxed while almost global synchronization can still be guaranteed.

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